An Efficient Method for the Evaluation of the Controllability and Observability Gramians of 2-D Digital Filters and Systems

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Abstract—**An efficient and general method for the evaluation of the gramians of causal, stable, 2-D recursive digital filters and systems is proposed. The method is based on a two-stage extension of the Åström–Jury–Agniel algorithm which was originally used for the evaluation of the scalar loss function of a stationary random process with rational spectral density. The new method is compared with other known methods for the evaluation of 2-D gramians with respect to accuracy and computational efficiency. The results obtained show that the new method leads to an accurate evaluation of the 2-D gramians and, furthermore, it requires only a small fraction of the computation required by existing methods.**

I. Introduction

The evaluation of the controllability and observability gramians of digital filters has been a key step in finding optimal digital-filter structures that minimize the output-noise power due to the roundoff of products [1]–[4]. In control theory, on the other hand, these gramians have been extensively used in the analysis and synthesis of linear dynamical systems [5]. For example, they have been used to obtain balanced approximations of one-dimensional (1-D) and two-dimensional (2-D) systems [6], [7], which are very useful in the design of 1-D and 2-D digital filters [8], [9].

It has long been known [5] that the most efficient method for the evaluation of the gramians for the 1-D case is to solve two relevant Lyapunov equations, and reliable algorithms for this purpose are available in the literature [10], [11]. For the 2-D case, the Lyapunov equations depend on a complex parameter, as is demonstrated in [3] and [7], which varies on the unit circle of a complex plane. In other words, should the Lyapunov approach be chosen for the evaluation of the 2-D gramians, one would need to solve a family of 1-D Lyapunov equations as opposed to two constant Lyapunov equations in the 1-D case. A Lyapunov approach for the 2-D case was described in [12] but the transfer function of the digital filters under consideration must have separable denominators. For general, 2-D, causal, stable, recursive digital filters, the most commonly used method is the truncation method described in [3] and [7], which provides numerical approximations of the gramians in terms of truncated double summations of two infinite series. While the method works well for many cases, a large number of terms must be included in the truncated double summations to guarantee acceptable numerical error. This is particularly the case when the filter under consideration has a small stability margin in which case the convergence of the infinite series is rather slow. An alternative method for the evaluation of the 2-D gramians was proposed by Premaratne, Jury, and Mansour in [13]. This method uses a matrix version of the well-known method of computing the infinite sum of the squares of discrete time signals described in [14] in conjunction with an approximation of an integral on the unit circle. Apparently the method is computationally more efficient than the truncation method, but it leads only to an approximate solution of the problem.

In this paper, we propose an efficient and general method for the evaluation of the gramians for the case of 2-D, causal, stable, recursive digital filters. The method is based on a two-stage extension of the Åström–Jury–Agniel (ÅJA) algorithm which was originally used in [15] and [16] for the evaluation of the scalar loss function of a stationary random process with rational spectral density. In Section III, the ÅJA algorithm is first extended to the vector case. It is shown that the modified ÅJA algorithm can be used to solve a 1-D Lyapunov equation in a recursive manner. In Section IV, the recursive algorithm obtained is further extended to the case where the vector rational function involved depends on two complex variables. It is shown that the two algorithms obtained can be combined to evaluate the diagonal block matrices of the 2-D gramians. In Section V, the algorithms presented in Section IV are applied for the computation of a) the entire 2-D gramians and b) the gramians for 2-D digital filters represented by the Fornasini–Marchesini model. In Section VI, the proposed method is compared with other known methods for the evaluation of the 2-D gramians with respect to accuracy and computational efficiency. In addition, it is illustrated by a numerical example. The

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results obtained show that the proposed method leads to an accurate evaluation of the 2-D gramians and, in addition, it requires only a small fraction of the computation required by existing methods.

II. Notation and Definitions

A 1-D digital filter of order n can be represented by the state-space equations

$$x(k+1) = Ax(k) + bu(k) \quad (1)$$
$$y(k) = cx(k) + du(k)$$

where $A \in R^{n\times n}$, $b \in R^{n\times 1}$, $c \in R^{1\times n}$, and $d \in R^{1\times 1}$. If the filter is stable, then the controllability and observability gramians of the filter are defined by

$$K_1 = \sum_{k=0}^{\infty} (A^T)^k bb^T (A^T)^k \quad (2)$$

and

$$W_1 = \sum_{k=0}^{\infty} A^k c^T c A^k \quad (3)$$

respectively. By writing

$$f_1(z) = (zI - A)^{-1} b \quad (4)$$

$$g_1(z) = c(zI - A)^{-1} \quad (5)$$

and applying Parseval's equality, K_1 and W_1 can be expressed as

$$K_1 = \frac{1}{2\pi j} \oint_{|z|=1} f_1(z) f_1^H(z) \frac{dz}{z} \quad (6)$$

$$W_1 = \frac{1}{2\pi j} \oint_{|z|=1} g_1^H(z) g_1(z) \frac{dz}{z} \quad (7)$$

where $f_1^H(z)$ denotes the conjugate transpose of $f_1(z)$. The integral representations of matrices K_1 and W_1 are of importance when an attempt is made to extend the concept of gramians for digital filters from the 1-D to the 2-D case.

A 2-D digital filter of order (n_1, n_2) can be represented by the Roesser state-space equation

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(i,j)$$
$$\equiv Ax + bu \quad (8)$$
$$y(i,j) = [c_1 \quad c_2] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + du(i,j)$$
$$\equiv cx + du$$

where $A_1 \in R^{n_1 \times n_1}$ and $A_4 \in R^{n_2 \times n_2}$. The controllability and observability gramians of the 2-D filter are defined as [3], [7]

$$K_2 = \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_1|=1} f_2(z_1, z_2) f_2^H(z_1, z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \quad (9)$$

and

$$W_2 = \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_1|=1} g_2^H(z_1, z_2) \cdot g_2(z_1 z_2) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \quad (10)$$

respectively, where

$$f_2(z_1, z_2) = [I(z_1, z_2) - A]^{-1} b \quad (11)$$

$$g_2(z_1, z_2) = c[I(z_1, z_2) - A]^{-1} \quad (12)$$

and

$$I(z_1, z_2) = \begin{bmatrix} z_1 I_{n_1} & 0 \\ 0 & z_2 I_{n_2} \end{bmatrix}.$$

III. New Recursive Algorithm for the Solution of 1-D Lyapunov Equations

It follows from (2) and (3) that matrices K_1 and W_1 can be evaluated by solving the 1-D discrete Lyapunov equations

$$AK_1A^T - K_1 = -bb^T \quad (13)$$

$$A^T W_1 A - W_1 = -c^T c. \quad (14)$$

Note that (13) and (14) are linear in K_1 and W_1 and a unique positive-definite solution K_1 (W_1) exists if the filter represented by (1) is a stable and controllable (observable) system.

Several algorithms are available for the solution of (13) and (14) [10], [11]. In this section, we present a new recursive algorithm for the solution of these equations based on the ÅJA algorithm [15], [16]. As will be demonstrated later, the new algorithm is much more efficient than other known algorithms and leads to an accurate solution of the problem at hand. Furthermore, it can readily be extended to the 2-D case and can be applied for the evaluation of the 2-D gramians given in (9) and (10).

In what follows, we concentrate our attention on the evaluation of matrix K_1 but the algorithm obtained can also be used for the evaluation of matrix W_1. If we let

$$f_1(z) = (zI - A)^{-1} b = \frac{n(z)}{d(z)}$$

where

$$n(z) = \operatorname{adj}(zI - A) b = n_0^{(n)} z^{n-1} + \cdots + n_{n-1}^{(n)}, \quad n_i^{(n)} \in R^{n\times 1} \quad (15)$$

$$d(z) = \det(zI - A) = d_0^{(n)} z^n + d_1^{(n)} z^{n-1} + \cdots + d_n^{(n)}, \quad d_0^{(n)} = 1 \quad (16)$$

equation (6) becomes

$$K_1 = \frac{1}{2\pi j}\oint_{|z|=1}\left[\frac{n(z)}{d(z)}\right]\left[\frac{n(z)}{d(z)}\right]^H\frac{dz}{z} = \frac{1}{2\pi j}\oint_{|z|=1}\left[\frac{n(z)}{d(z)}\right]\left[\frac{n(z^{-1})}{d(z^{-1})}\right]^T\frac{dz}{z} \tag{17}$$

where we have used the fact that $\bar{z} = z^{-1}$ for $z \in T = \{z: |z| = 1\}$.

The original ÅJA algorithm provides a recursive formula for the evaluation of the scalar loss function of a stationary stochastic process with a rational spectral density $\phi = |H(e^{j\omega})|^2$. More specifically, it can be used to compute the loss function L defined by

$$L = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(\omega)\,d\omega = \frac{1}{2\pi j}\oint_{|z|=1}H(z)H(z^{-1})\frac{dz}{z}$$

where $H(z)$ is a scalar rational function. On the other hand, the new algorithm evaluates the series of integrals

$$Y_k = \frac{1}{2\pi j}\oint_{|z|=1}\left[\frac{n_k(z)}{d_k(z)}\right]\left[\frac{n_k(z)}{d_k(z)}\right]^H\frac{dz}{z} \tag{18}$$

for $k = 1, 2, \cdots, n$ recursively. Functions $n_k(z)$ and $d_k(z)$ for $k = n, n-1, \cdots, 1, 0$ are vector and scalar polynomials, respectively, defined as

$$n_n(z) = n(z) \tag{19}$$

$$d_n(z) = d(z) \tag{20}$$

$$n_{k-1}(z) = z^{-1}\left[n_k(z) - \xi_k \hat{d}_k(z)\right] \tag{21}$$

$$d_{k-1}(z) = z^{-1}\left[d_k(z) - \eta_k \hat{d}_k(z)\right] \tag{22}$$

where

$$n_k(z) = n_0^{(k)}z^k + n_1^{(k)}z^{k-1} + \cdots + n_k^{(k)} \tag{23}$$

$$d_k(z) = d_0^{(k)}z^k + d_1^{(k)}z^{k-1} + \cdots + d_k^{(k)} \tag{24}$$

$$\hat{d}_k(z) = \bar{d}_0^{(k)} + \bar{d}_1^{(k)}z + \cdots + \bar{d}_k^{(k)}z^k \tag{25}$$

$$\xi_k = n_k^{(k)}/d_0^{(k)} \tag{26}$$

$$\eta_k = d_k^{(k)}/d_0^{(k)} \tag{27}$$

and $\bar{d}$ denotes the complex conjugate of d. Polynomial $\hat{d}_k(z)$ defined in (25) is usually referred to as the reciprocal polynomial of $d_k(z)$.

From (17) and (18) we note that $K_1 = Y_n$ and, therefore, K_1 can be evaluated in a recursive manner through the following steps.

Algorithm 1 (1-D Modified ÅJA Algorithm):

Step 1: Compute $n_n(z) = n(z)$ and $d_n(z) = d(z)$ defined by (15) and (16), respectively.

Step 2: Compute $n_{k-1}(z)$ and $d_{k-1}(z)$ for $k = n, \cdots, 1$ and ξ_k, β_k for $k = n, \cdots, 0$ using (21)–(27).

Step 3: Form

$$Y_0 = \xi_0 \xi_0^H. \tag{28}$$

Step 4: Compute

$$Y_k = \left(1 - |\eta_k|^2\right)Y_{k-1} + \xi_k \xi_k^H \tag{29}$$

for $k = 1, 2, \cdots, n$.

A closed-form solution of the Lyapunov equation in (13) can be obtained as

$$K_1 = \frac{1}{d_0^{(n)}}\sum_{i=0}^{n}\left[n_i^{(i)}(n_i^{(i)})^H\right]/d_0^{(i)} \tag{30}$$

by using (23)–(27), (28), and (29). This is the counterpart of corollary 2.1 of [16, chap. 5, theorem 2.3] when $k = n$.

It is noted that $d(z)$ in (16) is the characteristic polynomial of A and can be evaluated by computing its eigenvalues in conjunction with a simple recursion for calculating the product of n first-order polynomials. Further, the polynomial vector $n(z)$ can be expressed as

$$n(z) = \begin{bmatrix} e_1^T \\ \vdots \\ e_n^T \end{bmatrix} \operatorname{adj}(zI - A)b = \begin{bmatrix} e_1^T \operatorname{adj}(zI - A)b \\ \vdots \\ e_n^T \operatorname{adj}(zI - A)b \end{bmatrix}$$

where e_i is the ith column vector of identify matrix I. Hence the ith entry in $n(z)$ can be evaluated by using the following formula [5]

$$e_i^T \operatorname{adj}(zI - A)b = \det(zI - A + be_i^T) - \det(zI - A).$$

It follows from (25) and (28) that Algorithm 1 is applicable in the case where A and b are complex. Moreover, by repeating the arguments presented in [16, chap. 5, theorem 2.3] it can be shown that Algorithm 1 is valid even for the case where b is a complex matrix of dimension $n \times m$ with $m > 1$. These properties will prove of significant importance when we attempt to extend Algorithm 1 to the 2-D case in the following section.

IV. A Recursive Algorithm for Evaluating the Diagonal Blocks of the 2-D Gramians

The important parts of the 2-D gramians K_2 and W_2 defined by (9) and (10) are their diagonal blocks of dimension $n_1 \times n_1$ and $n_2 \times n_2$, namely, matrices K_{11}, K_{22}, W_{11}, and W_{22} in [3], [7], [9]:

$$K_2 = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^H & K_{22} \end{bmatrix}, \quad W_2 = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^H & W_{22} \end{bmatrix}.$$

In what follows we shall focus our attention on the evaluation of matrix K_{11} but with some straightforward modifications, the recursive algorithm obtained can also be used to evaluate matrix K_{22}. Matrices W_{11} and W_{22} can be evaluated by using the same algorithm in conjunction with the concept of duality [13].

From [7, section 5], we can write matrix K_{11} as

$$K_{11} = \frac{1}{2\pi j}\oint_{|z_2|=1}K_p(z_2)\frac{dz_2}{z_2} \tag{31}$$

where $z_2 \in T_2 = \{z_2 \colon |z_2| = 1\}$ and $K_p(z_2)$ is the unique positive-definite Hermitian solution of the parametric Lyapunov equation

$$A_1(z_2)K_p(z_2)A_1^H(z_2) - K_p(z_2) = -b_1(z_2)b_1^H(z_2), \quad z_2 \in T_2 \quad (32)$$

with

$$A_1(z_2) = A_1 + A_2(z_2 I - A_4)^{-1}A_3 \quad (33)$$

$$b_1(z_2) = b_1 + A_2(z_2 I - A_4)^{-1}b_2. \quad (34)$$

It follows that the evaluation of K_{11} can be carried out by first solving the *parametric* Lyapunov equation in (32) and then computing the complex integral in (31). Since the resulting $K_p(z_2)$ has the form

$$K_p(z_2) = \sum_{k=0}^{n} \tilde{\beta}_k(z_2)\tilde{\beta}_k^H(z_2) \quad (35)$$

where each $\tilde{\beta}_k(z_2)$ is a stable rational function of dimension $n_1 \times 1$ (i.e., its poles are located inside the unit circle $|z_2| = 1$), the evaluation of the complex integral in (31) can readily be carried out by applying Algorithm 1 described in Section III, assuming that $K_p(z_2)$ is known. We shall now develop an algorithm that can be used for the evaluation of $K_p(z_2)$.

The Hermitian solution of equation (32) can be expressed as the complex integral

$$K_p(z_2) = \frac{1}{2\pi j}\oint_{|z_1|=1} [z_1 I - A_1(z_2)]^{-1} b_1(z_2) b_1^H(z_2) \cdot [z_1 I - A_1(z_2)]^{-H} \frac{dz_1}{z_1} \quad (36)$$

where $[z_1 I - A_1(z_2)]^{-H}$ denotes the complex-conjugate transpose of $[z_1 I - A_1(z_2)]^{-1}$. We can write $[z_1 I - A_1(z_2)]^{-1} b_1(z_2)$ in the form

$$[z_1 I - A_1(z_2)]^{-1} b_1(z_2) = \frac{n(z_1, z_2)}{d(z_1, z_2)} \quad (37)$$

where

$$\begin{aligned} n(z_1, z_2) &= \operatorname{adj}[z_1 I - A_1(z_2)] b_1(z_2) \\ &= n_1^{(n)}(z_2) z_1^{n-1} + \cdots \\ &\quad + n_{n-1}^{(n)}(z_2) z_1 + n_n^{(n)}(z_2) \end{aligned} \quad (38)$$

$$\begin{aligned} d(z_1, z_2) &= \det[z_1 I - A_1(z_2)] \\ &= d_0^{(n)}(z_2) z_1^n + \cdots + d_{n-1}^{(n)}(z_2) z_1 \\ &\quad + d_n^{(n)}(z_2), \qquad d_0^{(n)}(z_2) \equiv 1. \end{aligned} \quad (39)$$

Further, we can define vector and scalar polynomials $n_k(z_1, z_2)$ and $d_k(z_1, z_2)$, respectively, for $k = n, n-1, \cdots, 0$ in terms of the recursive relations

$$n_n(z_1, z_2) = n(z_1, z_2) \quad (40)$$

$$d_n(z_1, z_2) = d(z_1, z_2) \quad (41)$$

$$n_{k-1}(z_1, z_2) = z_1^{-1}\left[n_k(z_1, z_2) - \beta_k(z_2)\hat{d}_k(z_1, z_2)\right] \quad (42)$$

$$d_{k-1}(z_1, z_2) = z_1^{-1}\left[d_k(z_1, z_2) - \alpha_k(z_2)\hat{d}_k(z_1, z_2)\right] \quad (43)$$

where

$$n_k(z_1, z_2) = n_0^{(k)}(z_2) z_1^k + \cdots + n_{k-1}^{(k)}(z_2) z_1 + n_k^{(k)}(z_2) \quad (44)$$

$$d_k(z_1, z_2) = d_0^{(k)}(z_2) z_1^k + \cdots + d_{k-1}^{(k)}(z_2) z_1 + d_k^{(k)}(z_2) \quad (45)$$

$$\alpha_k(z_2) = d_k^{(k)}(z_2)/d_0^{(k)}(z_2) \quad (46)$$

$$\beta_k(z_2) = n_k^{(k)}(z_2)/d_0^{(k)}(z_2) \quad (47)$$

and

$$\hat{d}_k(z_1, z_2) = z_1^k d_k\left(z_1^{-1}, \bar{z}_2\right) \quad (48)$$

$$= z_1^k d_k(z_1^{-1}, z_2^{-1}) \text{ for } z_2 \in T_2. \quad (49)$$

As in (25), $\hat{d}_k(z_1, z_2)$ is called the reciprocal polynomial of $d_k(z_1, z_2)$.

Function $K_p(z_2)$ can be determined by evaluating the series of integrals

$$J_k(z_2) = \frac{1}{2\pi j}\oint_{|z_1|=1} \left[\frac{n_k(z_1, z_2)}{d_k(z_1, z_2)}\right]\left[\frac{n_k(z_1, z_2)}{d_k(z_1, z_2)}\right]^H \frac{dz_1}{z_1} \quad (50)$$

for $k = 0, 1, \cdots, n$ and then noting from (36)–(37), (40)–(41), and (50) that $K_p(z_2) = J_n(z_2)$. The steps involved are detailed in Algorithm 2 below.

Algorithm 2 (2-D ÅJA Algorithm):

Step 1: Compute $n_n(z_1, z_2) = n(z_1, z_2)$ and $d_n(z_1, z_2) = d(z_1, z_2)$ defined by (38) and (39).

Step 2: Compute $n_{k-1}(z_1, z_2)$ and $d_{k-1}(z_1, z_2)$ for $k = n, \cdots, 1$ and $\alpha_k(z_2), \beta_k(z_2)$ for $k = n, \cdots, 0$ using (40)–(49).

Step 3: Form

$$J_0(z_2) = \beta_0(z_2)\beta_0^H(z_2). \quad (51)$$

Step 4: Compute

$$J_k(z_2) = \left[1 - |\alpha_k(z_2)|^2\right] J_{k-1}(z_2) + \beta_k(z_2)\beta_k^H(z_2) \quad (52)$$

for $k = 1, 2, \cdots, n$.

The computation of $n(z_1, z_2)$ and $d(z_1, z_2)$ can be carried out by noting that

$$[z_1 I - A_1(z_2)]^{-1} b_1(z_2) = [I_{n_1} \quad 0_{n_1 \times n_2}] \cdot [I(z_1, z_2) - A]^{-1} b.$$

Hence $[z_1 I - A_1(z_2)]^{-1} b_1(z_2)$ can be viewed as the transfer function of a single-input, multi-output 2-D filter with the same matrices A and b as in the original filter but with the matrix c replaced by $[I_{n_1}\ \mathbf{0}_{n_1 \times n_2}]$. Available algorithms [17] can then be used for the evaluation of $n(z_1, z_2)$ and $d(z_1, z_2)$.

The derivation of formulas (51) and (52) is based on the following theorems.

Theorem 1: If $d_0^{(k)}(z_2) > 0$ for $z_2 \in T_2$ where $T_2 = \{z_2\colon |z_2| = 1\}$, then polynomial $d_k(z_1, z_2)$ is stable with respect to (w.r.t.) z_1 for $z_2 \in T_2$ if and only if $d_{k-1}(z_1, z_2)$ is stable w.r.t. z_1 for $z_2 \in T_2$ and $d_0^{(k-1)}(z_2) > 0$ for $z_2 \in T_2$.

Theorem 2: If $d_0^{(n)}(z_2) > 0$ for $z_2 \in T_2$, then $d_n(z_1, z_2)$ is stable w.r.t. z_1 for $z_2 \in T_2$ if and only if $d_0^{(k)}(z_2) > 0$ for $z_2 \in T_2$ and for $k = 0, 1, \cdots, n-1$.

With the 2-D reciprocal polynomial $\hat{d}_k(z_1, z_2)$ given by (48) properly defined, Theorems 1 and 2 can be proved by using the arguments adopted in the proofs of theorems 2.1 and 2.2 of [16, chap. 5, section 2]. Now if we regard the complex variable z_2 in $J_k(z_2)$ as an arbitrarily *fixed* parameter on the unit circle, then $J_k(z_2)$ is very much the same as the integral Y_k defined by (18). Consequently, the argument presented in the proof of theorem 2.3 of [16, chap. 5, section 2] in conjunction with Theorems 1 and 2 leads to formulas (51) and (52).

The following analysis is of usefulness in the computation of the integral in (31). From (51) and (52), we have

$$\begin{aligned}
K_{11} &= \frac{1}{2\pi j} \oint_{|z_2|=1} J_n(z_2) \frac{dz_2}{z_2} \\
&= \frac{1}{2\pi j} \oint_{|z_2|=1} \left[1 - |\alpha_n(z_2)|^2\right] J_{n-1}(z_2) \frac{dz_2}{z_2} \\
&\quad + \frac{1}{2\pi j} \oint_{|z_2|=1} \boldsymbol{\beta}_n(z_2) \boldsymbol{\beta}_n^H(z_2) \frac{dz_2}{z_2} \\
&= \frac{1}{2\pi j} \oint_{|z_2|=1} \left[1 - |\alpha_{n-1}(z_2)|^2\right] \\
&\quad \cdot \left[1 - |\alpha_n(z_2)|^2\right] J_{n-2}(z_2) \frac{dz_2}{z_2} \\
&\quad + \frac{1}{2\pi j} \oint_{|z_2|=1} \left[1 - |\alpha_n(z_2)|^2\right] \\
&\quad \cdot \boldsymbol{\beta}_{n-1}(z_2) \boldsymbol{\beta}_{n-1}^H(z_2) \frac{dz_2}{z_2} \\
&\quad + \frac{1}{2\pi j} \oint_{|z_2|=1} \boldsymbol{\beta}_n(z_2) \boldsymbol{\beta}_n^H(z_2) \frac{dz_2}{z_2} \\
&= \cdots\cdots \\
&= \frac{1}{2\pi j} \sum_{k=0}^{n} \oint_{|z_2|=1} \left[\prod_{l=k+1}^{n+1} \left(1 - |\alpha_l(z_2)|^2\right) \right] \\
&\quad \cdot \boldsymbol{\beta}_k(z_2) \boldsymbol{\beta}_k^H(z_2) \frac{dz_2}{z_2}
\end{aligned} \tag{53}$$

where $\alpha_{n+1}(z_2) \equiv 0$ is assumed. Further notice that as in the proof of theorem 2.1 of [16, chap. 5, section 2, equation (2.17)], it can be shown that

$$|\alpha_k(z_2)| = \left| \frac{d_k^{(k)}(z_2)}{d_0^{(k)}(z_2)} \right| < 1 \text{ for all } z_2 \in T_2 \text{ and } k = 0, 1, \cdots, n. \tag{54}$$

Consequently, the scalar factor

$$\prod_{l=k+1}^{n+1} \left[1 - |\alpha_l(z_2)|^2\right]$$

in (53) is strictly positive for $z_2 \in T_2$ and $k = 0, 1, \cdots, n$, and, therefore, has the spectral factorization [16, chap. 4]

$$\prod_{l=k+1}^{n+1} \left[1 - |\alpha_l(z_2)|^2\right] = r_k(z_2) r_k(z_2^{-1}) \text{ for } k = 0, 1, \cdots, n \tag{55}$$

where $r_k(z_2)$ is a stable rational function. Since $z_2^{-1} = \bar{z}_2$ for $z_2 \in T_2$, (53) and (55) imply that

$$K_{11} = \sum_{k=0}^{n} K_1^{(k)} \tag{56}$$

where

$$K_1^{(k)} = \frac{1}{2\pi j} \oint_{|z_2|=1} \tilde{\boldsymbol{\beta}}_k(z_2) \tilde{\boldsymbol{\beta}}_k^H(z_2) \frac{dz_2}{z_2} \tag{57}$$

$$\tilde{\boldsymbol{\beta}}_k(z_2) = r_k(z_2) \boldsymbol{\beta}_k(z_2). \tag{58}$$

Note that $\tilde{\boldsymbol{\beta}}_k(z_2)$ defined by (58) is a stable rational function of dimension $n_1 \times 1$ and, therefore, Algorithm 1 can be applied to the integral in (57).

In summary, we have developed a two-stage method for the evaluation of the controllability and observability gramians of 2-D digital filters and systems. In the first stage, Algorithm 2 is applied to obtain the positive Hermitian solution of the parametric Lyapunov equation (32), and in the second stage the 1-D spectral factorization technique is used to express the resulting $K_p(z_2)$ in the form of (35) and Algorithm 1 is applied for the evaluation of the integral in (31).

V. Computation of the 2-D Gramians

Although in many applications only the diagonal block matrices of the 2-D gramians are needed [3], [4], [7], [9], [12], the entire 2-D gramian is sometimes required to provide useful information with regard to interrelations between the horizontal and vertical states. In addition, the problem of evaluating the gramians for 2-D digital filters represented by the Fornasini–Marchesini model is sometimes required [18], [19]. This section is devoted to these two issues.

A. Evaluation of K_2 and W_2

By (9) and (11), the controllability gramian can be expressed as

$$K_2 = \frac{1}{2\pi j} \oint_{|z_2|=1} K_p(z_2) \frac{dz_2}{z_2} \tag{31'}$$

where

$$K_p(z_2) = \frac{1}{2\pi j} \oint_{|z_1|=1} [I(z_1, z_2) - A]^{-1} bb^T \cdot [I(z_1, z_2) - A]^{-H} \frac{dz_1}{z_1}. \quad (36')$$

So if we replace (36) and (37) by (36′) and

$$[I(z_1, z_2) - A]^{-1} b = \frac{n(z_1, z_2)}{d(z_1, z_2)} \quad (37')$$

respectively, Algorithm 2 in Section IV can be applied to compute $K_p(z_2)$ as

$$K_p(z_2) = J_n(z_2) = \sum_{k=0}^{n} \tilde{\beta}_k(z_2)\tilde{\beta}_k^H(z_2)$$

and, therefore,

$$K_2 = \sum_{k=0}^{n} \frac{1}{2\pi j} \oint_{|z_2|=1} \tilde{\beta}_k(z_2)\tilde{\beta}_k^H(z_2)\frac{dz_2}{z_2} \quad (56')$$

where each integral can be computed by applying Algorithm 1. On comparing (37′) with (37), it is noted that vector $n(z_1, z_2)$ in (37′) has dimension $n_1 + n_2$ rather than n_1 in (37). Further note that $n(z_1, z_2)$ and $d(z_1, z_2)$ can be found by applying the algorithm in [17] to the transfer function matrix

$$I_{n_1+n_2}[I(z_1, z_2) - A]^{-1} b = \frac{n(z_1, z_2)}{d(z_1, z_2)}.$$

B. Evaluation of 2-D Gramians for Filters Represented by Fornasini–Marchesini State-Space Model

The local Fornasini–Marchesini model of a recursive 2-D digital filter is given by

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\ &\quad + b_1 u(i, j+1) + b_2 u(i+1, j) \\ y(i,j) &= cx(i,j) + du(i,j) \end{aligned}$$

and the associated controllability and observability gramians are defined by [18]

$$K_{fm} = \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_1|=1} f(z_1, z_2) \cdot f^H(z_1, z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \quad (9')$$

$$W_{fm} = \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_1|=1} g(z_1, z_2) \cdot g^H(z_1, z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2} \quad (10')$$

respectively, where

$$\begin{aligned} f(z_1, z_2) &= (I - z_1^{-1}A_1 - z_2^{-1}A_2)^{-1} \cdot (z_1^{-1}b_1 + z_2^{-1}b) \\ g(z_1, z_2) &= c(I - z_1^{-1}A_1 - z_2^{-1}A_2)^{-1}. \end{aligned}$$

Let $z = z_1^{-1}z_2$ and rewrite $f(z_1, z_2)$ as

$$\begin{aligned} f(z_1, z_2) &= \hat{f}(z_1, z) \\ &= (z_1 I - A_1 - z^{-1}A_2)^{-1}(b_1 + z^{-1}b_2) \\ &= I\left[z_1 I - A_1 - I(zI - 0)^{-1}A_2\right]^{-1} \\ &\quad \cdot\left[b_1 + I(zI - 0)^{-1}b_2\right] \\ &= C[I(z_1, z) - A]^{-1} b \end{aligned}$$

where matrices A, b, C in the last equality are associated with the Roesser model with

$$A = \begin{bmatrix} A_1 & I \\ A_2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad C = [I \quad 0].$$

Hence the algorithm in [17] can be used to obtain

$$\hat{f}(z_1, z) = \frac{n(z_1, z)}{d(z_1, z)}. \quad (37'')$$

Further notice that z is on the unit circle whenever both z_1 and z_2 are on the unit circle, and

$$K_{fm} = \frac{1}{(2\pi j)^2} \oint_{|z|=1} \oint_{|z_1|=1} \hat{f}(z_1, z)\hat{f}^H(z_1, z)\frac{dz_1}{z_1}\frac{dz}{z}. \quad (9'')$$

Therefore, Algorithms 2 and 1 can be applied to evaluate K_{fm}.

Likewise, W_{fm} can be written as

$$W_{fm} = \frac{1}{(2\pi j)^2} \oint_{|z|=1} \oint_{|z_1|=1} \hat{g}(z_1, z)\hat{g}^H(z_1, z)\frac{dz_1}{z_1}\frac{dz}{z} \quad (10'')$$

where

$$\begin{aligned} \hat{g}(z_1, z) &= c(z_1 I - A_1 - z^{-1}A_2)^{-1} \\ &= \left[c + 0(zI - 0)^{-1}A_2\right]\left[z_1 I - A_1 \right. \\ &\quad \left. - I(zI - 0)^{-1}A_2\right]^{-1} I \\ &= C[I(z_1, z) - A]^{-1} B \end{aligned}$$

with

$$A = \begin{bmatrix} A_1 & I \\ A_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad C = [c \quad 0].$$

Hence $\hat{g}(z_1, z)$ can be expressed as

$$\hat{g}(z_1, z) = \frac{n(z_1, z)}{d(z_1, z)}$$

by using the algorithm in [17]. Therefore, Algorithms 2 and 1 can be applied to compute W_{fm}.

VI. Computational Issues

The most commonly used approach to the numerical evaluation of K_2 and W_2 has been the truncation method [3], [7], [18], [19]. In this approach, K_2 and W_2 are approximated by the truncated double summations

$$K_2 \approx \sum_{l=0}^{N_1} \sum_{k=0}^{N_2} q(l,k) q^T(l,k) \tag{59a}$$

and

$$W_2 \approx \sum_{l=0}^{N_1} \sum_{k=0}^{N_2} \begin{bmatrix} (A_{lk}^T c^T c A_{lk})_{11} & (A_{lk}^T c^T c A_{l+1,k-1})_{12} \\ (A_{lk}^T c^T c A_{l-1,k+1})_{21} & (A_{lk}^T c^T c A_{lk})_{22} \end{bmatrix} \tag{59b}$$

respectively, where $(\cdot)_{11}$, $(\cdot)_{12}$, $(\cdot)_{21}$, and $(\cdot)_{22}$ are the upper-left, upper-right, lower-left, and lower-right blocks of the matrix involved, respectively,

$$q(l,k) = A_{l-1,k} \begin{bmatrix} b_1 \\ 0 \end{bmatrix} + A_{l,k-1} \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \tag{60}$$

with A_{lk} defined by

$$A_{lk} = A_{10} A_{l-1,k} + A_{01} A_{l,k-1} \tag{61}$$

$$A_{00} = I_{n_1+n_2}, \quad A_{10} = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix},$$

$$A_{01} = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$$

and N_1, N_2 are positive integers sufficiently large to guarantee a small approximation error. A problem with the truncation method is its low computational efficiency. In particular, when the 2-D digital filter under consideration has a small stability margin, the indexes N_1 and N_2 need to be large to obtain a satisfactory approximation. Consequently, a very large amount of computation is required. Another approach is to express K_2 and W_2 as double integrals over the rectangle $R = \{(\theta_1, \theta_2)\colon -\pi \le \theta_1 \le \pi, -\pi \le \theta_2 \le \pi\}$, i.e.,

$$K_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f_2(e^{j\theta_1}, e^{j\theta_2}) \cdot f_2^T(e^{-j\theta_1}, e^{-j\theta_2}) \, d\theta_1 \, d\theta_2$$

$$W_2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} g_2^T(e^{-j\theta_1}, e^{-j\theta_2}) \cdot g_2(e^{j\theta_1}, e^{j\theta_2}) \, d\theta_1 \, d\theta_2$$

and then approximate them by the double summations

$$K_2 \approx \frac{1}{N_1 N_2} \sum_{l=0}^{N_1} \sum_{k=0}^{N_2} f_2(e^{jl\delta_1}, e^{jk\delta_2}) \cdot f_2^T(e^{-jl\delta_1}, e^{-jk\delta_2}) \tag{62}$$

$$W_2 \approx \frac{1}{N_1 N_2} \sum_{l=0}^{N_1} \sum_{k=0}^{N_2} g_2^T(e^{-jl\delta_1}, e^{-jk\delta_2}) \cdot g_2(e^{jl\delta_1}, e^{jk\delta_2}) \tag{63}$$

where $\delta_1 = 2\pi/N_1$, $\delta_2 = 2\pi/N_2$. This method is less sensitive to the stability margin of the filter than the truncation method but it also requires a very large amount of computation since each term at the right-hand side of (62) and (63) involves the inversion of a complex matrix of dimension $(n_1 + n_2) \times (n_1 + n_2)$.

By contrast, the proposed approach provides an *exact* solution to the problem of evaluating matrices K_2 and W_2 with a much improved computation efficiency as compared to the truncation and the numerical integration methods. As will be demonstrated by an illustrative example below, the basic types of operation used in Algorithms 1 and 2 and the 1-D spectral factorization include constant matrix–matrix addition and multiplication, polynomial addition and multiplication, computation of the characteristic polynomial of a square matrix and computation of the roots of an algebraic equation containing one unknown. Furthermore, all these operations can be programmed in terms of numerically reliable subroutines. Multiplication of two 1-D polynomials can be performed by computing the convolution of two finite sequences formed by the coefficients of the polynomials involved, the roots of an algebraic equation can be obtained by computing the eigenvalues of the corresponding companion matrix, and the product of an adjoint matrix and a vector as required in equations (15) and (38) can be formed by using the formula

$$c \operatorname{adj}(zI - A) b = \det[zI - (A - bc)] - \det(zI - A) \tag{64}$$

which requires the computation of two characteristic polynomials.

Example: Let us consider a 2-D, stable, state-space digital filter of order $(n_1, n_2) = (2, 2)$ characterized by (8) with

$$A_1 = \begin{bmatrix} -0.5583 & 0.5825 \\ -0.0558 & 0.0583 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.3744 & 0.7525 \\ -0.0374 & 0.0753 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -0.1185 & -0.0356 \\ -0.0047 & -0.0014 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -0.4527 & -0.1665 \\ 0.1037 & -0.0723 \end{bmatrix}$$

$$b_1 = \begin{bmatrix} 1.0 \\ 1.2 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -1.1 \\ 2.0 \end{bmatrix},$$

$$c_1^T = \begin{bmatrix} 0.5 \\ -1.0 \end{bmatrix}, \quad c_2^T = \begin{bmatrix} -0.5 \\ 2.0 \end{bmatrix}.$$

Following Algorithm 2, we first compute $\boldsymbol{n}(z_1, z_2)$ and $d(z_1, z_2)$ defined by (38) and (39), respectively, where $\boldsymbol{A}_1(z_2)$ and $\boldsymbol{b}_1(z_2)$ are given by (33) and (34). By repeatedly applying (64), we find that

$$\boldsymbol{n}(z_1, z_2) = \boldsymbol{n}_1^{(2)}(z_2)z_1^{n-1} + \boldsymbol{n}_2^{(2)}(z_2)$$
$$d(z_1, z_2) = d_0^{(2)}(z_2)z_1^2 + d_1^{(2)}(z_2)z_1 + d_2^{(2)}(z_2)$$

with

$$\mathbf{n}_2^{(2)}(z_2) = \begin{bmatrix} 0.6407 & 0.3498 & 0.0294 \\ 0.6142 & 0.2776 & 0.0395 \end{bmatrix} \frac{\mathbf{z}_2}{(\Delta(z_2))} \equiv \begin{bmatrix} q_3(z_2)/\Delta(z_2) \\ q_4(z_2)/\Delta(z_2) \end{bmatrix}$$

$$d_0^{(2)}(z_2) = 1$$

$$d_1^{(2)}(z_2) = [0.0500 \quad 0.2204 \quad 0.0332] \frac{\mathbf{z}_2}{(\Delta(z_2))} \equiv \frac{p(z_2)}{\Delta(z_2)}$$

$$d_2^{(2)}(z_2) = 0$$

$$\Delta(z_2) = [1.0000 \quad 0.5250 \quad 0.0500]\mathbf{z}_2$$

$$\mathbf{z}_2 = \begin{bmatrix} z_2^2 & z_2 & 1 \end{bmatrix}^T$$

Using (40)–(49), we compute

$$\alpha_2(z_2) = 0$$
$$\boldsymbol{\beta}_2(z_2) = \boldsymbol{n}_2^{(2)}(z_2)$$
$$\boldsymbol{n}_1(z_1, z_2) = \boldsymbol{n}_0^{(1)}(z_2)z_1 + \boldsymbol{n}_1^{(1)}(z_2)$$
$$d_1(z_1, z_2) = d_0^{(1)}(z_2)z_1 + d_1^{(1)}(z_2)$$

where

$$\boldsymbol{n}_0^{(1)}(z_2) = \mathbf{0}$$

$$\boldsymbol{n}_1^{(1)}(z_2) = \boldsymbol{n}_1^{(2)}(z_2) - \boldsymbol{n}_2^{(2)}(z_2)d_1^{(2)}(\bar{z}_2) = \begin{bmatrix} |\Delta(z_2)|q_1(z_2) - q_3(z_2)p(\bar{z}_2) \\ |\Delta(z_2)|q_2(z_2) - q_4(z_2)p(\bar{z}_2) \end{bmatrix} \cdot \frac{1}{\Delta(z_2)|\Delta(z_2)|}$$

$$d_0^{(1)}(z_2) = 1$$

$$d_1^{(1)}(z_2) = p(z_2)/\Delta(z_2).$$

Hence

$$\alpha_1(z_2) = p(z_2)/\Delta(z_2)$$
$$\boldsymbol{\beta}_1(z_2) = \boldsymbol{n}_1^{(1)}(z_2)$$
$$\boldsymbol{n}_0(z_1, z_2) = \boldsymbol{n}_0^{(0)}(z_2) = -d_1^{(2)}(\bar{z}_2)\boldsymbol{n}_1^{(1)}(z_2)$$

$$d_0(z_1, z_2) = d_0^{(0)}(z_2) = \frac{|\Delta(z_2)|^2 - |p(z_2)|^2}{|\Delta(z_2)|^2}$$

$$\alpha_0(z_2) = 1$$

and

$$\boldsymbol{\beta}_0(z_2) = \boldsymbol{n}_0^{(0)}(z_2)/d_0^{(0)}(z_2).$$

By applying (51) and (46), we have

$$J_2(z_2) = \frac{|\Delta(z_2)|^2}{|\Delta(z_2)|^2 - |p(z_2)|^2}\boldsymbol{\beta}_1(z_2)\boldsymbol{\beta}_1^H(z_2) + \boldsymbol{n}_2^{(2)}(z_2)\left[\boldsymbol{n}_2^{(2)}(z_2)\right]^H \quad (65)$$

where the second term can be written as

$$\boldsymbol{n}_2^{(2)}(z_2)\left[\boldsymbol{n}_2^{(2)}(z_2)\right]^H = \left(\begin{bmatrix} q_3(z_2) \\ q_4(z_2) \end{bmatrix} \frac{1}{\Delta^2(z_2)}\right)\left(\begin{bmatrix} q_3(z_2) \\ q_4(z_2) \end{bmatrix} \frac{1}{\Delta^2(z_2)}\right)^H. \quad (66)$$

On applying Algorithm 1 to (66), we obtain

$$\frac{1}{2\pi j} \oint_{|z_2|=1} \boldsymbol{n}_2^{(2)}(z_2)\left[\boldsymbol{n}_2^{(2)}(z_2)\right]^H \frac{dz_2}{z_2} = \begin{bmatrix} 0.4108 & 0.3925 \\ 0.3925 & 0.3804 \end{bmatrix}. \quad (67)$$

Note that on the unit circle T_2, the spectral factorization of $|\Delta(z_2)|^2 - |p(z_2)|^2$ leads to

$$|\Delta(z_2)|^2 - |p(z_2)|^2 = \Delta(z_2)\Delta(z_2^{-1}) - p(z_2)p(z_2^{-1}) = v(z_2)v(z_2^{-1})$$

where

$$v(z_2) = 0.8637z_2^2 + 0.4807z_2 + 0.0387.$$

Hence on T_2 the first term on the right-hand side of (65) becomes

$$\frac{|\Delta(z_2)|^2}{|\Delta(z_2)|^2 - |p(z_2)|^2}\boldsymbol{\beta}_1(z_2)\boldsymbol{\beta}_1^H(z_2) = \left[\frac{\boldsymbol{q}_{12}(z_2)}{v(z_2)\Delta^2(z_2)}\right]\left[\frac{\boldsymbol{q}_{12}(z_2)}{v(z_2)\Delta^2(z_2)}\right]^H \quad (68)$$

where

$$\boldsymbol{q}_{12}(z_2) = \begin{bmatrix} 0.0287 & 0.5093 & 2.1844 & 3.7150 & 2.2886 & 0.5463 & 0.0393 \\ 0.0396 & 0.5473 & 1.5470 & 1.4374 & 0.5699 & 0.0978 & 0.0058 \end{bmatrix}\tilde{\boldsymbol{Z}}_2$$

$$\tilde{\boldsymbol{Z}}_2 = \begin{bmatrix} z_2^6 & z_2^5 & z_2^4 & z_2^3 & z_2^2 & z_2 & 1 \end{bmatrix}^T$$

and the application of Algorithm 1 to (68) gives

$$\frac{1}{2\pi j} \oint_{|z_2|=1} \frac{|\Delta(z_2)|^2}{|\Delta(z_2)|^2 - |p(z_2)|^2}\boldsymbol{\beta}_1(z_2)\boldsymbol{\beta}_1^H(z_2) = \begin{bmatrix} 5.5320 & 1.3018 \\ 1.3018 & 1.4894 \end{bmatrix}. \quad (69)$$

Equations (53), (65), (66), and (69) now imply that

$$\boldsymbol{K}_{11} = \begin{bmatrix} 5.9428 & 1.6943 \\ 1.6943 & 1.4894 \end{bmatrix}.$$

As pointed out in Section V-A, with several straightforward modifications, Algorithms 2 and 1 can be used to obtain the complete gramians $\boldsymbol{K}_2$ and $\boldsymbol{W}_2$ as

$$\boldsymbol{K}_2 = \left[\begin{array}{cc|cc} 5.9428 & 1.6943 & -0.1963 & -0.0070 \\ 1.6943 & 1.4894 & -0.0196 & -0.0007 \\ \hline -0.1963 & -0.0196 & 1.3363 & -2.2425 \\ -0.0070 & -0.0007 & -2.2425 & 4.0696 \end{array}\right]$$

and

$$\boldsymbol{W}_2 = \left[\begin{array}{cc|cc} 0.3233 & -0.5691 & -0.1714 & 0.9097 \\ -0.5691 & 1.0758 & 0.4672 & -1.9018 \\ \hline -0.1714 & 0.4672 & 0.5282 & -1.0648 \\ 0.9097 & -1.9018 & -1.0648 & 4.1374 \end{array}\right].$$

In order to compare the proposed method with the truncation and integration methods with respect to computational efficiency and accuracy, the three methods were programmed in MATLAB (version 3.5e) on a Sun workstation using double-precision floating-point arithmetic. The total number of floating-point operations (flops) required to compute $\boldsymbol{K}_{ii}$ and $\boldsymbol{W}_{ii}$ $(i = 1, 2)$ and to compute the entire $\boldsymbol{K}_2$ and $\boldsymbol{W}_2$ by the proposed method were found to be 3.82×10^3 and 9.86×10^3, respectively. In order to compute the gramians to an accuracy of 4 significant digits, the truncation method described in (59) required $N_1 = N_2 = 15$ and a total of 7.2×10^4 flops while the integration method described in (62) and (63) required $N_1 = N_2 = 20$ and a total of 2.96×10^5 flops. Note that in these methods, the entire gramians *must* be computed even in the case where matrices $\boldsymbol{K}_{ij}$ and $\boldsymbol{W}_{ij}$ $(i \neq j)$ are not required.

VII. Conclusions

A general and computationally efficient method for the evaluation of the controllability and observability gramians of 2-D digital filters and systems has been proposed. The new method yields a high-accuracy closed-form solution of the problem at hand and, in addition, it requires only a fraction of the computation required by existing methods. In the case of a 2-D digital filter of order (2, 2), the new method required only 5.3 (13.7)% of the computation required by the truncation method and only 1.3 (3.4)% that required by the numerical integration method to compute $\boldsymbol{K}_{ii}$ and $\boldsymbol{W}_{ii}$ $(i = 1, 2)$ $(\boldsymbol{K}_2$ and $\boldsymbol{W}_2)$. The theoretical foundation of the method is the ÅJA algorithm which was originally proposed for the evaluation of scalar loss functions of stationary random processes with rational spectral density. Through a two-stage extension of the ÅJA algorithm, the evaluation of 2-D gramians can be carried out by recursively computing the positive Hermitian solution of the parametric 1-D Lyapunov equation given in (32) and then recursively computing the complex integral in (31) by means of the 1-D spectral factorization technique.

References

[1] C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 551–562, Sept. 1976.

[2] S. Y. Hwang, "Minimum uncorrelated unit noise in state-space digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, pp. 256–262, Aug. 1977.

[3] W.-S. Lu and A. Antoniou, "Synthesis of 2-D state-space fixed-point digital filter structures with minimum roundoff noise," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 965–973, Oct. 1986.

[4] T. Lin, M. Kawamata, and T. Higuchi, "A unified study on the roundoff noise in 2-D state space digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 724–730, July 1986.

[5] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice Hall, 1980.

[6] B. C. Moore, "Principal component analysis in linear systems: Controllability, observability, and model reduction," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 17–32, Feb. 1981.

[7] W.-S. Lu, E. B. Lee, and Q.-T. Zhang, "Balanced approximation of two-dimensional and delay-differential systems," *Int. J. Contr.*, vol. 46, pp. 2199–2218, 1987.

[8] H. Kimura and Y. Honoki, "Balanced approximation of digital FIR filter with linear phase characteristic," in *Proc. Int. Symp. Circuits Syst.*, pp. 283–286, June 1985.

[9] W.-S. Lu, H.-P. Wang, and A. Antoniou, "Design of two-dimensional digital filters using singular-value decomposition and balanced approximation method," *IEEE Trans. Signal Processing*, vol. ASSP-39, pp. 2253–2262, Oct. 1991.

[10] S. J. Hammarling, "Numerical solution of the stable non-negative definite Lyapunov equation," *IMA J. Numer. Anal.*, vol. 2, pp. 303–323, 1982.

[11] A. J. Laub, M. T. Heath, C. C. Paige, and R. C. Ward, "Computation of system balancing transformations and other applications of simultaneous diagonalization algorithms," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 115–122, Feb. 1987.

[12] M. Kawamata and T. Higuchi, "Synthesis of 2-D separable denominator digital filters with minimum roundoff noise and no overflow oscillations," in *Proc. Int. Symp. Circuits Syst.*, pp. 1087–1091, June 1985.

[13] K. Premaratne, E. I. Jury, and M. Mansour, "An algorithm for model reduction of 2-D discrete time systems," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 1116–1132, Sept. 1990.

[14] E. I. Jury, *Inners and Stability of Dynamic Systems*. Robert E. Krieger, 1982.

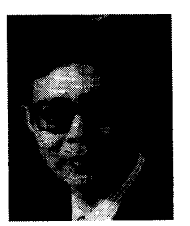
[15] K. J. Åström, E. I. Jury, and R. G. Agniel, "A numerical method for the evaluation of complex integrals," *IEEE Trans. Automat. Contr.*, vol. AC-15, pp. 468–471, Aug. 1970.

[16] K. J. Åström, *Introduction to Stochastic Control Theory*. New York: Academic Press, 1970.

[17] B. G. Mertzios, "An algorithm for the computation of the transfer function matrix of two-dimensional systems," *J. Franklin Inst.*, vol. 321, pp. 73–80, Feb. 1986.

[18] T. Hinamoto, T. Hamanaka, and S. Maekawa, "A generalized study on the synthesis of 2-D state-space digital filters with minimum roundoff noise," *IEEE Trans. Circuits Syst.*, vol. CAS-35, pp. 1037–1042, Aug. 1988.

[19] T. Hinamoto, T. Hamanaka, and S. Maekawa, "Synthesis of 2-D state-space digital filters with low sensitivity based on the Fornasini–Marchesini model," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-38, pp. 1587–1594, Sept. 1990.



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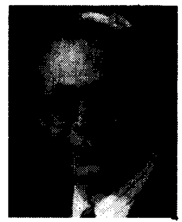

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