Design of digital differentiators satisfying prescribed specifications using optimisation techniques

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Abstract: A procedure is described which can be used to design digital differentiators satisfying prescribed specifications. The procedure, which entails two distinct but similar methods, involves formulating the error between the practical and ideal digital differentiators as a quadratic function which can be readily minimised using standard optimisation algorithms or routines. The new methods are more efficient than methods based on the Remez algorithm, and the designs obtained are more precise or economical than designs based on the window method.

1 Introduction

Digital differentiators can be designed by using the many available numerical differentiation formulas such as the Gregory-Newton forward- and backward-difference formulas or the Bessel, Everett and Stirling central-difference formulas. Alternatively, these discrete systems can be designed as nonrecursive digital filters on the basis of the frequency response of an ideal differentiator. Digital differentiators of this class can be designed by using the equiripple method advanced by Rabiner *et al.* [1], the Fourier-series method in conjunction with the Kaiser window function [2] or the eigenfilter method [3, 4]. In the first approach, the passband error is minimised using the Remez exchange algorithm so as to achieve an equiripple weighted error. In the second approach, the passband error is minimised with respect to a single independent parameter α which is inherent in the Kaiser window function. In the eigenfilter approach, the difference between the actual and desired frequency responses is used to formulate a quadratic error function, and an optimal design is achieved by computing the eigenvector that corresponds to the smallest eigenvalue of the matrix involved. Digital differentiators satisfying prescribed specifications can be designed by using the methods in References 1 or 2 in conjunction with the prediction technique for the differentiator length described in Reference 5.

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This contribution describes an alternative procedure for the design of digital differentiators. The new procedure entails two distinct but similar methods and involves the formulation of a quadratic error function similar to that in Reference 4. The design can readily be accomplished by using standard optimisation algorithms. The new methods are computationally more efficient than the method in Reference 1 and lead to better designs than the methods in References 2 and 5. In addition, the methods can be used in conjunction with a prediction technique for the differentiator length described in Reference 5 to design differentiators satisfying a prescribed specification.

2 Problem formulation

A nonrecursive digital filter of length N can be represented by the transfer function [6]

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

where $h(n)$ is the impulse response of the filter. In the design of digital differentiators the impulse response is required to be antisymmetrical and N can be odd or even.

For odd N, we have

$$h(n) = -h(N-1-n) \quad \text{for } n = 0, 1, \ldots, (N-3)/2$$

and

$$h\left(\frac{N-1}{2}\right) = 0$$

Consequently,

$$\begin{aligned} H(e^{j\omega}) &= M(\omega) e^{j\theta(\omega)} \\ &= e^{-j(N-1)\omega/2} e^{j\pi/2} \sum_{k=1}^{(N-1)/2} c(k) \sin(k\omega) \end{aligned} \tag{1}$$

where

$$M(\omega) = \sum_{k=1}^{(N-1)/2} c(k) \sin(k\omega) \tag{2}$$

$$\theta(\omega) = \frac{\pi}{2} - \frac{(N-1)}{2}\omega \tag{3}$$

and

$$c(k) = 2h[(N-1)/2 - k] \tag{4}$$

On the other hand, for even N, we have

$$h(n) = -h(N-1-n) \quad \text{for } n = 1, \ldots, N/2 - 1$$

and so

$$H(e^{j\omega}) = M(\omega)e^{j\theta(\omega)}$$

$$= e^{-j(N-1)\omega/2} e^{j\pi/2} \sum_{k=1}^{N/2} c(k) \sin [(k - \tfrac{1}{2})\omega] \tag{5}$$

where

$$M(\omega) = \sum_{k=1}^{N/2} c(k) \sin [(k - \tfrac{1}{2})\omega] \tag{6}$$

$$\theta(\omega) = \frac{\pi}{2} - \frac{(N-1)}{2}\omega \tag{7}$$

and

$$c(k) = 2h[N/2 - k] \tag{8}$$

An ideal digital differentiator has a frequency response

$$H_I(e^{j\omega}) = M_I(\omega)e^{j\theta_I(\omega)} = j\omega \quad \text{for } 0 \leqslant \omega \leqslant \omega_s/2 \tag{9}$$

where

$$M_I(\omega) = \omega$$

$$\theta_I(\omega) = \pi/2 \tag{10}$$

and $\omega_s = 2\pi f_s$; f_s is the sampling rate in samples/s. On comparing eqns. 3 and 7 with eqn. 10, we note that an antisymmetrical impulse response yields the required phase shift except for the phase-shift component $(N-1)\omega/2$ which represents a constant delay. If coefficients $c(k)$ in eqns. 1 or 5 are chosen such that the error function

$$e(\boldsymbol{c}) = \int_0^{\omega_s/2} [W(\omega)E(\omega)]^2 \, d\omega \tag{11}$$

is minimised, where $W(\omega)$ is a weighting function,

$$E(\omega) = M(\omega) - M_I(\omega)$$

and

$$\boldsymbol{c} = \begin{cases} \left[c(1)\; c(2) \cdots c\left(\dfrac{N-1}{2}\right) \right]^T & N \text{ odd} \\ \left[c(1)\; c(2) \cdots c\left(\dfrac{N}{2}\right) \right]^T & N \text{ even} \end{cases}$$

then a digital differentiator can be designed.

In practice, a digital differentiator is required to have a specific degree of accuracy over a specific range of frequencies, depending on the application and the spectrum of the signal to be processed. Hence the main specifications of a differentiator are its bandwidth ω_p and the maximum passband error δ. The weighting function $W(\omega)$ can be used to emphasise the error function in one or more frequency bands so as to achieve lower error in these bands. In order to minimise the relative error defined as $E(\omega)/M_I(\omega)$, the weighting function [7]

$$W(\omega) = 1/\omega \quad \text{for } 0 \leqslant \omega \leqslant \omega_p \tag{12}$$

can be used in eqn. 11. Two design methods are possible depending on whether an approximate or exact evaluation of $e(\boldsymbol{c})$ is utilised.

2.1 Method 1

The error function $e(\boldsymbol{c})$ can be minimised by minimising the function

$$e_1(\boldsymbol{c}) = \sum_{k=1}^{m} [W(\omega_k)E(\omega_k)]^2 \tag{13}$$

where $0 \leqslant \omega_k \leqslant \omega_p$ and m is the number of points at which the error is sampled. For $\omega_k > 0$

$$E(\omega_k) = M(\omega_k) - \omega_k \tag{14}$$

and if the weighting function in eqn. 12 is used, eqns. 13 and 14 yield

$$e_1(\boldsymbol{c}) = \sum_{k=1}^{m} \left[\frac{M(\omega_k)}{\omega_k} - 1 \right]^2$$

Using matrix notation, the error function may be written as

$$e_1(\boldsymbol{c}) = \sum_{k=1}^{m} \left[\frac{\boldsymbol{c}^T \boldsymbol{p}_k}{\omega_k} - 1 \right]^2$$

where $\boldsymbol{p}_k$ is the vector given by

$$\boldsymbol{p}_k = \begin{cases} \left[\sin(\omega_k) \sin(2\omega_k) \cdots \sin\left(\dfrac{(N-1)}{2}\omega_k\right) \right]^T & N \text{ odd} \\ \left[\sin(\tfrac{1}{2}\omega_k) \sin(\tfrac{3}{2}\omega_k) \cdots \sin\left(\dfrac{(N-1)}{2}\omega_k\right) \right]^T & N \text{ even} \end{cases}$$

Thus

$$e_1(\boldsymbol{c}) = \sum_{k=1}^{m} (\boldsymbol{c}^T \boldsymbol{s}_k \boldsymbol{s}_k^T \boldsymbol{c} - 2\boldsymbol{c}^T \boldsymbol{s}_k) + m \tag{15}$$

where $\boldsymbol{s}_k = \boldsymbol{p}_k/\omega_k$. Evidently, $e_1(\boldsymbol{c})$ can be minimised by minimising the modified error function

$$\bar{e}_1(\boldsymbol{c}) = \sum_{k=1}^{m} (\boldsymbol{c}^T \boldsymbol{s}_k \boldsymbol{s}_k^T \boldsymbol{c} - 2\boldsymbol{c}^T \boldsymbol{s}_k) \tag{16}$$

It can readily be seen that the function in eqn. 16 represents a quadratic function of the form [8]

$$f(\boldsymbol{x}) = \tfrac{1}{2}\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} - \boldsymbol{x}^T \boldsymbol{b} \tag{17}$$

where

$$\boldsymbol{Q} = 2 \sum_{k=1}^{m} \boldsymbol{s}_k \boldsymbol{s}_k^T \tag{18}$$

$$\boldsymbol{b} = 2 \sum_{k=1}^{m} \boldsymbol{s}_k \tag{19}$$

and

$$\boldsymbol{x} = \boldsymbol{c}$$

It follows from eqn. 18 that matrix $\boldsymbol{Q}$ is always positive semidefinite. It was found by experiment that if $m > N/2$, $\boldsymbol{Q}$ becomes positive definite. Thus $f(\boldsymbol{x})$ defined by eqn. 17 is a convex function and therefore has a strong global minimum. If N is small ($N < 15$), $\bar{e}_1(\boldsymbol{c})$ can be minimised exactly by obtaining the inverse of $\boldsymbol{Q}$ and then multiplying it by vector $\boldsymbol{b}$. On the other hand, if N is large $e_1(\boldsymbol{c})$ can be minimised in at most $N/2$ iterations by using the well known conjugate gradient unconstrained optimisation method [8]. These two algorithms are as follows:

Algorithm 1:

(i) Evaluate matrix $\boldsymbol{Q}$ in eqn. 18
(ii) Compute $\boldsymbol{Q}^{-1}$
(iii) Obtain the solution as $\boldsymbol{c} = \boldsymbol{Q}^{-1}\boldsymbol{b}$.

Algorithm 2:

(i) Pick an initial point $\boldsymbol{c}_0 \in E^{N/2}$, compute $\boldsymbol{g}_0 = \boldsymbol{Q}\boldsymbol{c}_0 - \boldsymbol{b}$ and set $\boldsymbol{r}_0 = -\boldsymbol{g}_0$ and $k = 0$

(ii) Compute

$$\alpha_k = -\frac{g_k^T r_k}{r_k^T Q r_k}$$

$$c_{k+1} = c_k + \alpha_k r_k$$

(iii) Compute

$$g_{k+1} = Qc_{k+1} - b$$

(iv) Compute

$$\beta_k = \frac{g_{k+1}^T Q r_k}{r_k^T Q r_k}$$

and

$$r_{k+1} = -g_{k+1} + \beta_k r_k$$

(v) If

$$\|g_{k+1}\| > \varepsilon$$

where $\varepsilon = 10^{-8}$ (say) replace k by $k + 1$ and return to step (2); otherwise output c_{k+1} as the required solution.

2.2 Method 2

In the second method, the error function $e(c)$ can be minimised by minimising the function

$$e_2(c) = \int_0^{\omega_p} [M(\omega) - \omega]^2 \, d\omega \tag{20}$$

Integration was considered to be an alternative for the reason that the functions involved are trigonometric and not complex. Thus integration gives exact results over a range of frequencies in the passband and stopband. Eqn. 20 can be expressed as

$$e_2(c) = \int_0^{\omega_p} [c^T p p^T c - 2c^T p \omega] \, d\omega + \xi$$

where

$$p = \begin{cases} \left[\sin(\omega)\sin(2\omega)\cdots\sin\left(\dfrac{(N-1)}{2}\omega\right)\right]^T & N \text{ odd} \\ \left[\sin(\tfrac{1}{2}\omega)\sin(\tfrac{3}{2}\omega)\cdots\sin\left(\dfrac{(N-1)}{2}\omega\right)\right]^T & N \text{ even} \end{cases}$$

and $\xi = \omega_p^3/3$ is a constant independent of c. Evidently, $e_2(c)$ can be minimised by minimising the modified error function

$$\bar{e}_2(c) = c^T\left[\int_0^{\omega_p} p p^T \, d\omega\right] c - 2c^T\left[\int_0^{\omega_p} p\omega \, d\omega\right] \tag{21}$$

On comparing eqn. 21 with eqn. 17, we have

$$Q = 2\int_0^{\omega_p} p p^T \, d\omega \tag{22}$$

$$b = 2\int_0^{\omega_p} p\omega \, d\omega \tag{23}$$

where the integration is carried out for each element in matrix Q and for each element in vector b.

In this method, the weighting function ω^{-1} is not used as the integration becomes very difficult. Instead piecewise weighting is used. The integration is broken into several parts between the limits 0 and ω_p and weights are assigned to each integral to emphasise a particular region in the passband. For example, if the integration is broken into three parts, eqns. 22 and 23 give

$$Q = 2\left[W_1 \int_0^{\omega_{p1}} p p^T \, d\omega + W_2 \int_{\omega_{p1}}^{\omega_{p2}} p p^T \, d\omega + W_3 \int_{\omega_{p2}}^{\omega_{p3}} p p^T \, d\omega\right]$$

$$b = 2\left[W_1 \int_0^{\omega_{p1}} p\omega \, d\omega + W_2 \int_{\omega_{p1}}^{\omega_{p2}} p\omega \, d\omega + W_3 \int_{\omega_{p2}}^{\omega_{p3}} p\omega \, d\omega\right]$$

where W_1, W_2, and W_3 are appropriate weights and $\omega_{p3} = \omega_p$.

The minimisation of the error function can be accomplished by using algorithms 1 and 2 described earlier.

3 Prescribed specifications

Fig. 1a shows plots of $\log \delta$ against N for various values of ω_p for the case of odd N using method 1, and Fig. 1b shows corresponding plots for method 2. As can be seen, the variation of $\log \delta$ with N is approximately linear.

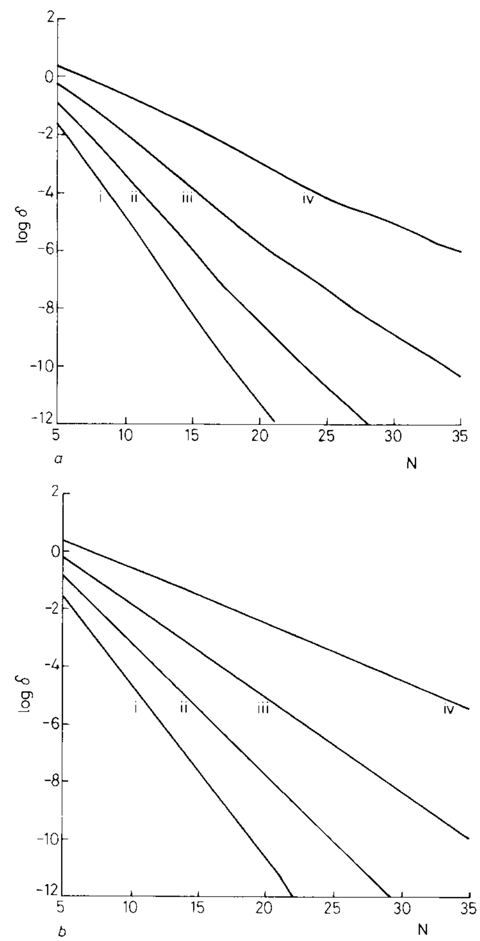

Fig. 1 *Variation of log (δ) with N*

a Method 1, b Method 2
(i) $\omega_p = 2.00$ rad/s
(ii) $\omega_p = 2.25$ rad/s
(iii) $\omega_p = 2.50$ rad/s
(iv) $\omega_p = 2.75$ rad/s

Consequently, given the maximum inband error, the required differentiator length can be determined and vice versa by using a technique described in Reference 5.

4 Design examples

The above methods were used to design three digital differentiators assuming a prescribed bandwidth $\omega_p = 2.0$ rad/s and prescribed maximum inband errors $\delta = 0.01, 0.005, 0.001$. The sampling rate was assumed to be 1.0 sample/s. In method 2, three weights were used, namely $W_1 = 1$, $W_2 = 2/3$, and $W_3 = 1/3$. The optimum values of N for the two methods are given in Table 1.

Table 1: Optimum *N* for design examples (ω_p = 2.0 rad/s)

Method	δ	Odd *N* without weighting	Odd *N* with weighting	Even *N* without weighting	Even *N* with weighting
1	0.010	9	11	6	6
	0.005	11	11	6	6
	0.001	15	15	8	8
2	0.010	9	11	6	6
	0.005	11	11	6	6
	0.001	13	15	8	8

The variation of $E(\omega)$ for $\delta = 0.005$ and 0.001 and $\omega_p = 2.0$ rad/s for the case of even N with weighting using method 1 is shown in Fig. 2.

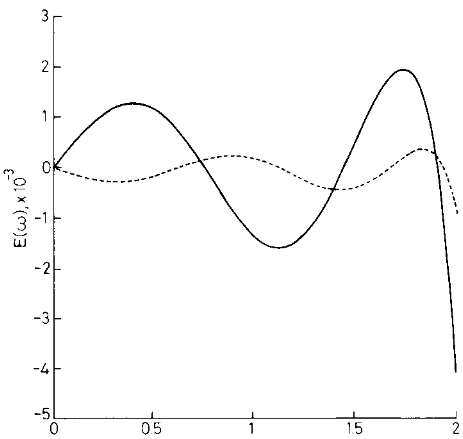

Fig. 2 *Variation of $E(\omega)$ with ω for $\omega_p = 2.0$ rad/s and different values of δ*

—— $\delta = 0.005$
- - - - $\delta = 0.001$

The same procedure was then used to design three digital differentiators assuming a prescribed maximum inband error $\delta = 0.01$ and prescribed bandwidths $\omega_p = 2.25, 2.5, 2.75$ rad/s. The optimum values of N for the two methods are given Table 2. The variation of $E(\omega)$ for $\omega_p = 2.25$, 2.50 and 2.75 rad/s with $\delta = 0.01$ with weighting for the case of odd N using method 2 are shown in Fig. 3.

Table 2: Optimum *N* for design examples (δ = 0.01)

Method	ω_p	Odd *N* without weighting	Odd *N* with weighting	Even *N* without weighting	Even *N* with weighting
1	2.25	13	13	6	6
	2.50	19	19	8	8
	2.75	31	33	10	12
2	2.25	13	13	6	6
	2.50	17	19	8	8
	2.75	31	31	10	10

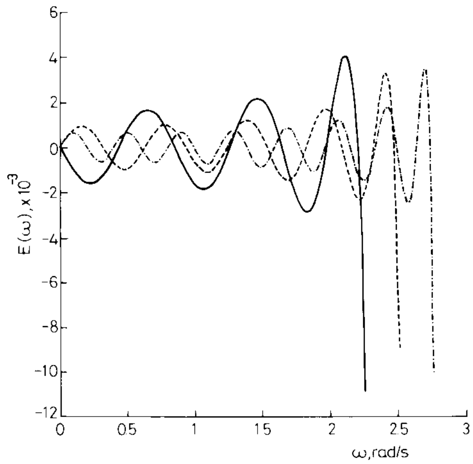

Fig. 3 *Variation of $E(\omega)$ with ω for $\delta = 0.01$ and different values ω_p*

—— $\omega_p = 2.25$ rad/s
- - - - $\omega_p = 2.5$ rad/s
· · · $\omega_p = 2.75$ rad/s

The procedure was also used to design a wideband differentiator having a prescribed bandwidth of $\omega_p = 3.0$ rad/s and a prescribed maximum inband error $\delta = 0.01$. The optimum values of N for the two methods are given in Table 3.

Table 3: Optimum *N* for wideband differentiator

Method	Odd *N* without weighting	Odd *N* with weighting	Even *N* without weighting	Even *N* with weighting
1	73	75	22	24
2	77	81	20	20

As can be observed, for a given inband error, even-length digital differentiators require a much lower value of N than odd-length differentiators. Conversely, for a given value of N, even-length differentiators yield a lower inband error than odd-length differentiators. The reason for this is that $M(\omega) = 0$ for $\omega = \pi$ for odd N and $M(\omega) \neq 0$ for $\omega = \pi$ for even N. Thus the error introduced in the case of odd N is larger near the passband edge, particularly in the design of wideband digital differentiators. It should be noted that, in the case of odd N, the impulse response is defined at instants n, whereas for the case of even N it is defined at $(n + \frac{1}{2})$. This can lead to problems in some applications.

In method 1, the weighting functions $\omega^{-1/2}$, $\omega^{-1/3}$ were also tried but did not yield better results than ω^{-1}.

5 Comparisons

For the sake of comparison, several differentiators were designed using methods 1 and 2 as well as the Remez approach of Reference 1, and the maximum relative inband error (δ_r) and the number of floating-point operations (flops) required were determined for each design. The results obtained for the cases of odd and even N are summarised in Tables 4 and 5, respectively. The variation of the relative error with frequency for the differentiator of length 15 is plotted in Fig. 4. As can be seen, Methods 1 and 2 yield a larger relative error (E_r) than the Remez approach but the amount of computation required to carry out a design is significantly reduced. On the other hand, for a fixed differentiator length these methods lead to reduced maximum inband error relative to the methods in References 2 and 5, and for a fixed maximum inband error to a reduced differentiator length.

We have not compared our procedure with that reported by Pei and Shyu [4] since this contribution did not come to our attention until recently. We note, however, that our problem formulation leads to a quadratic objective function which is similar to that obtained in Reference 4; hence we expect the two approaches to give similar results. However, further work is necessary before some definitive statements can be made on the pros and cons of the two families of differentiators.

An important property of differentiators obtained with our approach, which may or may not apply to differentiators obtained with the procedure in Reference 4, is that the logarithm of the maximum inband error is approximately linear with respect to the differentiator length (see Figs. 1*a* and *b*). Hence differentiators satisfying prescribed specifications can readily be designed as described in Section 3.

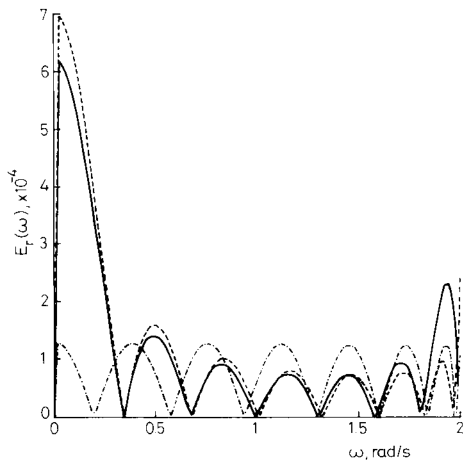

Fig. 4 *Variation of the relative error E_r with frequency for a differentiator of length 15 designed using method 1 with m = 11, method 2 and the Remez method*

—— method 1
····· method 2
–·–·– Remez method

6 Conclusion

Two methods for the design of digital differentiators that are based on standard optimisation methods have been described. The methods can be used in conjunction with a prediction technique for the differentiator length described in Reference 5 to design differentiators satisfying prescribed specifications.

The methods were used to design differentiators for varying prescribed bandwidths and maximum inband errors and also to design a wideband differentiator. The results obtained show that even-length differentiators require lower length for the same prescribed specifications. As was demonstrated by the examples of Section 4, method 2 yields better results and is more efficient than method 1 since the integrals involved provide an exact closed-form expression for the error function which can be evaluated with less computational effort.

The comparisons of Section 5 show that the new methods require less computation than the method in Reference 1 and are easier to program. On the other hand, they lead to reduced differentiator error relative to

Table 4: Comparison of method 1, method 2 and the Remez method for the case of odd N

Length	Method 1			Method 2		Remez method	
	m	flops	δ_r	flops	δ_r	flops	δ_r
9	6	1396	2.30e-03	544	2.06e-02	8924	6.20e-03
11	11	3484	8.40e-03	885	6.80e-03	20904	1.70e-03
15	11	6817	5.36e-04	1887	6.97e-04	25807	1.29e-04
31	26	65523	2.56e-04	12079	7.64e-04	149426	6.27e-04

ω_p = 2.0 rad/s.

Table 5: Comparison of method 1, method 2 and the Remez method for the case of even N

Length	Method 1			Method 2		Remez method	
	m	flops	δ_r	flops	δ_r	flops	δ_r
6	6	838	6.50e-03	343	5.00e-03	5825	1.90e-03
10	11	3585	3.88e-04	972	3.33e-04	16406	8.22e-05
20	16	19350	4.24e-07	4642	5.08e-07	64280	7.11e-07

ω_p = 2.0 rad/s

Table 6: Comparison of method 1, method 2 and the Remez method for the case of a wideband differentiator

Length	Method 1			Method 2		Remez method	
	m	flops	δ_r	flops	δ_r	flops	δ_r
75	71	1213183	3.87e-03	241597	4.50e-03	1764741	5.23e-04
79	81	1634778	1.81e-03	324364	2.10e-03	2161765	2.45e-04
81	91	1850971	1.56e-03	381623	1.30e-03	2505339	1.56e-04

the methods in References 2 and 5 for a given differentiator length, or to a lower differentiator length for a given inband error. Moreover, unlike the methods in Reference 2, they are applicable to differentiators of even length.

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