Design of 2-D nonrecursive filters using the window method

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Abstract: The discrete version of a theorem due to Huang is presented and is then used to demonstrate that the window method is effective for the design of 2-D circularly symmetric nonrecursive filters with abitrary piecewise-constant amplitude responses. The window method is then shown to be applicable to the design of quadrantally symmetric 2-D filters, such as fan filters. The paper concludes with an example that illustrates that fairly good results are obtained, with minimal design effort and computation.

1 Introduction

The design of two-dimensional (2-D) nonrecursive digital filters can be carried out by using the McClellan transformation [1–3], by applying the singular-value decomposition [4] or by using the window method [5–7]. While the window method does not lead to optimal designs, it is often preferred due to its versatility and simplicity and the minimal computational effort needed for the design.

The window method involves the application of the Fourier series in conjunction with a class of functions known as window functions. Two-dimensional window functions can be derived by forming the product of two 1-D window functions or by replacing t in a 1-D continuous window function with $\sqrt{(t_1^2 + t_2^2)}$, as suggested by Huang [5]. Prescribed specifications can be achieved by using the Kaiser window function [8–9] in conjunction with certain empirical formulas for the parameter of the window function and the order of the filter due to Speake and Mersereau [6].

The methods presented in References 5, 6 and 7 are concerned with the design of circularly symmetric lowpass filters. Nevertheless, as will be shown in this paper, the window method is quite effective for the design of circularly symmetric filters with arbitrary piecewise-constant amplitude responses, such as highpass, bandpass and bandstop filters, and also for the design of other types of quadrantally symmetric filters, such as fan filters. The design of fan filters is illustrated by an example that demonstrates that fairly good results can be obtained, with minimal design effort and computation.

2 Window method

The window method for the design of 1-D nonrecursive filters involves obtaining the impulse response of the filter, designated as $h_1(nT)$, by applying the Fourier series to an idealised periodic frequency response, and then multiplying the impluse response by a window function $w_1(nT)$ to obtain a modified impulse response as [9]

$$h_{W1}(nT) = w_1(nT)h_1(nT) \tag{1}$$

The window function is an even function of nT and is of finite duration such that

$$w_1(nT) = 0 \quad \text{for } |n| > (N-1)/2$$

where N is odd. Hence, a finite-order transfer function given by

$$H_{W1}(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} w_1(nT)h_1(nT)z^{-n} \tag{2}$$

is obtained. If $h_1(nT)$ is an even function with respect to nT, then $w_1(nT)h_1(nT)$ is also an even function with respect to nT and, therefore, the filter obtained has zero phase response.

Through the application of the 1-D complex convolution, the frequency response of the modified filter can be expressed as

$$H_{W1}(e^{j\omega T}) = \frac{T}{2\pi}\int_0^{2\pi/T} H_1(e^{j\Omega T})W_1(e^{j(\omega-\Omega)T})\, d\Omega \tag{3}$$

The application of window functions to the design of 2-D filters involves the same steps. The impulse response of the filter, designated as $h_2(n_1T_1, n_2T_2)$, is obtained by applying the 2-D Fourier series to an idealised frequency response and is then multiplied by a 2-D window function $w_2(n_1T_1, n_2T_2)$ to obtain a modified impulse response as

$$h_{W2}(n_1T_1, n_2T_2) = w_2(n_1T_1, n_2T_2)h_2(n_1T_1, n_2T_2) \tag{4}$$

A 2-D window function should have a finite region of support, i.e.

$$w_2(n_1T_1, n_2T_2) = 0 \qquad \text{for } |n_1| > (N_1-1)/2 \text{ or } |n_2| > (N_2-1)/2$$

with N_1 and N_2 odd; furthermore, it should be symmetrical with respect to the n_1T_1 and n_2T_2 axes, i.e.

$$w_2(-n_1T_1, -n_2T_2) = w_2(n_1T_1, n_2T_2)$$

In this way, a symmetrical impulse response would be achieved that would, in turn, lead to an economical filter implementation. The amplitude spectrum of a 2-D window function should consist of a 3-D main lobe centred at the origin of the (ω_1, ω_2) plane and a number of 3-D sidelobes such that the volume of the sidelobes is a small proportion of the volume of the main lobe.

Owing to the finite support of the window function, the transfer function of the 2-D filter is of finite order and

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is given by

$$H_{W2}(z_1, z_2) = \sum_{n_1=-(N_1-1)/2}^{(N_1-1)/2} \sum_{n_2=-(N_2-1)/2}^{(N_2-1)/2} w_2(n_1T_1, n_2T_2) \times h_2(n_1T_1, n_2T_2) z_1^{-n_1} z_2^{-n_2} \quad (5)$$

If $h_2(n_1T_1, n_2T_2)$ is symmetrical with respect to the n_1T_1 and n_2T_2 axes, then $h_{W2}(n_1T_1, n_2T_2)$ is also symmetrical with respect to the n_1T_1 and n_2T_2 axes and, consequently, the filter represented by $H_{W2}(z_1, z_2)$ has zero phase response.

Through the application of the 2-D complex convolution, the frequency response of the modified filter can be expressed as

$$H_{W2}(e^{j\omega_1T_1}, e^{j\omega_2T_2}) = \frac{T_1T_2}{4\pi^2} \times \int_0^{2\pi/T_2} \int_0^{2\pi/T_1} H_2(e^{j\Omega_1T_1}, e^{j\Omega_2T_2}) \times W_2(e^{j(\omega_1-\Omega_1)T_1}, e^{j(\omega_2-\Omega_2)T_2})\, d\Omega_1\, d\Omega_2 \quad (6)$$

If $w_{1A}(n_1T_1)$ and $w_{1B}(n_2T_2)$ are discrete 1-D windows, then the function

$$w_2(n_1T_1, n_2T_2) = w_{1A}(n_1T_1) w_{1B}(n_2T_2) \quad (7)$$

has the required properties and is, therefore, a 2-D window function. On the other hand, if $w_1(t)$ is a continuous 1-*D* window function, it can be transformed as suggested by Huang [5] to yield the continuous 2-D window function

$$w_2(t_1, t_2) = w_1(t)\Big|_{t=\sqrt{(t_1^2+t_2^2)}}$$

By letting $t = nT$, $t_1 = n_1T_1$ and $t_2 = n_2T_2$, the discrete 2-D window function

$$w_2(n_1T_1, n_2T_2) = w_1(nT)\Big|_{nT=\sqrt{[(n_1T_1)^2+(n_2T_2)^2]}} \quad (8)$$

can be obtained.

By proving a theorem involving the Fourier transforms of the continuous 1-D and 2-D windows and then applying it to the case of circularly symmetric lowpass filters, Huang has shown that if $w_1(t)$ is a good continuous 1-D window, then $w_1(\sqrt{(t_1^2+t_2^2)})$ is a good continuous 2-D window. This result also holds for the corresponding discrete 2-D window $w_2(n_1T_1, n_2T_2)$ by virtue of the sampling theorem.

The above conclusion can also be reached by proving and applying the following theorem. This is the discrete version of the Huang theorem and offers the advantage that the result can be deduced directly, without invoking the sampling theorem.

Theorem: Let $W_1(z)$ and $W_2(z_1, z_2)$ be the z transforms of $w_1(nT)$ and $w_2(n_1T_1, n_2T_2)$, respectively, and assume that $w_2(n_1T_1, n_2T_2)$ is obtained from $w_1(nT)$ by using eqn. 8. If $A_1(z_1)$ and $A_2(z_1, z_2)$ are allpass functions such that

$$A_1(z_1) = A_2(z_1, z_2) = 1$$

for all z_1 and z_2, then

$$\frac{T_2}{2\pi} W_2(e^{j\omega_1T_1}, e^{j\omega_2T_2}) \otimes A_2(e^{j\omega_1T_1}, e^{j\omega_2T_2}) = W_1(e^{j\omega_1T_1}) \otimes A_1(e^{j\omega_1T_1})$$

where the symbol $\otimes$ represents convolution.

Proof: From eqn. 8, we note that

$$w_2(n_1T_1, 0) = w_1(n_1T_1) \quad (9)$$

and since

$$w_2(n_1T_1, n_2T_2) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_2} \oint_{\Gamma_1} W_2(z_1, z_2) \times z_1^{n_1-1} z_2^{n_2-1}\, dz_1\, dz_2 \quad (10)$$

and

$$w_1(n_1T_1) = \frac{1}{2\pi j} \oint_{\Gamma_1} W_1(z_1) z_1^{n_1-1}\, dz_1 \quad (11)$$

Eqns. 9 to 11 give

$$\frac{1}{(2\pi j)^2} \oint_{\Gamma_2} \oint_{\Gamma_1} W_2(z_1, z_2) z_1^{n_1-1} z_2^{-1}\, dz_1\, dz_2 = \frac{1}{2\pi j} \oint_{\Gamma_1} W_1(z_1) z_1^{n_1-1}\, dz_1$$

or

$$\oint_{\Gamma_1} \left[\frac{1}{2\pi j} \oint_{\Gamma_2} W_2(z_1, z_2) z_2^{-1}\, dz_2\right] z_1^{n_1-1}\, dz_1 = \oint_{\Gamma_1} W_1(z_1) z_1^{n_1-1}\, dz_1$$

and so

$$W_1(z_1) = \frac{1}{2\pi j} \oint_{\Gamma_2} W_2(z_1, z_2) z_2^{-1}\, dz_2$$

From the 1-D complex convolution, we can write

$$W_1(z_1) \otimes A_1(z_1) = \frac{1}{2\pi j} \oint_{\Gamma_1} W_1(v_1) A_1\left(\frac{z_1}{v_1}\right) \frac{dv_1}{v_1} = \frac{1}{2\pi j} \oint_{\Gamma_1} W_1(v_1) v_1^{-1}\, dv_1$$

and, similarly, from the 2-D complex convolution, we have

$$\begin{aligned} W_2(z_1, z_2) \otimes A_2(z_1, z_2) &= \frac{1}{(2\pi j)^2} \oint_{\Gamma_2} \oint_{\Gamma_1} W_2(v_1, v_2) A_2\left(\frac{z_1}{v_1}, \frac{z_2}{v_2}\right) \frac{dv_1}{v_1} \frac{dv_2}{v_2} \\ &= \frac{1}{2\pi j} \oint_{\Gamma_1} \left[\frac{1}{2\pi j} \oint_{\Gamma_2} W_2(v_1, v_2) A_2\left(\frac{z_1}{v_1}, \frac{z_2}{v_2}\right) \frac{dv_2}{v_2}\right] \frac{dv_1}{v_1} \\ &= \frac{1}{2\pi j} \oint_{\Gamma_1} \left[\frac{1}{2\pi j} \oint_{\Gamma_2} W_2(v_1, v_2) v_2^{-1}\, dv_2\right] v_1^{-1}\, dv_1 \\ &= \frac{1}{2\pi j} \oint_{\Gamma_1} W_1(v_1) v_1^{-1}\, dv_1 \\ &= W_1(z_1) \otimes A_1(z_1) \end{aligned}$$

Hence, if we let $z_1 = e^{j\omega_1T_1}$, $z_2 = e^{j\omega_2T_2}$, $v_1 = e^{j\Omega_1T_2}$ and $v_2 = e^{j\Omega_2T_2}$, we obtain

$$\frac{T_2}{2\pi} \int_0^{2\pi/T_2} \int_0^{2\pi/T_1} W_2(e^{j\Omega_1T_1}, e^{j\Omega_2T_2}) \times A_2(e^{j(\omega_1-\Omega_1)T_1}, e^{j(\omega_2-\Omega_2)T_2})\, d\Omega_1\, d\Omega_2 = \int_0^{2\pi/T_1} W_1(e^{j\Omega_1T_1}) A_1(e^{j(\omega_1-\Omega_1)T_1})\, d\Omega_1$$

or

$$\frac{T_2}{2\pi} W_2(e^{j\omega_1 T_1}, e^{j\omega_2 T_2}) \otimes A_2(e^{j\omega_1 T_1}, e^{j\omega_2 T_2})$$

$$= W_1(e^{j\omega_1 T_1}) \otimes A_1(e^{j\omega_1 T_1})$$

It will now be demonstrated that eqn. 8 provides a mechanism for generating good 2-D windows for the broader class of 2-D circularly symmetric filters with arbitrary piecewise-constant amplitude responses. Assume that $T_1 = T_2 = T$ and let

$$H_1(e^{j\omega_1 T}) = \begin{cases} g & \text{for } \omega_{C1} \leqslant |\omega_1| \leqslant \omega_{C2} \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

be the frequency response of an arbitrary 1-D ideal bandpass digital filter, where g is a constant, $\omega_{C1} \geqslant 0$ and $\omega_{C2} \leqslant \pi/T$, and let

$$H_2(e^{j\omega_1 T}, e^{j\omega_2 T}) = \begin{cases} g & \text{for } \omega_{C1} \leqslant \sqrt{(\omega_1^2 + \omega_2^2)} \leqslant \omega_{C2} \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

be the frequency response of a corresponding 2-D circularly symmetric bandpass digital filter. If $w_1(nT)$ is a 1-D window function with a main-lobe width B such that $B \ll \omega_{C2} - \omega_{C1}$ and $w_2(n_1 T, n_2 T)$ is the 2-D window obtained by using eqn. 8, then eqns. 1 to 3 and eqns. 4 to 6 yield the modified 1-D and 2-D frequency responses

$$H_{W1}(e^{j\omega_1 T}) = W_1(e^{j\omega_1 T}) \otimes H_1(e^{j\omega_1 T})$$

and

$$H_{W2}(e^{j\omega_1 T}, e^{j\omega_2 T}) = W_2(e^{j\omega_1 T}, e^{j\omega_2 T}) \otimes H_2(e^{j\omega_1 T}, e^{j\omega_2 T})$$

respectively. Under these circumstances, if $\omega_1 = \omega_{1k}$ is a frequency in the passband of the 1-D filter such that $\omega_{C1} + B/2 \leqslant \omega_{1k} \leqslant \omega_{C2} - B/2$, one can readily show that

$$H_{W1}(e^{j\omega_1 T}) \simeq g W_1(e^{j\omega_1 T}) \otimes A_1(e^{j\omega_1 T})$$

(see Fig. 9.6 of Reference 9). Similarly, for any pair of frequencies (ω_1, ω_2) such that $\sqrt{(\omega_1^2 + \omega_2^2)} = \omega_{1k}$ and $\omega_{C1} + B/2 \leqslant \omega_{1k} \leqslant \omega_{C2} - B/2$, one can show that

$$H_{W2}(e^{j\omega_1 T}, e^{j\omega_2 T}) \simeq g W_2(e^{j\omega_1 T}, e^{j\omega_2 T}) \otimes A_2(e^{j\omega_1 T}, e^{j\omega_2 T})$$

Therefore, from the above theorem, we obtain

$$\frac{T}{2\pi} H_{W2}(e^{j\omega_1 T}, e^{j\omega_2 T}) \simeq H_{W1}(e^{j\omega_{1k} T})\Big|_{\omega_{1k} = \sqrt{(\omega_1^2 + \omega_2^2)}}$$

In other words, if the 1-D window $w_1(nT)$ yields a good 1-D bandpass filter, then the 2-D window obtained by using eqn. 8 yields a good 2-D bandpass filter. Note that eqns. 12 and 13 hold for lowpass filters (i.e. $\omega_{C1} = 0$ and $\omega_{C2} = \omega_C$) and, as can be readily demonstrated, they are also applicable to each passband in the case of filters with several passbands. Consequently, the window method can be used to design circularly symmetric filters with arbitrary piecewise-constant amplitude responses.

3 Design of fan filters

The window method is also very useful for the design of other types of quadrantally symmetric filters such as fan filters, as will now be demonstrated.

Consider an ideal fan filter characterised by the amplitude response illustrated in Fig. 1. If $v_1 = \omega_1 T_1$ and $v_2 = \omega_2 T_2$, then the impulse response of the filter is given by

$$h_I(n_1, n_2) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H_I(e^{jv_1}, e^{jv_2}) e^{j(n_1 v_1 + n_2 v_2)}\, dv_1\, dv_2$$

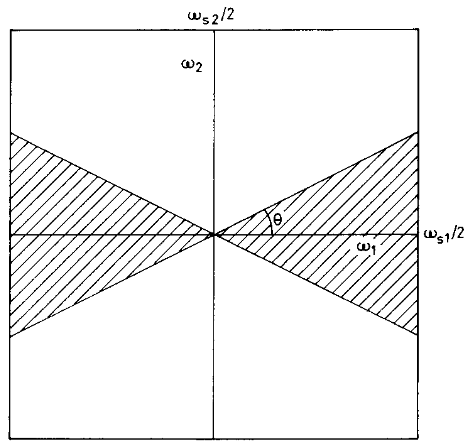

Fig. 1 *Contour plot of ideal fan filter*
passband
stopband

where

$$H_I(e^{jv_1}, e^{jv_2}) = \begin{cases} 1 & \text{for } -kv_1 \leqslant v_2 \leqslant kv_1,\ v_1 \geqslant 0 \\ & \text{or } kv_1 \leqslant v_2 \leqslant -kv_1,\ v_1 < 0 \\ 0 & \text{otherwise} \end{cases}$$

with $k = \tan\theta$. Thus

$$h_I(n_1, n_2) = \frac{1}{(2\pi)^2} \left[\int_{-\pi}^{0} e^{jn_1 v_1} \left(\int_{kv_1}^{-kv_1} e^{jn_2 v_2}\, dv_2 \right) dv_1 + \int_{0}^{\pi} e^{jn_1 v_1} \left(\int_{-kv_1}^{kv_1} e^{jn_2 v_2}\, dv_2 \right) dv_1 \right]$$

$$= \begin{cases} \dfrac{k}{2} & \text{for } (n_1, n_2) = (0, 0) \\[2ex] \dfrac{k[\cos(n_1\pi) - 1]}{n_1^2 \pi^2} & \text{for } n_1 \neq 0,\ n_2 = 0 \\[2ex] 0 & \text{for } n_2 \neq 0,\ n_1^2 - n_2^2 k^2 = 0 \\[2ex] \dfrac{k}{(n_2^2 k^2 - n_1^2)\pi^2} - \dfrac{1}{2 n_2 \pi^2} & \\ \quad \times \left\{ \dfrac{\cos[(n_2 k - n_1)\pi]}{n_2 k - n_1} + \dfrac{\cos[(n_2 k + n_1)\pi]}{n_2 k + n_1} \right\} & \text{otherwise} \end{cases}$$

Since the filter being designed is quadrantally symmetric, it appears appropriate to use the rectangular 2-D window of eqn. (7). Hence we can write

$$h_F(n_1 T_1, n_2 T_2) = w_{1A}(n_1 T_1) w_{1B}(n_2 T_2) h_I(n_1 T_1, n_2 T_2)$$

This formulation was used with the Kaiser window function for the design of a 29 × 29 fan filter with $k = 0.35$. The parameter α of the Kaiser window as assumed to be 4.0 and the sampling frequencies were 10 rad/s. The amplitude response achieved is illustrated in Fig. 2. The maximum passband and stopband errors are 4.38×10^{-2} and 2.33×10^{-2}, respectively.

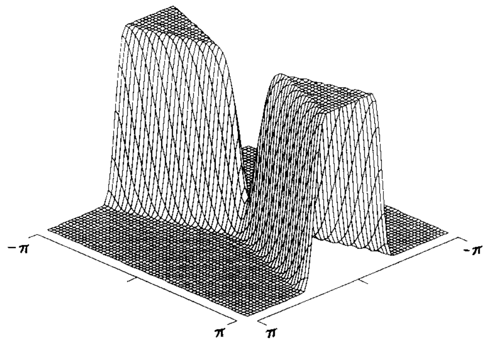

Fig. 2 *Amplitude response of fan filter*

4 Conclusions

The discrete version of a theorem due to Huang has been presented and used to demonstrate that, given a good discrete 1-D window function, a good discrete 2-D window function that can be used to design nonrecursive filters with arbitrary piecewise-constant amplitude responses can be obtained. Then, the design of fan filters was considered and formulas for the impulse response were obtained. A design example showed that the window method yields fairly good designs with minimal design effort and computation.

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