plot of RPSD versus frequency is shown in Fig. 6. The GIC structure has lower RPSD than the canonic structure.

V. Conclusions

A new complex biquad recursive digital filter structure based on the GIC concept has been presented. The zero-input stability and the forced response stability of the proposed filter have been investigated. It has been shown that the proposed filter can be designed free of limit cycles using known techniques. The sensitivity and noise properties of the proposed filter were shown to be better than those of the structures given in [6]. However, the proposed structure is more complex compared to the direct form structure in the sense that it requires more number of operations.

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Stability Robustness of Two-Dimensional Discrete Systems and Its Computation

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***Abstract* — Through the use of DeCarlo's stability criterion, we extend the measure of stability robustness proposed in [6] to the two-dimensional case. Moreover, an easy-to-use linear algebra type algorithm is developed to accomplish the numerical computation of the proposed robustness measure. It turns out that in the Roesser's local state-space setting, the measure provides a least conservative upper bound for system's stable perturbations.**

I. Introduction

When a two-dimensional (2-D) discrete system is designed by certain synthesis methodology, one often needs to know how robust it is in terms of stability under parameter variations due to model inaccuracy, measurement noise, or in the content of digital filter, quantization error in implementation. It is therefore desired to define a quantitative measure of the perturbations that will not cause system instability and to develop a feasible procedure to compute it. 2-D stability under parameter variation was addressed in [1] using the concept of minimum norm realization. It follows that such a stability problem has a close connection with the concept of 2-D stability margins which have been studied in [2] and [3]. For a recent development along this direction, see [4].

In this paper, measure of stability robustness is proposed for 2-D shift-invariant systems. While stability robustness analysis may be carried out through sensitivity analysis of the stability margin in particular, Kharitonov's theorem [14], [15], the result presented here, as an alternative, provides the tightest 2-norm bound for the unstructured complex perturbations in the Roesser's local state-space setting. The main tool to be used to tackle our problem is the Gastinel–Kahan theorem [5], [6] which was also employed in [6] to study the finite-dimensional case. It is shown that the numerical computation of the robustness measure can be accomplished by a linear algebra type algorithm. In comparison with the results obtained in [1], the present robust measure is far less conservative and is computationally more feasible. The results obtained in this paper may be extended to multidimensional cases (> 2) in a straightforward manner.

Throughout the paper, the Euclidean norm of vector x is denoted by $\|x\|$. The corresponding induced norm of a matrix $M \in C^{n \times m}$ is denoted by $\|M\|$, and is equal to its largest singular value. The singular value decomposition (SVD) of M is a decomposition of M in the form of

$$M = U \sum V^H \tag{1}$$

where U and V are unitary matrices of dimension n and dimension m, respectively, V^H denotes the complex-conjugate transpose of V, and (assume $n \leqslant m$)

$$\sum = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]$$

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contains the singular values of $M, \sigma_1 \geqslant \cdots \geqslant \sigma_n \geqslant 0$. It follows from (1) that $\|M\| = \sigma_1$ and, if $n = m$ and if M is invertible then $\|M^{-1}\| = \sigma_n^{-1}$. For a detailed discussion on SVD, the interested reader is referred to [7], [8].

In this paper we shall use z_1^{-1} and z_2^{-1} for the unit delay operators while in [9]–[12] the unit delay operators were denoted by z_1 and z_2. Consequently, the stability conditions (5) as well as (6) described below are the modified yet equivalent versions of their original statements.

The following notation will also be used in the sequel:

$$\Omega = \{(\omega_1, \omega_2): 0 \leqslant \omega_1 \leqslant 2\pi.\ 0 \leqslant \omega_2 \leqslant 2\pi\}$$

$$\overline{U} = \{z: |z| \leqslant 1\}$$

$$\overline{U}_c = \{z: |z| \geqslant 1\}$$

$$\overline{U}^2 = \{z_1, z_2): |z_1| \leqslant 1, |z_2| \leqslant 1\}$$

$$\overline{U}_c^2 = \{(z_1, z_2): |z_1| \geqslant 1, |z_2| \geqslant 1\}$$

$$T^2 = \{(z_1, z_2): |z_1| = 1, |z_2| = 1\}$$

I = identity matrix of appropriate dimension,

$I(z_1, z_2) = z_1 I \oplus z_2 I$, where $\oplus$ denotes the direct sum, i.e.,

$$I(z_1, z_2) = \begin{bmatrix} z_1 I & 0 \\ 0 & z_2 I \end{bmatrix}.$$

II. Measure for Stability Robustness

Consider a 2-D shift-invariant system characterized by a Roesser's local state-space model [13] as

$$\begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(i, j)$$

$$\equiv Ax + bu \tag{2a}$$

$$y(i, j) = [c_1 \quad c_2] \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} \equiv cx \tag{2b}$$

where $x^h(i, j) \in R^{n_1}$ and $x^v(i, j) \in R^{n_2}$ form a local state of the system at position (i, j) and the initial conditions of system (2) are given by $x^h(o, j) = w(j)$ and $x^v(i, o) = v(i)$ for $i, j = 0, 1, \cdots$. By applying the 2-D z transform to (2) while assuming zero initial conditions, one obtains the transfer function of system (2) as

$$H(z_1, z_2) = c(I(z_1, z_2) - A)^{-1} b = \frac{n(z_1, z_2)}{h(z_1, z_2)} \tag{3}$$

where the denominator polynomial $h(z_1, z_2)$ is often called the characteristic polynomial of the system and is given by

$$h(z_1, z_2) = \det[I(z_1, z_2) - A]. \tag{4}$$

It is assumed throughout that system (2) is free of nonessential singularities of the second kind on T^2 (i.e., there is no point on T^2 at which both $n(z_1, z_2)$ and $h(z_1, z_2)$ vanish) and that system (2) is bounded-input-bounded-output (BIBO) stable, that is, every bounded input signal yields a bounded output signal. By [12], the above assumptions imply that

$$h(z_1, z_2) \neq 0, \qquad \text{for } (z_1, z_2) \in \overline{U}_c^2 \tag{5}$$

It then follows from the DeCarlo criterion [9, th. 5] [10, p. 1022] [11, pp. 105–106] that condition (5) is equivalent to

$$h(z, z) \neq 0, \qquad \text{for } z \in \overline{U}_c \tag{6a}$$

and

$$h(z_1, z_2) \neq 0, \qquad \text{for } (z_1, z_2) \in T^2. \tag{6b$'$}$$

Note that condition (6b′) may also be stated as

$$h(e^{jw_1}, e^{jw_2}) \neq 0, \qquad \text{for } (w_1, w_2) \in \Omega. \tag{6b}$$

Let ΔA be a complex perturbation to the system matrix A. The main objective of this paper is to provide a computationally feasible algorithm for the minimum norm of perturbation ΔA that makes the perturbed system unstable.

To begin with, we assume $m = n$ and write U and V of (1) as $U = [u_1 \quad u_2 \cdots u_n]$ and $V = [v_1 \quad v_2 \cdots v_n]$, (1) then becomes

$$M = \sum_{i=1}^{n} \sigma_i u_i v_i^H.$$

The following lemma is an immediate consequence of a more general result known as the Gastinel–Kahan theorem [5].

Lemma 1: Let $M \in C^{n \times n}$ be nonsingular, then

$$\min_{\Delta M} \{\|\Delta M\|: M + \Delta M \text{ singular}\} = \sigma_n \tag{7}$$

where σ_n is the smallest singular value of M. The above minimum can be achieved by the perturbation ΔM given by

$$\Delta M = -\sigma_n u_n v_n^H. \tag{8}$$

Let

$$A_p = A + \Delta A$$

$$\Phi(z_1, z_2) = [I(z_1, z_2) - A]^{-1}$$

and

$$\Phi_p(z_1, z_2) = [I(z_1, z_2) - A - \Delta A]^{-1} = [I(z_1, z_2) - A_p]^{-1}.$$

A sufficient condition for ΔA being a stable perturbation is presented below.

Lemma 2: Assume system (2) is BIBO stable then any perturbation ΔA satisfying

$$\|\Delta A\| < \|\Phi(e^{j\omega_1}, e^{j\omega_2})\|^{-1}, \qquad \text{for all } 0 \leqslant \omega_1, \omega_2, \leqslant 2\pi \tag{9}$$

will not cause system instability.

Proof: Assume there exists ΔA such that ΔA satisfies (9) and $A + \Delta A$ is 2-D unstable. By DeCarlo's criterion, at least one of (6a) and (6b) does not hold for $h_p(z_1, z_2) = \det[I(z_1, z_2) - A_p]$. If $h_p(z, z) = 0$ for some $z \in \overline{U}_c$, then $A + \Delta A$ is *1-D* unstable. Hence there exist $\delta \in (0,1]$ and $0 \leqslant \omega \leqslant 2\pi$ such that $\Phi^{-1}(e^{j\omega}, e^{j\omega}) - \delta \Delta A$ is singular [6, lemma 2.1]. By Lemma 1 this means that

$$\|\delta \Delta A\| \geqslant \|\Phi(e^{j\omega}, e^{j\omega})\|^{-1} \quad \text{and} \quad \|\Delta A\| \geqslant \|\Phi(e^{j\omega}, e^{j\omega})\|^{-1}$$

which is in contradiction with (9). If $h_p(z_1, z_2) = 0$ for some $(z_1, z_2) \in T^2$, then by Lemma 1 we conclude that for some $(\omega_1, \omega_2) \in \Omega$, $\|\Delta A\| \geqslant \|\Phi(e^{j\omega_1}, e^{j\omega_2})\|^{-1}$ which once again contradicts (9). □

By the definition of $\Phi(e^{j\omega_1}, e^{j\omega_2})$, one may write

$$\|\Phi(e^{j\omega_1}, e^{j\omega_2})\|^{-1} = \sigma_n[I(e^{j\omega_1}, e^{j\omega_2}) - A].$$

Since the singular values of a given matrix are continuous functions of the matrix entries, $\sigma_n[I(e^{j\omega_1}, e^{j\omega_2}) - A]$ achieves its minimum on $\Omega = \{(\omega_1, \omega_2): 0 \leqslant \omega_1 \leqslant 2\pi, 0 \leqslant \omega_2 \leqslant 2\pi\}$. Thus condition (9) may be stated equivalently as

$$\|\Delta A\| < \nu \tag{9$'$}$$

where ν is defined by

$$\nu = \min_{\Omega} \sigma_n \left[I(e^{j\omega_1}, e^{j\omega_2}) - A \right]. \tag{10}$$

It turns out that ν defined above provides an appropriate measure for the stability robustness. As a matter of fact, let

$$S_u = \{ \Delta A : A + \Delta A \text{ is 2-D unstable} \} \tag{11}$$

and

$$\mu = \min_{\Delta A \in S_u} \|\Delta A\| \tag{12}$$

the following theorem claims that ν and μ are equal.

Theorem: Assume system (2) is BIBO stable. Then

$$\mu = \nu. \tag{13}$$

Proof: From Lemma 2 it follows that $\mu \geqslant \nu$. On the other hand, there exists $(\omega_1^0, \omega_2^0) \in \Omega$ such that

$$\nu = \sigma_n \left[I(e^{j\omega_1^0, j\omega_2^0}) - A \right].$$

Let $I(e^{j\omega_1^0}, e^{j\omega_2^0}) - A = U_0 \Sigma_0 V_0^H$ be the SVD of $I(e^{j\omega_1^0}, e^{j\omega_2^0}) - A$. Since system (2) is stable, $I(e^{j\omega_1^0}, e^{j\omega_2^0}) - A$ is nonsingular. Thus Lemma 1 implies that the perturbation defined by $\Delta A_0 = -\nu u_n^0 (v_n^0)^H$, where u_n^0 and v_n^0 are the last column vectors of U_0 and V_0, respectively, will make $I(e^{j\omega_1^0}, e^{j\omega_2^0}) - (A + \Delta A_0)$ singular. This means the instability of $A + \Delta A_0$. Therefore $\Delta A_0 \in S_u$ and $\mu \leqslant \|\Delta A_0\| = \|-\nu u_n^0 (v_n^0)^H\| = \nu$. □

Remark: It follows that ν defined by (10) provides the tightest upper bound for unstructured complex perturbations that will not cause system instability. In comparison with the result obtained in [1, eq. (4.4)], (9′) is least conservative and, as will be seen in the next section, is computationally more feasible.

III. Computational Issues

In this section we discuss how to numerically find the robustness measure ν defined by (10).

Let $0 = \omega_{10} < \omega_{11} < \cdots < \omega_{1K} = 2\pi$ and $0 = \omega_{20} < \omega_{21} < \cdots < \omega_{2L} = 2\pi$ be two sets of points that equally partition intervals $0 \leqslant \omega_1 \leqslant 2\pi$ and $0 \leqslant \omega_2 \leqslant 2\pi$, respectively, and let $\Delta\theta_1 = \pi/K$, $\Delta\theta_2 = \pi/L$. For each pair $(\omega_{1k}, \omega_{2l})$, define

$$d_{kl} = \sigma_n \left[I(e^{j\omega_{1k}}, e^{j\omega_{2l}}) - A \right], \qquad 0 \leqslant k \leqslant K; 0 \leqslant l \leqslant L \tag{14}$$

and

$$\nu_{KL} = \min_{k,l} d_{kl} \tag{15}$$

Note that for any $(\omega_1, \omega_2) \in \Omega$ there exists $(\omega_{1k}, \omega_{2l})$ such that $|\omega_1 - \omega_{1k}| \leqslant \Delta\theta_1$, $|\omega_2 - \omega_{2l}| \leqslant \Delta\theta_2$, and thus

$$\begin{aligned} |e^{j\omega_1} - e^{j\omega_{1k}}| &\leqslant \Delta\theta_1 \\ |e^{j\omega_2} - e^{j\omega_{2l}}| &\leqslant \Delta\theta_2. \end{aligned} \tag{16}$$

By (16) and a property of SVD [8, p. 286], we have

$$\begin{aligned} &\left| \sigma_n \left[I(e^{j\omega_1}, e^{j\omega_2}) - A \right] - \sigma_n \left[I(e^{j\omega_{1k}}, e^{j\omega_{2l}}) - A \right] \right| \\ &\quad \leqslant \| I(e^{j\omega_1}, e^{j\omega_2}) - I(e^{j\omega_{1k}}, e^{j\omega_{2l}}) \| \\ &\quad \leqslant \max(\Delta\theta_1, \Delta\theta_2). \end{aligned} \tag{17}$$

It then follows from (10), (15), and (17) that

$$\nu \leqslant \nu_{KL} \leqslant \nu + \pi \max\left(\frac{1}{K}, \frac{1}{L} \right). \tag{18}$$

By (18) it is seen that ν_{KL} defined by (15) will provide a good approximation of ν if both K and L are large enough. The absolute and relative error of such an approximation can also be estimated based on (18) as

$$|\nu_{KL} - \nu| \leqslant \pi \max\left(\frac{1}{K}, \frac{1}{L} \right) \tag{19}$$

and

$$\left| \frac{\nu_{KL} - \nu}{\nu} \right| \leqslant \frac{\pi \max\left(\frac{1}{K}, \frac{1}{L} \right)}{\nu_{KL} - \pi \max\left(\frac{1}{K}, \frac{1}{L} \right)} \tag{20}$$

respectively. Since ν_{KL} approaches to ν as K and L go to infinity, and ν is positive, we conclude that both absolute and relative errors approach to zero as K and L tend to infinity.

We are now in a position to propose an algorithm to compute the measure of stability robustness as follows

Algorithm (Computation of ν)

1) Set $L = K$ and determine the smallest positive integer K that satisfies
$$\frac{\pi}{K} \leqslant \epsilon$$
where $\epsilon > 0$ is the prescribed tolerance of ν.
2) For $0 \leqslant k \leqslant K$, $0 \leqslant l \leqslant L$, compute d_{kl} defined by (14).
3) Compute ν_{KL} defined by (15).
4) Take $\nu = \nu_{KL}$.

By (19) it follows that the robustness measure computed through the above algorithm differs from the true ν at most by ϵ. The above algorithm can be easily implemented by using the software package MATLAB. As an example, consider a 2-D system modeled by (2) with

$$A = \begin{bmatrix} -0.5 & -1 \\ 0.395 & -0.01 \end{bmatrix}.$$

Applying the algorithm to this system yields $\nu = 0.070694$ which occurs at $(\omega_1, \omega_2) = (\pi, 0)$. In comparison with [1], the upper bound of stable perturbations given by (4.4) of [1] was 0.0483972.

IV. Conclusions

Through the use of DeCarlo's stability criterion and the Gastinel–Kahan theorem, a reasonable measure of stability robustness has been proposed in the Roesser's 2-D state-space setting. The results presented in Section II might be viewed as an natural extension of its 1-D counterpart [6]. Moreover, a linear algebra type algorithm to compute the proposed measure is given, which makes it feasible to use the measure in practical design or analysis procedures.

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Electronic "Neural" Net Algorithm for Maximum Entropy Solutions of Ill-Posed Problems

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***Abstract*—An ill-posed problem does not provide sufficient information to obtain a unique solution. In such cases, a predetermined strategy must be used to choose a particular solution. A powerful and widely used technique is to select the solution with the maximum informational entropy. We describe here an algorithm suitable for solving ill-posed problems using the entropy of the solution as a regularizer. The algorithm has been simulated on a microcomputer as if implemented in a "neural", i.e., multiply connected, net type electronic circuit. Two types of problem are considered. First, where the constraints on the solution are hard as in the loaded dice problem described by Jaynes [1], the maximum entropy solution is shown to be achieved in the high gain limit of the net. Second, where the constraints are soft as in the deconvolution of data corrupted by noise, a maximum entropy solution is obtained directly. Prior knowledge of the solution is not required but can be introduced to the net so that the cross entropy is used as the regularizer. Finally, issues pertinent to the building of an actual circuit are discussed.**

I. Introduction

A problem is considered ill-posed if there is insufficient information available to achieve a unique solution. For example, the loaded dice problem described by Jaynes [1] requires the best solution for the biases on the six sided dice given only the information that the average throw is observed to be 4.5 rather than the 3.5 expected from an unbiased dice. In general, a deconvolution problem can be termed ill-posed if there are any singularities in the transfer function or if the observation data to be deconvolved are corrupted by noise. This is often the case in signal processing related to spectroscopy and imperfect imaging systems. The goal of the signal processing is to reconstruct the original object, i.e., to deconvolve the observations and to cancel noise. In other words, the problem is to invert (1) to find the unknown object O from the known convolution function T and known observation data I:

$$I = O * T \tag{1}$$

where $*$ represents the convolution operation.

Numerous algorithms exist for such a deconvolution (see, for example, [2], [3] which can readily be implemented on a digital computer. A simple example is that of Fourier deconvolution. As convolution can be expressed as the product of the transforms of the object and the transfer function, the transform of the object can be found by dividing the transform of the observations by that of the transfer function. However, this approach breaks down when there is noise present in the observations or the transfer function transform has zeros. In such cases the deconvolution can be solved by using a "regularizer" which is a function of successive estimates made of the object. The technique is iterative in nature as the successive estimates O are convolved and compared to the observations, I. One then minimizes

$$|O * T - I|^2 + |G(O)|$$

where $G(O)$ is the regularizer. The regularizer is best chosen to reflect an important characteristic of the specific problem at hand. Examples of such choices are given in [4]. The problem, however, remains as to the best choice for a regularizer in the absence of specific prior knowledge of the object being reconstructed.

The technique of maximum entropy (MaxEnt) [5]–[11] is often described as the preferred method of recreating a positive distribution, i.e., containing only nonnegative values, with well-defined moments from incomplete data. MaxEnt has been demonstrated to be extremely powerful in several fields such as optical image enhancement [7], [8] deconvolution [9], spectral analysis [10] and diffraction tomography [11]. The essence of the MaxEnt method is to maximize the entropy of the reconstructed distribution subject to satisfying constraints on the distribution. These constraints are often defined by a set of observations such as, for example, a moment (e.g., average) or convolution (e.g., blurred image) of the true distribution. Thus entropy is a meaningful choice for a regularizer when the only specific knowledge about the object being reconstructed is positivity. Furthermore, the regularized reconstructions have error term distributions that are of the exponential family. That is, they have well-defined means and variances, which is what one would expect from a real physical system. In contrast least squares estimates do not assure this.

MaxEnt methods are computationally intensive and require at least a minicomputer and the necessary software. As a result it is difficult to achieve a MaxEnt deconvolution, for example, in real time which would be of great use in many applications. This type of problem would appear to be suited to computation in a multiply connected or "neural" electronic net [12]. Such a net can be designed so that its operation is characterized by a stability (Lyapunov) function which is a well-defined function of the net parameters (i.e., inputs, outputs, interconnects etc.). The output from the net evolves with time until a minimum in its Lyapunov function is reached. By designing a net with specific cost function to be minimized contained within its Lyapunov function, powerful computational powers have been demonstrated, see, for exam-

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