An Efficient Implementation Scheme for Quadrantally Symmetric 2-D Digital Filters

WU-SHENG LU AND BRUCE LEE

Abstract —**A parallel structure is proposed for implementation of a general quadrant-plane causal 2-D transfer function where each channel is realized such that spatial delays in the two directions are minimal. The appropriate framework is provided by the singular value decomposition technique in conjunction with recent results from 2-D realization theory. The class of quadrantally symmetric filters is especially discussed since it would gain most from the proposed realization structure.**

I. Introduction

Realizations and implementations of two-dimensional (2-D) discrete systems and digital filters have been studied for some time, see [1]–[5] and references therein. It is known that given an arbitrary, single-input–single-output, causal, 2-D transfer function $H(z_1, z_2)$ of order (n, m), the existence of an absolutely minimal realization in Roesser's local state-space form over the field of complex numbers C is still not resolved [2], [3]. In the digital filter setting, because of the widespread use of microprocessors in implementation of digital filters and the rapid growth of VLSI techniques, it is often significant to find realization schemes that minimize the output noise power due to finite precision in the implementation of the 2-D filter and that provide filter structures suitable for VLSI.

Using the singular value decomposition (SVD) technique in conjunction with 2-D realization theory [2] and recent development in synthesis of 2-D state-space filters [6], we present a parallel structure realizing a general quadrant-plane (QP) causal 2-D transfer function in which each channel is realized such that the numbers of delay elements used in both directions are minimal and the output noise power due to the roundoff of products is also minimal. This fast-parallel processing structure is quite suitable for VLSI implementation. Some important classes of 2-D filters such as circularly symmetric filters and fan filters are further discussed since they would gain most from the proposed realization structure. A discussion of various decomposition schemes for multidimensional filters can be found in a paper by Venetsanopoulos and Mertzios [8].

II. SVD and Parallel Realization

Let

$$H(z_1, z_2) = \frac{f(z_1, z_2)}{q(z_1, z_2)} = \frac{\sum_{i=0}^{n}\sum_{j=0}^{m} f_{ij} z_1^i z_2^j}{\sum_{i=0}^{n}\sum_{j=0}^{m} q_{ij} z_1^i z_2^j}, \qquad q_{00} = 1 \quad (1)$$

be the transfer function of a QP causal 2-D digital filter of order (n, m). Using matrix notation, (1) can be rewritten as

$$H(z_1, z_2) = \frac{Z_1^T F Z_2}{Z_1^T Q Z_2} \tag{2}$$

where

$$F = (f_{ij}) \in \mathbf{R}^{(n+1)\times(m+1)}, \quad Q = (q_{ij}) \in \mathbf{R}^{(n+1)\times(m+1)}$$

$$Z_1 = \begin{bmatrix} 1 \\ z_1 \\ \vdots \\ z_1^n \end{bmatrix} \in \mathbf{C}^{n+1}$$

and

$$Z_2 = \begin{bmatrix} 1 \\ z_2 \\ \vdots \\ z_2^m \end{bmatrix} \in \mathbf{C}^{m+1}.$$

For simplicity $n \geqslant m$ is always assumed. It is well known [7] that the SVD of matrix F is given as

$$F = U \Sigma V^T \tag{3}$$

where U and V are orthogonal matrices of dimension $n+1$ and $m+1$, respectively, and

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$$

with $\Sigma_1 = \operatorname{diag}\{\sigma_1, \sigma_2, \cdots \sigma_r, 0, \cdots 0\} \in \mathbf{R}^{(m+1)\times(m+1)}$, $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$, and $r = \operatorname{rank} F$.

Next let

$$U = [u_1 \quad u_2 \quad \cdots \quad u_{n+1}] \quad \text{and} \quad V = [v_1 \quad v_2 \quad \cdots \quad v_{m+1}] \tag{4}$$

with $u_i \in \mathbf{R}^{n+1}$ and $v_i \in \mathbf{R}^{m+1}$, and let

$$\alpha_i(z_1) = Z_1^T u_i, \qquad 1 \leqslant i \leqslant n \tag{5a}$$

and

$$\beta_j(z_2) = v_j^T Z_2, \qquad 1 \leqslant j \leqslant m. \tag{5b}$$

It follows from (2) and (3) that

$$H(z_1, z_2) = \sum_{k=1}^{r} H_k(z_1, z_2) \tag{6a}$$

with

$$H_k(z_1, z_2) = \frac{\sigma_k \alpha_k(z_1) \beta_k(z_2)}{q(z_1, z_2)} \triangleq \frac{f_k(z_1, z_2)}{q(z_1, z_2)}, \qquad 1 \leqslant k \leqslant r. \tag{6b}$$

A remarkable feature of (6) is that $H(z_1, z_2)$ is constructed by connecting in parallel r subfilters $H_k(z_1, z_2)$, as shown in Fig. 1(a), each of which has separable numerator so that $H_k(z_1, z_2)$ can be absolutely minimally realized using n delay elements z_1 and m delay elements z_2 in conjunction with several multipliers, see [2, pp. 956–957] for a detailed discussion.

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The authors are with the Department of Electrical Engineering, University of Minnesota, Minneapolis, MN 55455.

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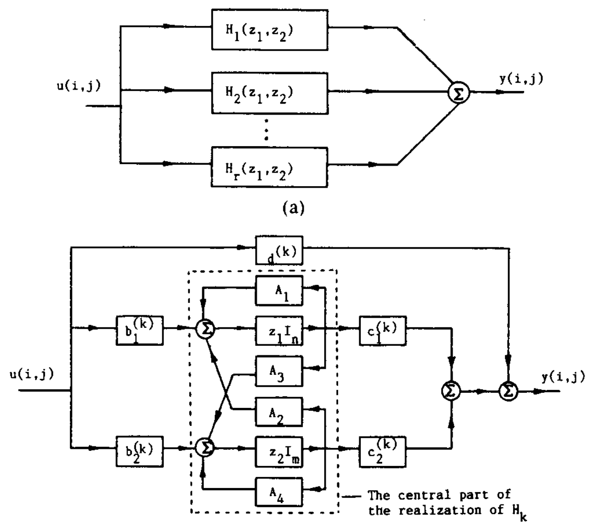

Fig. 1. (a) A parallel realization of $H(z_1, z_2)$. (b) An Absolutely minimal realization of $H_k(z_1, z_2)$ in Roesser's local state space.

Note that $H_k(z_1, z_2)$ can also be realized in Roesser's local state-space form as

$$\begin{bmatrix} x_k^h(i+1,j) \\ x_k^v(i,j+1) \end{bmatrix} = \left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] \begin{bmatrix} x_k^h(i,j) \\ x_k^v(i,j) \end{bmatrix} + \left[\begin{array}{c} b_1^{(k)} \\ \hline b_2^{(k)} \end{array}\right] u(i,j) \triangleq A x_k + b^{(k)} u$$

$$y^{(k)}(i,j) = \left[\begin{array}{c|c} c_1^{(k)} & c_2^{(k)} \end{array}\right] \begin{bmatrix} x_k^h(i,j) \\ x_k^v(i,j) \end{bmatrix} + d^{(k)} u(i,j) \triangleq c^{(k)} x_k + d^{(k)} u \quad (7)$$

where $A_1 \in \boldsymbol{R}^{n \times n}$ and $A_4 \in \boldsymbol{R}^{m \times m}$ are of smallest sizes, see Fig. 1(b). As an example, let us take a look at a transfer function, say $H_1(z_1, z_2)$, of order (3,2) as follows:

$$H_1(z_1, z_2) = \frac{\sigma_1(\alpha_0 + \alpha_1 z_1 + \alpha_2 z_1^2 + \alpha_3 z_1^3)(\beta_0 + \beta_1 z_2 + \beta_2 z_2^2)}{\sum_{i=0}^{3} \sum_{j=0}^{2} q_{ij} z_1^i z_2^j}$$

$$q_{00} = 1 \quad (8)$$

one of whose absolutely minimal realizations can readily be found [2, pp. 956–957] as

$$A = \left[\begin{array}{ccc|cc} -q_{10} & 1 & 0 & -q_{10}q_{01} + q_{11} & -q_{10}q_{02} + q_{12} \\ -q_{20} & 0 & 1 & -q_{20}q_{01} + q_{21} & -q_{20}q_{02} + q_{22} \\ -q_{30} & 0 & 0 & -q_{30}q_{01} + q_{31} & -q_{30}q_{02} + q_{32} \\ \hline -1 & 0 & 0 & -q_{01} & -q_{02} \\ 0 & 0 & 1 & 0 & \end{array}\right]$$

$$b^{(1)} = \sigma_1 \left[\begin{array}{c} q_{10}\alpha_0 - \alpha_1 \\ q_{20}\alpha_0 - \sigma_2 \\ q_{30}\alpha_0 - \alpha_3 \\ \hline \alpha_0 \\ 0 \end{array}\right]$$

$$c^{(1)} = \left[\begin{array}{ccc|cc} -\beta_0 & 0 & 0 & -q_{01}\beta_0 + \beta_1 & -q_{02}\beta_0 + \beta_2 \end{array}\right]$$

$$d^{(1)} = \sigma_1 \alpha_0 \beta_0. \quad (9)$$

Since the system matrix of realization (9) is entirely characterized by polynomial $q(z_1, z_2)$, it follows from (6b) that the whole transfer function can be implemented via a parallel structure, as sketched in Fig. 1, where each channel can be realized using the *same* central part, as shown in Fig. 1(b), in conjunction with its "input/output interface" implemented by $b^{(k)}$, $c^{(k)}$, and $d^{(k)}$. It is now clear that the proposed realization scheme is suitable when VLSI chips for implementation are considered.

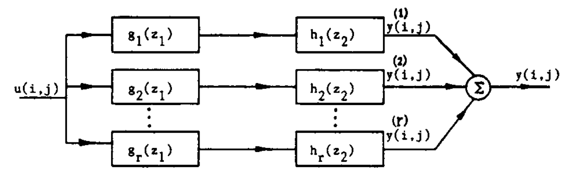

Fig. 2. A parallel-cascade implementation scheme for quadrantally symmetric 2-D filters.

We would now like to point out that the above proposed realization scheme appears to be quite attractive in terms of structure simplicity and throughput efficiency for a broad class of 2-D filters whose amplitude responses are quadrantally symmetric. Fan filters, circularly symmetric low-pass, bandpass and high-pass 2-D filters are examples in this class that have been widely used in practice.

To see this, notice that every QP causal, stable, quadrantally symmetric 2-D filter has separable denominator [10]. Thus for such a filter (1) becomes

$$H(z_1, z_2) = \frac{f(z_1, z_2)}{q(z_1, z_2)} = \frac{f(z_1, z_2)}{q_1(z_1) q_2(z_2)} \quad (10)$$

and, correspondingly, (6) becomes

$$H(z_1, z_2) = \sum_{k=1}^{r} H_k(z_1, z_2) = \sum_{k=1}^{r} g_k(z_1) h_k(z_2) \quad (11a)$$

where

$$H_k(z_1, z_2) = \left(\sqrt{\sigma_k} \frac{\alpha_k(z_1)}{q_1(z_1)}\right)\left(\sqrt{\sigma_k} \frac{\beta_k(z_2)}{q_2(z_2)}\right) \triangleq g_k(z_1) h_k(z_2) \quad (11b)$$

and $g_k(z_1)$, $h_k(z_2)$ are stable 1-D filters of order n and order m, respectively. We therefore conclude that any stable quadrantally symmetric 2-D filters can be implemented using a parallel-cascase scheme, as shown in Fig. 2, where each channel has smallest possible spatial delays in both directions.

As a signal processing system, one may also consider the throughput rate of the above implementation scheme. It follows from Fig. 2 that

$$y(i,j) = \sum_{k=1}^{r} y^{(k)}(i,j) \quad (12)$$

where $y^{(k)}(i,j)$ is the output of the kth subsystem described by $H_k(z_1, z_2) = g_k(z_1) h_k(z_2)$. By (11) we can write $y^{(k)}(i,j)$ as

$$y^{(k)}(i,j) = [1 - q_2(z_2)] y^{(k)}(i,j) + \sqrt{\sigma_k} \beta_k(z_2) x^{(k)}(i,j) \quad (13a)$$

where $x^{(k)}(i,j)$ is the intermediate output in the kth subsystem, as shown in Fig. 2 and is given by

$$x^{(k)}(i,j) = [1 - q_1(z_1)] x^{(k)}(i,j) + \sqrt{\sigma_k} \alpha_k(z_1) u(i,j). \quad (13b)$$

Note that in 2-D difference equations (13a) and (13b), z_1 and z_2 represent the spatial delay operators in the horizontal and vertical directions, respectively, i.e., $z_1 \cdot x(i,j) = x(i-1,j)$ and $z_2 \cdot x(i,j) = x(i,j-1)$, etc. On the other hand, (10) implies a direct implementation scheme as

$$y(i,j) = [1 - q_1(z_1) q_2(z_2)] y(i,j) + f(z_1, z_2) u(i,j). \quad (14)$$

For the sake of simplicity, in the following we assume $m = n$. Then to compute output $y(i,j)$ the direct realization (14) requires $O(n^2)$ multiplications and $O(n^2)$ additions while the proposed parallel-cascade implementation scheme (12) in conjunction with (13) requires $O(rn)$ multiplications and $O(rn)$ additions. Since

$$r = \operatorname{rank} F \leqslant \min(n, m) = n$$

the parallel-cascade implementation requires at most the same amount of computation as the direct realization scheme. Moreover, design experience has indicated (see, e.g. [9]) that for many important types of quadrantally symmetric 2-D filters such as circularly symmetric filters and fan filters with small stopband or passband angles, the rank r is often significantly less than the order of the design filter. In such cases the above comparison implies that the proposed implementation is computationally more efficient.

III. A Remark

In microprocessor-based implementations of 2-D digital filters, one of the major concerns is to minimize output-noise power due to the roundoff of products. To prevent overflow in a digital filter, signal scaling should also be considered in the above minimization, which accordingly gives a set of dynamic range constraints on the local state variables. In recent work [6], it has been shown that given a local state-space realization (A, b, c, d) of a stable 2-D filter that is both locally controllable and locally observable [1], it then is always possible to find a 2-D transformation $T = T_1 \oplus T_2$ with $T_1 \in \boldsymbol{R}^{n \times n}$, $T_2 \in \boldsymbol{R}^{m \times m}$ nonsingular such that the equivalent realization $(\hat{A}, \hat{b}, \hat{c}, \hat{d}) \triangleq (T^{-1}AT, T^{-1}b, cT, d)$ minimizes the output-noise power due to roundoff subject to a l_2-norm dynamic range constraint.

The results of [6] may immediately apply to the current case if the parallel structure (6a) is preserved. Indeed, let

$$x(i,j) = \begin{bmatrix} x_1^h(i,j) \\ \vdots \\ x_r^h(i,j) \\ \hline x_1^v(i,j) \\ \vdots \\ x_r^v(i,j) \end{bmatrix}$$

$$A = \left[\begin{array}{ccc|ccc} A_1^{(1)} & & 0 & A_2^{(1)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & A_1^{(r)} & 0 & & A_2^{(r)} \\ \hline A_3^{(1)} & & 0 & A_4^{(1)} & & 0 \\ & \ddots & & & \ddots & \\ 0 & & A_3^{(r)} & 0 & & A_4^{(r)} \end{array}\right]$$

$$b = \begin{bmatrix} b_1^{(1)} \\ \vdots \\ b_1^{(r)} \\ \hline b_2^{(1)} \\ \vdots \\ b_2^{(r)} \end{bmatrix}$$

$$c = \left[c_1^{(1)} \cdots c_1^{(r)} \vdots c_2^{(1)} \cdots c_2^{(r)} \right]$$

$$d = \sum_{k-1}^{r} d^{(k)}$$

then (A, b, c, d) gives a realization of $H(z_1, z_2)$. If we want to maintain the parallel structure when a 2-D coordinate transformation T is applied, then obviously T has to be of the form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} = \left[\begin{array}{cccc|cccc} T_1^{(1)} & & 0 & & & & & \\ & T_1^{(2)} & & & & & 0 & \\ & & \ddots & & & & & \\ 0 & & & T_1^{(r)} & & & & \\ \hline & & & & T_2^{(1)} & & & 0 \\ & & & & & T_2^{(2)} & & \\ & 0 & & & & & \ddots & \\ & & & & 0 & & & T_2^{(r)} \end{array}\right]. \tag{15}$$

It is now straightforward to see that preserving the parallel realization structure (6a), an optimal 2-D transformation T with form (15) which minimizes the output-noise power in each channel due to roundoff can be found by applying the results of [6] to subsystem (7) for each k, $1 \leqslant k \leqslant r$, yielding an optimal $T^{(k)} = T_1^{(k)} \oplus T_2^{(k)}$.

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Convergent and Roundoff Error Properties of Reflection Coefficients in Adaptive Spatial Recursive Least Squares Lattice Algorithm

S. H. AN AND K. YAO

Abstract —**Recursive least squares lattice (RLSL) algorithms are used in many adaptive filtering and array processing problems. Reflection coefficients are important parameters that characterize the behaviors of these**

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The authors are with the Electrical Engineering Department, University of California, Los Angeles, CA 90024.

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