Transactions Briefs

Realizations of 2-D Filters and Time Delay Systems

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Abstract —**Possible techniques for achieving an absolutely minimal realization of an input/output system described by a matrix of rational functions in two indeterminates are described.**

I. Introduction

Dynamical systems which can be modeled by a $p \times r$ proper transfer function matrix, say $H(s,z)$ as a proper rational matrix in two indeterminates s and z are considered. Quarter-plane causal 2-D digital filters, retarded or neutral delay systems, are examples of such dynamical systems (see [1] and [2] for detailed interpretation). Write $H(s,z)$ as

$$H \triangleq H(s,z) = \frac{N(s,z)}{a(s,z)}$$

where $N(s,z) \in R^{p \times r}[s,z]$, $a(s,z) \in R[s,z]$. The properness of H means that i) $\deg_s a(s,z) \geqslant \deg_s N(s,z)$, ii) $\deg_z a(s,z) \geqslant \deg_z N(s,z)$, and iii) the leading monomial $a_{nm}s^n z^m$ of

$$a(s,z) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} s^i z^j$$

is not zero.

One of the fundamental issues studied in systems theory is the realization of a given transfer function by a system of dynamical equations. Here, we are interested in finding a realization of a given $H(s,z)$ with such property that the numbers of both types of dynamical elements (i.e., integrators s^{-1} and delay lines z^{-1}) required in an implementation of H are minimized simultaneously. In the sequel, such a realization will be referred to as an absolutely minimal realization.

It should be mentioned that the concept of absolutely minimal realization in the 2-D setting is not new [3]–[7]. The main purpose of this paper is to further explore the main difficulties in obtaining such a realization. Possible techniques for achieving a minimal realization will also be addressed. In the next section, some preliminaries which enable us to tackle the central issues are given. Based on the use of a class of admissible transformations at the first level and the use of the system equivalent operation of an augmented system matrix, respectively, Section III contains a description of two procedures leading (possibly) to absolutely minimal realizations of a given 2-D transfer function matrix. These two methods are illustrated by examples.

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II. Preliminaries

Given a proper 2-D transfer function matrix $H(s,z) \in R^{p \times r}(s,z)$, it is always possible [5], [6] to find a 4-tuple $\{A, B, C, D\}$ such that the following Roesser model (system of dynamical equations) realizes $H(s,z)$:

$$\begin{bmatrix} sX^1 \\ zX^2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} X^1 \\ X^2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \equiv Ax + Bu$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} X^1 \\ X^2 \end{bmatrix} + Du \equiv Cx + Du. \quad (1)$$

That is

$$H(s,z) = C\begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & zI - A_4 \end{bmatrix}^{-1} B + D$$

and the dimensions of X^1 and X^2 determine the number of s^{-1} and z^{-1} dynamical elements in the implementation of (1).

Similar to the 1-D case [8], the system matrix $\pi(s,z)$ of (1) is defined as

$$\pi(s,z) = \begin{bmatrix} sI - A_1 & -A_2 & B_1 \\ -A_3 & zI - A_4 & B_2 \\ -C_1 & -C_2 & D \end{bmatrix}. \quad (2)$$

Two system matrices π_1 and π_2 are said to be system equivalent (SE), to be denoted by $\pi_1 \sim \pi_2$, if one can be obtained from the other by the elementary operations described in [8, p. 59]. It is easy to check that

$$\begin{aligned}\pi(s,z) \sim & \left\{ C_2 + C_1[sI - A_1]^{-1}A_2 \right\} \\ & \cdot \left\{ zI - A_4 - A_3[sI - A_1]^{-1}A_2 \right\}^{-1} \\ & \cdot \left\{ B_2 + A_3[sI - A_1]^{-1}B_1 \right\} \\ & + \left\{ D + C_1[sI - A_1]^{-1}B_1 \right\} \\ \triangleq & \bar{C}(s)[zI - \bar{A}(s)]^{-1}\bar{B}(s) + \bar{J}(s) = H(s,z) \quad (3a)\end{aligned}$$

and

$$\begin{aligned}\pi(s,z) \sim & \left\{ C_1 + C_2[zI - A_4]^{-1}A_3 \right\} \\ & \cdot \left\{ sI - A_1 - A_2[zI - A_4]^{-1}A_3 \right\}^{-1} \\ & \cdot \left\{ B_1 + A_2[zI - A_4]^{-1}B_2 \right\} \\ & + \left\{ D + C_2[zI - A_4]^{-1}B_2 \right\} \\ \triangleq & \tilde{C}(s)[sI - \tilde{A}(z)]^{-1}\tilde{B}(z) + \tilde{J}(z) = H(s,z) \quad (3a)\end{aligned}$$

where $\{\bar{A}(s), \bar{B}(s), \bar{C}(s), \bar{J}(s)\}$ and $\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{J}(z)\}$ are known as the first-level realizations of $H(s,z)$ [6]. By (3), it can be observed [5] that

$$P(s) \triangleq \begin{bmatrix} \bar{A}(s) & \bar{B}(s) \\ \bar{C}(s) & \bar{J}(s) \end{bmatrix} = \begin{bmatrix} A_3 \\ C_1 \end{bmatrix}[sI - A_1]^{-1}[A_2 \quad B_1] + \begin{bmatrix} A_4 & B_2 \\ C_2 & D \end{bmatrix} \quad (4a)$$

and

$$Q(z) \triangleq \begin{bmatrix} \tilde{A}(z) & \tilde{B}(z) \\ \tilde{C}(z) & \tilde{J}(z) \end{bmatrix} = \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} [zI - A_4]^{-1} [A_3 \quad B_2] + \begin{bmatrix} A_1 & B_1 \\ C_1 & D \end{bmatrix}. \quad (4b)$$

Therefore, in order to realize $H(s,z)$, one may find a minimal first-level realization $\{\bar{A}(s), \bar{B}(s), \bar{C}(s), \bar{J}(s)\}$ over $R(s)$ ($\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{J}(z)\}$ over $R(z)$) and then form $P(s)$ ($Q(z)$) and find its minimal realization

$$\left\{ A_1, [A_2 \quad B_1], \begin{bmatrix} A_3 \\ C_1 \end{bmatrix}, \begin{bmatrix} A_4 & B_2 \\ C_2 & D \end{bmatrix} \right\}$$

$$\left(\left\{ A_4, [A_3 \quad B_2], \begin{bmatrix} A_2 \\ C_2 \end{bmatrix}, \begin{bmatrix} A_1 & B_1 \\ C_1 & D \end{bmatrix} \right\} \right).$$

Clearly, the use of the above realization approach implies that

$$\text{size of } A_4 = \text{size of } \bar{A}(s) \text{ and size of } A_1 = \delta_M(P(s)) \quad (5a)$$

and

$$\text{size of } A_1 = \text{size of } \tilde{A}(z) \text{ and size of } A_4 = \delta_M(Q(z)) \quad (5b)$$

where $\delta_M(P(s))$ denotes the McMillan degree of $P(s)$.

The following controllability and observability concepts are needed in the realization theory. Given a first-level realization of H, say $(A(s), B(s), C(s), J(s))$ of order n over $R(s)$, pair $(A(s), B(s))$ is said to be $R(s)$-controllable if

$$\text{Span}[B(s) \quad A(s)B(s) \cdots A^{n-1}(s)B(s)] = R^n(s)$$

and pair $(C(s), A(s))$ is said to be $R(s)$-observable if $(A^T(s), C^T(s))$ is $R(s)$-controllable. At the second-level, pair (A, B) of (1) is said to be modally controllable if

$$\begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & zI - A_4 \end{bmatrix} \text{ and } B \text{ are left-coprime} \quad (6a)$$

and pair (C, A) of (1) is said to be modally observable if

$$C \text{ and } \begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & zI - A_4 \end{bmatrix} \text{ are right-coprime.} \quad (6b)$$

It turns out that the minimality of a 2-D state-space model is closely related to the concept of modal controllability and observability. In fact, for a scalar 2-D proper rational function $h(s,z)$ of order (n,m), it has been shown [4] that only a state-space realization with order (n,m), i.e., the same order as the transfer function, can be both modally controllable and modally observable. Therefore, a formal definition of absolutely minimal realization can be given as below.

Definition: Realization $\{A, B, C, D\}$ in (1) is said to be absolutely minimal if (1) is modally controllable as well as modally observable.

With $\bar{A}(s)$, $\bar{B}(s)$, $\bar{C}(s)$, $\tilde{A}(z)$, $\tilde{B}(z)$, and $\tilde{C}(z)$ defined as in (3), the following theorem relates ring controllability (ring observability) to modal controllability (modal observability).

Theorem [9]:

1) Equation (1) is modal controllable iff $(\bar{A}(s), \bar{B}(s))$ is $R(s)$-controllable and $(\tilde{A},(z), \tilde{B}(z))$ is $R(z)$-controllable;

2) Equation (1) is modal observable iff $(\bar{C}(s), \bar{A}(s))$ is $R(s)$-observable and $(\tilde{C}(z), \tilde{A}(z))$ is $R(z)$-observable.

As an immediate consequence of the theorem, we have the following corollary.

Corollary: Assume that $\{\bar{A}(s), \bar{B}(s), \bar{C}(s), \bar{J}(s)\}$ and $\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{J}(z)\}$ are minimal first-level realizations of $H(s,z)$,

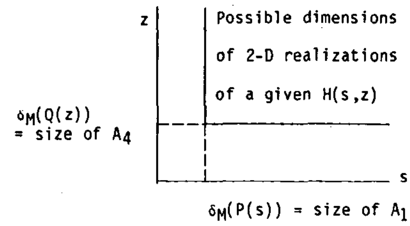

Fig. 1. Graphical representation of the Corollary.

then (1) is absolutely minimal iff

$$\delta_M(P(s)) = \text{size of } \tilde{A}(z) \quad (7a)$$

$$\delta_M(Q(z)) = \text{size of } \bar{A}(s). \quad (7b)$$

The above corollary can be illustrated by Fig. 1 (suggested to the authors by Prof. L. Markus).

III. Realization Procedures

The problem of finding an absolutely minimal realization of $H(s,z)$ still remains open [4]. In this section, two procedures leading possibly to an absolutely minimal realization for a given proper 2-D transfer function matrix are considered.

Procedure 1: The first suggested procedure is based on the fact that the McMillan degree of $Q(z)$ ($P(s)$) defined as in (4) may be different among the equivalent first-level realizations. To be more precise, assume that $\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{D}(z)\}$ is a minimal first-level realization of H and that size of $\tilde{A}(z) = n$. A transformation $T(z) \in C^{n \times n}(z)$ is said to be admissible if $T\tilde{A}T^{-1}$, $T\tilde{B}$ and $\tilde{C}T^{-1}$ are all proper rational matrices. Clearly, such a similarity transformation leads to an equivalent and minimal first-level realization. Note that

$$Q_T(z) \triangleq \begin{bmatrix} T(z)\tilde{A}(z)T^{-1}(z) & T(z)\tilde{B}(z) \\ \tilde{C}(z)T^{-1}(z) & \tilde{D}(z) \end{bmatrix} = \begin{bmatrix} T(z) & 0 \\ 0 & I \end{bmatrix} Q(z) \begin{bmatrix} T(z) & 0 \\ 0 & I \end{bmatrix}^{-1} \quad (8)$$

and $\delta_M(Q_T(z))$ might be different from $\delta_M(Q(z))$. Thus, one may try to use an admissible transformation adequately such that $\delta_M(Q_T(z))$ is less than $\delta_M(Q(z))$.

As an example, let us consider a SISO retarded delay-differential system with the transfer function

$$h(s,z) = \frac{sz+1}{s^2z^2+1}. \quad (9)$$

Obviously

$$\tilde{A}_1(z) = \begin{bmatrix} 0 & 1 \\ -z^{-2} & 0 \end{bmatrix}, \quad \tilde{b}_1(z) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and}$$

$$\tilde{c}_1(z) = [z^{-2} \quad z^{-1}] \quad (10)$$

is a first-level realization of $h(s,z)$, which minimizes the number of integrators. Note that

$$\delta_M \begin{bmatrix} \tilde{A}_1(z) & \tilde{b}_1(z) \\ \tilde{c}_1(z) & 0 \end{bmatrix} = 3.$$

Thus, (10) generates a Roesser model requiring two integrators and three delay elements. Further notice that with

$$T(z) = \begin{bmatrix} z^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$\tilde{A}_2(z) = T\tilde{A}_1T^{-1} = \begin{bmatrix} 0 & z^{-2} \\ -1 & 0 \end{bmatrix}, \qquad \tilde{b}_2(z) = T\tilde{b}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\tilde{c}_2(z) = \tilde{c}_1T^{-1} = [1 \quad z^{-1}] \tag{11}$$

which leads to

$$\delta_M \begin{bmatrix} \tilde{A}_2(z) & \tilde{b}_2(z) \\ \tilde{c}_2(z) & 0 \end{bmatrix} = 2.$$

Thus, $\{\tilde{A}_2(z), \tilde{b}_2(z), \tilde{c}_2(z)\}$ generates an absolutely minimal realization of (9).

It was shown [4], [5] that for the following 2-D transfer function

$$h(s,z) = \frac{s+z}{sz-1}$$

no real 2-D state-space realizations of dimension (1,1) exist. Therefore, it is necessary to let $T(z)$ belong to $C^{n \times n}(z)$ instead of $R^{n \times n}(z)$.

We now suggest a method for possible reduction of the size of a realization of a 2-D transfer matrix. The method consists of the following steps: 1) Find a first level minimal realization of $H(s,z)$, say $\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{J}(z)\}$. 2) Find the size of a minimal realization w.r.t. the second variable. Note the size of m. 3) If $\delta_M(Q(z)) = m$, we are done. If not, try to use an admissible $T(z)$ such that $\delta_M(Q_T(z))$ given by (8) is reduced. Definitely, effort is needed to give a systematic way yielding such an admissible transformation (if any).

Procedure 2: The second suggested procedure of finding an absolutely minimal realization is dependent upon the use of SE operation of an augmented system matrix, as explained by the following example.

Example: Consider the transfer function matrix of a neutral delay-differential system

$$H(s,z) = \frac{\begin{bmatrix} s/z & z/(z-1) \\ 1/z(z-1) & sz/(z-1) \end{bmatrix}}{s^2 - \dfrac{1}{z-1}}. \tag{12}$$

It is easy to check that

$$\tilde{A}(z) = \begin{bmatrix} 0 & \dfrac{1}{z-1} \\ 1 & 0 \end{bmatrix}, \quad \tilde{B}(z) = \begin{bmatrix} 0 & 1 \\ (z-1)/z^2 & 0 \end{bmatrix}$$

$$\tilde{C}(z) = \begin{bmatrix} 0 & \dfrac{z}{z-1} \\ \dfrac{z}{z-1} & 0 \end{bmatrix}$$

and

$$\tilde{J}(z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{13}$$

constitute a minimal first-level realization of (12). Now express $H(s,z)$, the transfer function of a system over $R(s)$, as

$$H(s,z) = \frac{\begin{bmatrix} \dfrac{1}{s}z - \dfrac{1}{s} & \dfrac{1}{s^2}z^2 \\ \dfrac{1}{s^2} & \dfrac{1}{s}z^2 \end{bmatrix}}{z^2 - \dfrac{s^2+1}{s^2}z}. \tag{14}$$

Using one of the standard methods [10], one can check that the order of a minimal realization of (14) is 3. Next, form the augmented first-level system matrix as

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & sI - \tilde{A}(z) & \tilde{B}(z) \\ 0 & -\tilde{C}(z) & \tilde{J}(z) \end{bmatrix}$$

$$= \left[\begin{array}{c|cc|cc} I_3 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline 0 & s & -\dfrac{1}{z-1} & 0 & 1 \\ & -1 & s & \dfrac{z-1}{z^2} & 0 \\ \hline 0 & 0 & -\dfrac{z}{z-1} & 0 & 0 \\ & -\dfrac{z}{z-1} & 0 & 0 & 0 \end{array}\right]. \tag{15}$$

Using elementary operations leads (15) to

$$\left[\begin{array}{cc|ccc|cc} s & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & s & 0 & 1 & 0 & 0 & -1 \\ \hline 0 & 1 & z-1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & & \\ 0 & 1 & -1 & 0 & 0 & \multicolumn{2}{c}{0} \end{array}\right]$$

$$\triangleq \begin{bmatrix} sI_2 - A_1 & -A_2 & B_1 \\ -A_3 & zI_3 - A_4 & B_2 \\ -C_1 & -C_2 & 0 \end{bmatrix}. \tag{16}$$

Since size of $A_1 = 2$ and size of $A_4 = 3$, 4-tuple $\{A, B, C, D\}$ given in (16) is absolutely minimal.

Thus, the second procedure uses the 2-D system matrix and consists of the following steps: 1) Find a first-level minimal realization, say $\{\tilde{A}(z), \tilde{B}(z), \tilde{C}(z), \tilde{J}(z)\}$; 2) Determine the order of a first-level minimal realization w.r.t. the second variable, say m; 3) Form the augmented first-level system matrix as

$$\begin{bmatrix} I_m & 0 & 0 \\ 0 & sI_n - \tilde{A}(z) & \tilde{B}_2(z) \\ 0 & -\tilde{C}(z) & \tilde{J}(z) \end{bmatrix};$$

4) Use SE operation to transform (17) to the form

$$\begin{bmatrix} sI_n - A_1 & -A_2 & B_1 \\ -A_3 & zI_m - A_4 & B_2 \\ -C_1 & -C_2 & D \end{bmatrix}$$

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On Complementarity and Sensitivity of Generalized Wave Digital Filters

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Abstract —**In recent literature, a general theory has been developed to design low-sensitivity digital filter structures in the z-domain. These structures are two-ports with two inputs and two outputs and wave digital filters belong to this general class. This paper explores the significance of the para-unitary or complementary condition on generalized wave digital filters. Satisfying this condition allows, for example, the simultaneous generation of low- and high-pass filter outputs. There is discussion too of sensitivity and its significance to those generalized wave digital filter structures that do not satisfy the para-unitary condition.**

I. Introduction

In recent literature, a systematic approach to the design of a low-sensitivity digital filter structure has been described [1]–[3]. This approach treats the digital filter as a two-input and two-output system, which, of course, is also a characteristic of wave digital filters [4]. One of the underlying assumptions used is that the para-unitary condition holds. That is, with reference to Fig. 1,

$$S^T S^* = E \tag{1}$$

where S is the two-port transfer or scattering matrix and E is the unit matrix and

$$\boldsymbol{Y} = S\boldsymbol{X}. \tag{2}$$

From (1), we can derive simply that

$$\left.\begin{aligned} |S_{11}|^2 + |S_{21}|^2 &= 1 \\ |S_{12}|^2 + |S_{22}|^2 &= 1 \end{aligned}\right\}. \tag{3}$$

Equation (1) holds for wave digital filters (WDF) that are pseudolossless, i.e., derived from lossless analog networks [5].

If the analog reference filter is also reciprocal, then

$$S_{21} = S_{12} \tag{4}$$

and, from (3), we have also that

$$|S_{11}|^2 = |S_{22}|^2.$$

The para-unitary or complementary property has the important characteristic of providing, for example, both low-pass and high-pass filter outputs or both bandpass and bandstop outputs [8]. This fact has been found useful in certain filtering applications [9].

In this paper, we will consider the implications of the para-unitary condition on generalized wave digital filters (GWDF) [6]. A brief review of GWDF is appropriate and will be given here.

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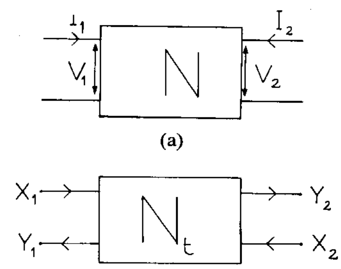

Fig. 1. (a) Two-port analog network. (b) Transformed two-port analog network.

The relationship between voltage, current, and wave variables can be generalized as follows:

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = P \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \tag{5}$$

$$\begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} = Q \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{6}$$

where X_1 and X_2 are input variables and Y_1 and Y_2 are output variables. P and Q are 2×2 nonsingular matrices. The realizability conditions that impose constraints on the elements of P and Q have been examined elsewhere [5]. We shall see that further constraints are necessary to ensure the complementary property holds.

For a two-port analog network, we may express the relationship between port voltages and currents by the $ABCD$ or transmission matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = T \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}. \tag{7}$$

This form is useful for cascading two-ports. Combining (5)–(7) to eliminate the voltages and currents gives

$$\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = R \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \tag{8}$$

where $R = PTQ^{-1}$ is the transmission matrix of the transformed two-port.

We can obtain finally

$$\boldsymbol{Y} = \sigma \boldsymbol{X} \tag{9}$$

where

$$\sigma_{11} = R_{22}/R_{12}$$
$$\sigma_{12} = -\det R/R_{12}$$
$$\sigma_{21} = 1/R_{12}$$
$$\sigma_{22} = -R_{11}/R_{12}$$

and $\det R = \det P \cdot \det T / \det Q$.

For reciprocal two-ports, $\det T = -1$.

Note that for the generalized scattering matrix, the symmetry condition $\sigma_{12} = \sigma_{21}$ holds if $\det R = -1$. For an analog reciprocal two-port as reference, the condition that $\sigma^T = \sigma$ is

$$\det P = \det Q.$$

This relationship is satisfied by voltage and current waves and indeed all transformations listed in [6].

A description of how signal-flow diagrams are derived for the various analog components by substituting for T and applying

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