However, from (7) it follows that range $[W]$ and null $[H]$ are equal and necessity has been shown. Sufficiency is obvious.

Lemma 2: rank $[H^T, B^T H^T] = n$ is both necessary and sufficient for (6).

Proof (Sufficiency): Suppose rank $[H^T, B^T H^T] = n$. Therefore, if $x \in$ null $[H]$, then $x \notin$ null $[B]$ and $x \notin$ null $[HB]$ and (6) holds.

Proof (Necessity): Suppose (6) holds. Therefore, if $Hx = 0$, then $HBx \neq 0$ and rank $[H^T, B^T H^T] = n$.

It is interesting to note that a necessary condition for the rank condition of Theorem 1 to be satisfied is $m \geq n/2$.

It remains to investigate conditions on the system triple (A, B, H) which ensure that T can also be chosen so the F is stable.

IV. Observer Stabilizability

Assume that the conditions of Theorem 1 are satisfied and that a basis for the state space has been chosen so that $\tilde{H}$ is given by (7). If an observer exists, then the basis for the observer's state space can be chosen so that T has the form, $T = [K, I_{n-m}]$. Therefore, assuming that T is in this form and that G is null, it is seen that (3) and (4) are satisfied with

$$F = \tilde{A}_{22} + KA_{12} \tag{11}$$

$$O = \tilde{B}_{22} + K\tilde{B}_{12} \tag{12}$$

$$E = [K\tilde{B}_{11} + \tilde{B}_{21}]\tilde{H}_1^{-1} \qquad D = [KA_{11} + A_{21} - FK]\tilde{H}_1^{-1} \tag{13}$$

where $\tilde{A}_{ij}$, $\tilde{B}_{ij}$ are partitions of $\tilde{A}$ and $\tilde{B}$ with $\tilde{B}_{12} = \tilde{H}_1 HBW$, $\tilde{B}_{22} = QBW$ (9).

Notice that E and D (13) can always be calculated once K is determined so that F (11) is stable and (12) is satisfied. Since the conditions of Theorem 1 are assumed, it follows from the proof of Lemma 1 that null $[\tilde{B}_{12}]$ is empty and therefore K can always be found to satisfy (12). The existence of an observer of the type being proposed here rests finally on the existence of a stable F matrix (11), for some K matrix satisfying (12). The condition which ensures that there is a potential observer which is stable is given now in the following.

Theorem 2: If the condition of Theorem 1 is satisfied, then the necessary and sufficient condition for the existence of a delayless observer is that all transmission zeros for (A, BQ_o^T, H) be stable where Q_o is any matrix satisfying range $[Q_o^T]$ = null $[H]$ and where s_o is a transmission zero of (A, BQ_o^T, H) if rank $[\Gamma(s_o)] < 2n - m$ where

$$\Gamma(s) = \left[\begin{array}{c:c} sI - A & BQ_o^T \\ \hdashline H & O \end{array}\right].$$

The proof of this theorem relies on the following lemma.

Lemma 3: If the condition of Theorem 1 is satisfied, then it is always possible to choose a basis for the given system (1) such that

$$\tilde{H} = [\tilde{H}_1, 0] \text{ and } \tilde{B} = [\tilde{B}_1, \tilde{B}_2]$$

where $\tilde{B}_2^T = [\bar{B}_2^T, 0]$ with $\bar{B}_2$ nonsingular.

Proof: There is no loss of generality [7] if Q in V^{-1}(8) is chosen as $Q = Q_0 + LH$ where $L \in R^{(n-m)\times m}$ and range $[Q_o^T]$ = null $[H]$. Then W in V (8) can be taken as $Q_o^T(Q_oQ_o^T)^{-1}$. Thus, $\tilde{B}_{22}$ (9) becomes $\tilde{B}_{22} = QBW = [Q_oBQ_o^T + LHBQ_o^T](Q_oQ_o)^{-1}$. Now L can be chosen so that $\tilde{B}_{22}$ is null only if null $[Q_oBQ_o^T \supset$ null $[HBQ_o^T]$. This requirement is satisfied since $z \in$ null $[HBQ_o^T]$ implies that $z \in$ null $[BQ_o^T]$. This implication follows from the facts that range $[Q_o^T]$ = null $[H]$ and range $[B(\text{null } [H])] \wedge$ null $[H] = \{0\}$, (6). Thus, L can be chosen to make $\tilde{B}_{22}$ null. Since the columns of $\tilde{B}_{12} = \tilde{H}_1 HBW$ are independent (Lemma 1), it follows that the arbitrary nonsingular matrix $\tilde{H}_1$ can be chosen so that $\tilde{B}_{12}^T = [\bar{B}_2^T, 0]$ with $\bar{B}_2$ nonsingular.

Proof of Theorem 2: Let the basis for the given system be chosen in the manner specified in Lemma 3. Then (12) is satisfied by K in the form $K = [0 \;\; K_2]$ where $K_2 \in R^{(n-m)\times 2m-n}$ is arbitrary and 0 is an $n - m$ by $n - m$ matrix of zeros. Then F becomes $\tilde{A}_{22} + K_2\tilde{A}_{122}$ where $\tilde{A}_{12}^T = [\tilde{A}_{121}^T, \tilde{A}_{122}^T]$, $\tilde{A}_{121} \in R^{(n-m)\times(n-m)}$, $\tilde{A}_{122} \in R^{(2m-n)\times(n-m)}$. Thus, the unobservable eigenvalues of $(\tilde{A}_{122}, \tilde{A}_{22})$ are eigenvalues of F which are fixed for all potential delayless observers. Therefore, the location of these invariant eigenvalues in the left half plane is seen to be necessary for the existence of delayless observers. It was shown in [6] that the necessary and sufficient condition for λ_o to be an unobservable eigenvalue of the pair $(\tilde{A}_{122}, \tilde{A}_{22})$ is that rank $[\tilde{\Gamma}(\lambda_o)] < 2n - m$ where in the present instance

$$\tilde{\Gamma}(s) = \begin{bmatrix} sI - \tilde{A} & \tilde{B}_{12} \\ \tilde{H} & 0 \end{bmatrix}.$$

Finally, notice that $Z^{-1}\Gamma(s)ZY \equiv \tilde{\Gamma}(s)$ where Z and Y are square nonsingular with $Z = \text{Diag}\,[V, I_m]$, $Y = \text{Diag}\,[I_{n-m}, (Q_oQ_o^T)^{-1}]$ with V taken in the manner specified in Lemma 3. Thus, rank $[\tilde{\Gamma}(s)] \equiv$ rank $[\Gamma(s)]$.

V. Conclusion

In this note it has been shown that delayless observers for linear systems having a time delay in the state can be constructed if and only if two conditions are satisfied. These conditions which are given in the theorems can readily be checked using available computer packages developed for use in automatic control.

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A Lyapunov Theory for Linear Time-Delay Systems

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***Abstract*—Some relationships between a Lyapunov based stability test and linear control system models with time delays are explored. The main result is a generalization of the result that if (A, B) is a reachable pair of matrices, then square A is a stability matrix if and only if there is a positive-definite matrix K such that $AK + KA' = -BB'$.**

I. Introduction

For a long time Lyapunov stability theory has played a central role in stability analysis of various dynamical systems. In the linear time-invariant case, the theory can be used to conclude that the null solution ($x(t) \equiv 0$) of the ordinary differential equation $\dot{x} = Ax$ is asymptotically stable if and only if for any given positive-definite symmetric matrix Q there exists a positive definite (symmetric) matrix P that satisfies the Lyapunov equation

$$A'P + PA = -Q. \tag{1}$$

Manuscript received July 22, 1985; revised October 22, 1985. This work was supported in part by the National Science Foundation under Grant DMS-8413129.

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IEEE Log Number 8406896.

0018-9286/86/0300-0259$01.00 © 1986 IEEE

As a variant of this, an important observation was made by Kalman [1] for the control system model

$$\dot{x}=Ax+Bu$$

$$y=Cx \tag{2}$$

which may be stated as follows.

Lemma 1.1: When (A, B) is a reachable pair, square A is a stability matrix (i.e., all its eigenvalues lie in the open left-half plane), if and only if there exists a positive-definite matrix K that satisfies

$$AK+KA'=-BB'. \tag{3}$$

When (C, A) is an observable pair, square A is a stability matrix if and only if there exists a positive-definite matrix W that satisfies

$$A'W+WA=-C'C. \tag{4}$$

Lemma 1.1 has been used extensively in various system-related applications. The recent evidence, among others, include the balanced-realization-based model reduction approaches of Moore [2] and Glover [3], and the synthesis of minimum roundoff noise fixed point digital filters by Mullis and Roberts [4] and Hwang [5].

For linear time-delay type systems, Brierley *et al.* [6] considered asymptotic stability independent of delay (i.o.d.) [7] of the null solution $x(t) \equiv 0$ of systems with differential-difference equation models $\dot{x}(t) = A(z)x(t)$ where z represents a delay (right shift) operator and $A(z)$ is an $n \times n$ polynomial matrix in z. They related stability i.o.d. to the existence of a positive-definite solution $P(z)$ of the generalized Lyapunov equation

$$A^*(z)P(z)+P(z)A(z)=-Q(z) \tag{5}$$

where $A^*(z)$ denotes the complex conjugate transpose (Hermitian transpose) of $A(z)$, $Q(z)$ is any given positive-definite Hermitian matrix, and $z = e^{j\omega}$ with $\omega \epsilon [0, 2\pi]$. However, the counterpart of the above-mentioned reachability-stability result for time-delay type systems is not available as yet.

The purpose of this note is to provide such a result for linear neutral type systems with commensurate delays. After a brief introduction (some notation and preliminaries in Section II), we present a Lyapunov criterion in Section III for retarded delay systems. The results are then extended to a class of linear neutral delay type systems with stable D-operator [10]. One of the potential applications of the results presented here is to derive a realization of a balanced type for a given time-delay model that would make it possible to do model reduction in a nearly optimal way.

II. Notation and Preliminaries

Let $R[z]$ be the set of all polynomials in z with real coefficients. A linear multivariable retarded system with commensurate delays can be represented by differential-difference equations

$$\dot{x}(t)=A(z)x(t)+B(z)u(t) \tag{6a}$$

$$y(t)=C(z)x(t) \tag{6b}$$

where z is a delay (right shift) operator, i.e., $zx(t) = x(t-h)$, $h \geqslant 0$, $A(z) \in R^{n\times n}[z]$, $B(z) \in R^{n\times m}[z]$, and $C(z) \in R^{r\times n}[z]$. Let $\Delta(s, e^{-sh}) = \det [sI - A(e^{-sh})]$, then the (null solution $x(t) \equiv 0$ of) system (6) (with $u(t) \equiv 0$) is asymptotically stable independent of delay if and only if [10]

$$\Delta(s, e^{-sh})\neq 0 \qquad \text{for Re}s\geqslant 0 \text{ and all } h\geqslant 0 \tag{7}$$

Kamen [7]–[9] has established that if

$$\Delta(0, z)\neq 0 \text{ for } |z|=1 \tag{8}$$

then condition (7) is equivalent to the two-variable criterion

$$\Delta(s, z)\neq 0 \qquad \text{for Re}s\geqslant 0,\ |z|=1. \tag{9}$$

A linear neutral type of control system with commensurate delays can be represented by the differential-difference equations

$$D(z)\dot{x}(t)=A(z)x(t)+B(z)u(t) \tag{10a}$$

$$y(t)=C(z)x(t) \tag{10b}$$

where $D(z) \in R^{n\times n}[z]$ with $D(0)$ nonsingular. The neutral equations considered in this note are those where $D(z)$ is formally stable [10], i.e.,

$$\det\ D(z)\neq 0, \qquad \text{for } |z|\leqslant 1+\delta \text{ for some } \delta>0 \tag{11}$$

and will be referred to as neutral delay systems with stable D-operator. Fix $\delta > 0$, and define $S_\delta = \{p(z) \in R[z] | p(z) \neq 0 \text{ for } |z| \geqslant 1 + \delta\}$ and define $R_\delta = S_\delta^{-1}R[z] \equiv \{q(z)/p(z) | q(z) \in R[z],\ p(z) \in S_\delta\}$. Given the neutral delay system (10) with stable D-operator, define the R_δ-associated differential-difference equations

$$\dot{x}(t)=F(z)x(t)+G(z)\mu(t) \tag{12a}$$

$$y(t)=C(z)x(t) \tag{12b}$$

where $F(z) = D^{-1}(z)A(z) \in R_\delta^{n\times n}$, $G(z) = D^{-1}(z)B(z) \in R_\delta^{n\times m}$. Since condition (11) implies that $\det D(e^{-sh}) \neq 0$ for Re$s \geqslant 0$ and all $h \geqslant 0$, the null solution of (10) (with $u(t) \equiv 0$) is asymptotically stable i.o.d., if and only if

$$\det\ [sI-F(e^{-sh})]\neq 0 \qquad \text{for Re}s\geqslant 0 \text{ for all } h\geqslant 0. \tag{13}$$

III. A Lyapunov Criterion for Retarded Systems

Let $[A(z)|B(z)]$ and $[A(z)/C(z)]$ denote the reachability and the observability matrices associated with the differential-difference equations (6).

Theorem 3.1: Assume that

$$\text{rank}\ [A(e^{j\omega})|B(e^{j\omega})]=n, \qquad \text{for all } \omega\epsilon[0,\ 2\pi]. \tag{14}$$

Then the null solution $x(t) \equiv 0$ of (6) with $u(t) \equiv 0$ is asymptotically stable i.o.d., if there exists a positive-definite Hermitian matrix $K(e^{j\omega})$ $\forall \omega\epsilon[0, 2\pi]$ that satisfies

$$A(e^{j\omega})K(e^{j\omega})+K(e^{j\omega})A^*(e^{j\omega})=-B(e^{j\omega})B^*(e^{j\omega}). \tag{15}$$

Assume that

$$\text{rank}\begin{bmatrix} A(e^{j\omega}) \\ \hline C(e^{j\omega})\end{bmatrix}=n \qquad \text{for } \omega\epsilon[0,\ 2\pi], \tag{16}$$

then the null solution $x(t) \equiv 0$ of (6) with $u(t) \equiv 0$ is asymptotically stable i.o.d., if there exists a positive-definite Hermitian matrix $W(e^{j\omega})$ $\forall \omega\epsilon[0, 2\pi]$ that satisfies

$$A^*(e^{j\omega})W(e^{j\omega})+W(e^{j\omega})A(e^{j\omega})=-C^*(e^{j\omega})C(e^{j\omega}). \tag{17}$$

Moreover, if

$$\det\ A(e^{j\omega})\neq 0, \qquad \forall\omega\epsilon[0,\ 2\pi] \tag{18}$$

then the above sufficient conditions are also necessary.

Proof: Suppose the Lyapunov equation (15) has a positive-definite solution $K(e^{j\omega})$ $\forall \omega\epsilon[0, 2\pi]$. Fixing any $\omega\epsilon[0, 2\pi]$ condition (14) implies $(A(e^{j\omega}), B(e^{j\omega}))$ is a (complex) reachable pair. Let $V[x] = x^*K(e^{j\omega})x$, then for any $x \neq 0$, $V[x] > 0$. It turns out that along any solution trajectory of the differential equations $\dot{x} = A(e^{j\omega})x(t)$,

$$\frac{d}{dt}V[x(t)]=-x^*(t)B(e^{j\omega})B^*(e^{j\omega})x(t)\leqslant 0 \tag{19}$$

which, along with the reachability of $(A(e^{j\omega}), B(e^{j\omega}))$, means that dV/dt is not identically zero along any trajectory of $\dot{x}(t) = A(e^{j\omega})x(t)$. Then LaSalle's well-known extension of the Lyapunov stability theorem [11] allows one to claim that $\det [sI - A(e^{j\omega})] \neq 0$ for Re$s \geqslant 0$, i.e., $\det [sI - A(z)] \neq 0$ for Re$s \geqslant 0$ and $|z| = 1$. Therefore, using results of [9] one concludes that the null solution of the equations (6) with $u(t) \equiv 0$ is asymptotically stable i.o.d. Conversely, notice that condition (18) is the same as condition (8). Thus, the asymptotic stability of null solution i.o.d. along with condition (18) implies (9). So one can define, for arbitrary fixed $\omega \in [0, 2\pi]$,

$$K(e^{j\omega}) = \int_0^{\infty} e^{A(e^{j\omega})t} B(e^{j\omega}) B^*(e^{j\omega}) e^{A^*(e^{j\omega})t}\, dt.$$

Straightforward manipulations then indicate that $K(e^{j\omega})$ satisfies (15). It is routine to show that $K(e^{j\omega})$ is positive-definite through the use of the reachability of $(A(e^{j\omega}), B(e^{j\omega}))$. A similar argument can be given for the other part of the theorem.

Remark 1: The asymptotic stability i.o.d. of the null solution of the equation (6) with $u(t) \equiv 0$ along with condition (17) also implies the uniqueness of the positive-definite solution of (15) as well as (17).

Remark 2: It can be observed that if the matrix $A(z)$ in (6) is independent of z then there is no difference between condition (7) and condition (9). Consequently in such a case, condition (18) in Theorem 3.1 is no longer needed. Theorem 3.1, therefore, is a natural extension of Kalman's result (Lemma 1.1) to the time-delay setting.

Remark 3: Reachability condition (14) and observability condition (16) are much weaker than $R[z]$-reachability and $R[z]$-observability, respectively. Moreover, (14) and (16), in the delay-free case become the ordinary notions of reachability and observability.

IV. Extension to Neutral Delay Systems

In this section we consider a class of neutral delay systems as modeled by differential difference equations (10) with stable D-operator.

Let $\hat{F}(z) = A(z)D^{-1}(z)$, then the Lyapunov result is stated as follows.

Theorem 4.1: Assume that

$$\text{rank } [\hat{F}(e^{j\omega}) | B(e^{j\omega})] = n \qquad \forall\ \omega \in [0, 2\pi]. \tag{20}$$

Then the null solution of homogeneous ($u(t) \equiv 0$) differential-difference equations (10) is asymptotically stable i.o.d. if there exists a positive-definite Hermitian matrix $\hat{K}(e^{j\omega})\ \forall\ \omega \in [0, 2\pi]$ that satisfies

$$F(e^{j\omega})\hat{K}(e^{j\omega}) + \hat{K}(e^{j\omega})F^*(e^{j\omega}) = -G(e^{j\omega})G^*(e^{j\omega}). \tag{21}$$

Assume that

$$\text{rank} \begin{bmatrix} \hat{F}(e^{j\omega}) \\ \overline{C(e^{j\omega})} \end{bmatrix} = n, \qquad \forall\ \omega \in [0, 2\pi] \tag{22}$$

then the null solution $x(t) \equiv 0$ of homogeneous ($u(t) \equiv 0$) differential-difference equations (10) is asymptotically stable i.o.d. if there exists a positive-definite Hermitian matrix $\hat{W}(e^{j\omega})\ \forall\ \omega \in [0, 2\pi]$ that satisfies

$$F^*(e^{j\omega})\hat{W}(e^{j\omega}) + \hat{W}(e^{j\omega})F(e^{j\omega}) = -C^*(e^{j\omega})C(e^{j\omega}). \tag{23}$$

If

$$\det A(e^{j\omega}) \neq 0 \qquad \forall \omega \in [0, 2\pi] \tag{24}$$

then the above sufficient conditions are also necessary conditions.

Proof: Note first that

$$\begin{aligned} D^{-1}(z)[\hat{F}(z)|B(z)] &= D^{-1}(z)[B(z)\ \hat{F}(z)B(z)\ \cdots\ \hat{F}^{n-1}(z)B(z)] \\ &= [D^{-1}B\ (D^{-1}\hat{F}D)D^{-1}B\ \cdots\ (D^{-1}\hat{F}D)^{n-1}D^{-1}B] \\ &= [G(z)\ F(z)G(z)\ \cdots\ F^{n-1}(z)G(z)] \end{aligned} \tag{25}$$

since the formal stability of $D(z)$ implies $\det D(e^{j\omega}) \neq 0\ \forall\ \omega \in [0, 2\pi]$ condition (20) along with (25) gives

$$\text{rank } [F(e^{j\omega}) | G(e^{j\omega})] = n \qquad \forall\ \omega \in [0, 2\pi].$$

If (21) has a positive-definite solution, one can now use the same argument as in the proof of Theorem 3.1 to conclude that for any $\omega \in [0, 2\pi]$

$$\det [sI - F(e^{j\omega})] \neq 0 \qquad \text{for } \operatorname{Re} s \geq 0$$

which, by the observation made in Section II, leads to the asymptotic stability i.o.d. associated with the null solution of the differential-difference equation (10) with $u(t) \equiv 0$. Conversely, write

$$\begin{aligned} \tilde{\Delta}(s, e^{-sh}) &= \det [D(e^{-sh})s - A(e^{-sh})] \\ &= \det D(e^{-sh})s^n + \tilde{a}_{n-1}(e^{-sh})s^{n-1} + \cdots + \tilde{a}_0(e^{-sh}), \end{aligned} \tag{26}$$

then the asymptotic stability i.o.d. of null solution of (10) with $u(t) \equiv 0$ means that

$$\tilde{\Delta}(s, e^{-sh}) \neq 0 \qquad \text{for } \operatorname{Re} s \geq 0 \text{ and all } h \geq 0 \tag{27}$$

and, condition (24) is the same as

$$\tilde{\Delta}(0, z) \neq 0 \qquad \text{for } |z| = 1. \tag{28}$$

In the retarded delay case the demonstration [7], [9] that condition (7) plus condition (8) imply condition (9) is based on Theorem 1 of [8] where the fact that $\Delta(s, e^{-sh})$ is monic in s is a key point (treating $e^{-sh} = z$ as another variable). Notice that in the present case $\tilde{\Delta}(s, e^{-sh})$ in (26) is no longer monic in s. However, since $|\det D(e^{j\omega})|$ is a continuous real function in ω, condition (10) implies that there exist $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq |\det D(^{j\omega})| \leq \beta,$$

i.e., the coefficient of s^n in $\tilde{\Delta}(s, e^{-sh})$ is always nonzero and uniformly bounded for all $\omega \in [0, 2\pi]$. It is easy to check for such a polynomial $\tilde{\Delta}(s, e^{-sh})$, [8, Theorem 1] still holds. Therefore, condition (27) plus condition (28) will lead to

$$\tilde{\Delta}(s, z) = 0 \qquad \text{for } \operatorname{Re} s \geq 0 \text{ and } |z| = 1.$$

The rest of the proof is exactly the same as that of Theorem 3.1 □

Similar remarks as at the end of the previous section can also be made here.

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