If $B \neq HQ$, the rows of B are not all in $Span\ row\ Q$. Without loss of generality we assume

$$Q = \begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix}, where\ Q_1 > 0$$

and partition y_2 as

$$y_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}.$$

Now (8) has the form

$$\frac{x^T M x}{x^T K^T K x}, \quad x \notin \mathrm{Ker}(K)$$
$$\frac{y_1^T (A) y) 1 + 2(y_{21}^T, y_{22}^T) B^T y_1 - t_{21}^T Q_1 y_{21}}{y_1^T S^T S y_1}.$$

The condition: "rows of B are not all in $Span\ row\ Q$" implies that By_{22} is not always zero. It is obvious that this term can make the value of the fraction be both positive and negative unbounded.

One can also see from (10) that the condition $B = HQ$ is independent of the linear transformation.

Q.E.D.

Case 5: Assume

$$x^T (M) x \geq 0; \forall x \in \mathrm{Ker}(K). \tag{19}$$

In this case replace $-Q$ by a positive semi-definite Q in (5). Similar to the proof of Case 4, we can show the following.

Corollary 4: Under condition (19), if there exists a suitable dimensional matrix H such that $B = HQ$, then

$$\underset{x}{sub} \left(\frac{x^T M x}{x^T K^T K x} \right) = \min \sigma (E^{-1} (A + HQH^T) E^{-1}),$$
$$x \notin \mathrm{Ker}(K).$$

Otherwise

$$\frac{x^T M x}{x^T K^T K x}, x \notin \mathrm{Ker}(K)$$

has neither upper bound nor lower bound.

IV. Conclusion

In this note we discussed the problem of finding the boundaries of conditional quadratic forms with all possible constraints over a subspace. In all cases, the necessary and sufficient conditions for the existence of upper and/or lower bounds are presented. The essential bounds are obtained whenever they exist.

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On Robust Stability of 2-D Discrete Systems

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***Abstract*— This note presents a study on robust stability of two-dimensional (2-D) discrete systems in the Fornasini–Marchesini (F-M) state space setting. A measure of stability robustness of a stable F-M model is defined. Relation of this measure to its counterpart in the Roessor state space and related computational issues are addressed. Three lower bounds of the stability-robustness measure defined are derived using an one-dimensional parameterization approach and a 2-D Lyapunov approach. A numerical example is included to illustrate the main results obtained.**

I. Introduction

In this note, we present a study on robust stability of two-dimensional (2-D) discrete systems under unstructured perturbations. Throughout the concerned system is modeled in the Fornasini-Marchesini (F-M) local state space [1] as

$$x(i+1, j+1) = A_1 x(i, j+1) + A_2 x(i+1, j) \tag{1}$$

where $x(i,j) \in R^{n \times 1}$, A_1, $A_2 \in R^{n \times n}$. Recall that system (1) is asymptotically stable if and only if

$$p(z_1, z_2) \equiv \det(I_n - z_1 A_1 - z_2 A_2) \neq 0 \ \text{ for } \ (z_1, z_2) \in \bar{U} \tag{2}$$

where $\bar{U} = \{(z_1, z_2) : |z_1| \leq 1, |z_2| \leq 1, \}$ [1]. To date, results on robust stability of 2-D discrete systems in a local state-space framework are only available for the Roesser model [2]–[6], and one might attribute this to the lack of a Lyapunov stability theory for the F-M model. The objectives of this note are twofold. First, we propose in Section II a quantitative measured, ν, for the unstructured, stable perturbations of a given stable 2-D F-M system, and derive in Section III a lower bound for ν. Issues on numerical evaluation of the bound obtained and the relation of the proposed stability robustness measure with its counterpart in the Roesser state space will also be addressed. Second, we propose in Section IV a Lyapunov approach to analyzing the robust stability of (1), leading to two lower bounds of ν. The proposed approach makes use of the 2-D Lyapunov equation [7] which is a generalization of the 2-D Lyapunov equation investigated recently by Hinamoto [8]. A numerical example is included in Section V to illustrate the main results of the paper.

In the rest of the paper we write $H > 0$ or $H \geq 0$ to mean that the symmetric matrix H is positive definite or positive semi-definite. For a real matrix $P \geq 0$, one can always write

$$P = U^T \Sigma U$$

where U is orthogonal and $\Sigma = \mathrm{diag}\{\sigma_1, \cdots, \sigma_n\}$ with $\sigma_k \geq 0$, for $1 \leq k \leq n$. If we denote $\Sigma^{1/2} = \mathrm{diag}\{\sigma_1^{1/2}, \cdots, \sigma_n^{1/2}\}$, then

$$P = P^{T/2} P^{1/2}$$

where

$$P^{1/2} = \Sigma^{1/2} U \ \text{ and } \ P^{T/2} = (P^{1/2})^T.$$

Such a $P^{1/2}$ is called the nonsymmetric square root of P. The largest and smallest singular value of matrix H is denoted by $\bar{\sigma}(H)$ and

Manuscript received September 1, 1993; revised February 28, 1994 and August 1, 1994. This work was supported in part by the Natural Sciences and Engineering Council of Canada and the Networks of Centres of Excellence program in Microelectronics (Micronet).

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IEEE Log Number 9407223.

$\underline{\sigma}(H)$, respectively, $||H||$ denotes the induced 2-norm of H, which is equal to $\bar{\sigma}(H)$. Throughout we shall use A to denote the $n \times 2n$ real matrix $[A_1 \quad A_2]$ where A_1 and A_2 are from (1) and call it the system matrix of (1). A is said to be stable if (1) is stable i.e., A satisfies (2).

II. A Measure of Robust Stability of System (1)

For asymptotically stable system (1), let $A = [A_1 \quad A_2]$ and denote by S_u the set of all complex perturbations $\Delta A = [\Delta A_1 \quad \Delta A_2] \in C^{n\times 2n}$ with $A + \Delta A$ unstable, i.e.,

$$S_u = \{\Delta A : [\Delta A_1 \quad \Delta A_2] \in C^{n\times 2n},\ A + \Delta A \text{ unstable}\}. \quad (3)$$

Two sets of matrices can be induced from set S_u as follows

$$S_{u_1} = \{\Delta A_1 : \Delta A_1 \in C^{n\times n},\ [\Delta A_1 \quad 0] \in S_u\} \quad (4)$$

$$S_{u_2} = \{\Delta A_2 : \Delta A_2 \in C^{n\times n},\ [0 \quad \Delta A_2] \in S_u\}. \quad (5)$$

In words, S_{u_1} and S_{u_2} are the collection of the first and the second $n \times n$ blocks, respectively, each of which itself causes system instability.

A measure for robust stability of system (1) is defined as

$$\nu = \inf_{\Delta A \in S_u} ||\Delta A||. \quad (6)$$

Obviously, a perturbed system with system matrix $A + \Delta A$ is stable if $||\Delta A|| < \nu$.

With the sets S_{u_1} and S_{u_2} defined in (4) and (5), two auxiliary quantities can be introduced as

$$\mu_1 = \inf_{\Delta A_1 \in S_{u_1}} ||\Delta A_1|| \text{ and } \mu_2 = \inf_{\Delta A_2 \in S_{u_2}} ||\Delta A_2||. \quad (7)$$

Since $S_{u_1} \subset S_u$ and $S_{u_2} \subset S_u$, we have $\nu \le \mu_1$, $\nu \le \mu_2$, and hence

$$\nu \le \min(\mu_1, \mu_2). \quad (8)$$

Shortly we shall see that a relation similar to (8) also holds when the system is modeled in Roesser state space.

Like the 1-D case, ν defined in (6) will be referred to as the stability radius for unstructured complex perturbations for state-space model (1). Since perturbations for a physical 2-D plant are always of real value, a more realistic robust stability measure can be defined as

$$\nu_r = \inf_{\Delta A \in T_u} ||\Delta A||$$

where T_u is the set of all real perturbations $\Delta A = [\Delta A_1 \quad \Delta A_2] \in R^{n\times 2n}$ with $A + \Delta A$ unstable, i.e.,

$$T_u = \{\Delta A : [\Delta A_1 \quad \Delta A_2] \in R^{n\times 2n},\ A + \Delta A \text{ unstable}\}.$$

Note that T_u is a proper subset of S_u and consequently the stability radius for unstructured real perturbations, ν_r, is always no less than ν. This means that ν_r is indeed is less conservative measure for stability robustness. For the 1-D case, the real stability radius has been investigated [17]–[22] and a complete solution has been obtained very recently by Qiu *et al.* [22]. The generalization of the approach of [22] to the 2-D case is however far from straightforward and seems adequate to leave it as a separate topic for research.

Concerning the relation of the proposed stability measure to that of Roesser model, we recall [1] that if a 2-D discrete system has been modeled in the Roesser state space as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \equiv \hat{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \quad (9a)$$

it can be remodeled in the Fornasini–Marchesini state-space as (1) with

$$x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, \quad \text{and } A_2 = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}. \quad (9b)$$

Assuming (9a) is stable, the commonly used stability robustness measure of the system (in the Roesser state space) is

$$\xi = \inf_{\Delta \hat{A} \in R_u} ||\Delta \hat{A}||$$

where

$$R_u = \{\Delta \hat{A} : \hat{A} + \Delta \hat{A} \text{ unstable}\}$$

with

$$\Delta \hat{A} = \begin{bmatrix} \Delta \hat{A}_1 & \Delta \hat{A}_2 \\ \Delta \hat{A}_3 & \Delta \hat{A}_4 \end{bmatrix}.$$

According to the structures of A_1 and A_2 in (9b), perturbation matrix $\Delta \hat{A}$ in the Roessor model is the sum of perturbations for A_1 and A_2 in the F-M model, i.e.,

$$\Delta \hat{A} = \begin{bmatrix} 0 & 0 \\ \Delta \hat{A}_3 & \Delta \hat{A}_4 \end{bmatrix} + \begin{bmatrix} \Delta \hat{A}_1 & \Delta \hat{A}_2 \\ 0 & 0 \end{bmatrix} \equiv \Delta A_1 + \Delta A_2. \quad (10)$$

Hence

$$\xi = \inf_{\Delta \hat{A} \in R_u} ||\Delta \hat{A}|| = \inf_{\Delta A \in S_u} ||\Delta A_1 + \Delta A_2||$$

where S_u is the set of unstable perturbations $\Delta A = [\Delta A_1 \quad \Delta A_2]$ with ΔA_1, ΔA_2 defined by (10). From $S_{u_1} \subset S_u$ and $S_{u_2} \subset S_u$, it follows that

$$\begin{aligned} \xi &= \inf_{\Delta A \in S_u} ||\Delta A_1 + \Delta A_2|| \le \inf_{\Delta A \in S_{u_1}} ||\Delta A_1 + \Delta A_2|| \\ &= \inf_{\Delta A_1 \in S_{u_1}} ||\Delta A_1|| = \mu_1 \end{aligned}$$

and

$$\begin{aligned} \xi &= \inf_{\Delta A \in S_u} ||\Delta A_1 + \Delta A_2|| \le \inf_{\Delta A \in S_{u_2}} ||\Delta A_1 + \Delta A_2|| \\ &= \inf_{\Delta A_1 \in S_{u_2}} ||\Delta A_2|| = \mu_2. \end{aligned}$$

Hence

$$\xi \le \min(\mu_1, \mu_2). \quad (11)$$

On comparing (11) with (8), we see that ν for the F-M model is in a certain sense similar to ξ in the Roesser model. In the rest of the paper, our attention will be focused on investigating bound ν.

III. A Lower Bound of ν and Its Evaluation

A. A Lower Bound of ν

Let $\Omega = \{(\omega_1, \omega_2) : -\pi \le \omega_1 \le \pi,\ -\pi \le \omega_2 \le \pi\}$, and define

$$\gamma = \inf_{(\omega_1,\omega_2)\in\Omega} \underline{\sigma}[I_n - e^{j\omega_1}A_1 - e^{j\omega_2}A_2] \quad (12)$$

where I_n is the identify matrix of dimension n. As

$$\underline{\sigma}[I_n - e^{j\omega_1}A_1 - e^{j\omega_2}A_2] = \underline{\sigma}[e^{-j\omega_1}I_n - A_1 + e^{j(\omega_2-\omega_1)}A_2]$$

where $\mu = \omega_2 - \omega_1$ and $\omega = -\omega_1$, we have

$$\begin{aligned} \gamma &= \inf_{-\pi\le\mu\le\pi} \inf_{-\pi\le\omega\le\pi} \underline{\sigma}[e^{j\omega}I_n - A(\mu)] \\ &= \inf_{0\le\mu\le\pi} \inf_{-\pi\le\omega\le\pi} \underline{\sigma}[e^{j\omega}I_n - A(\mu)] \end{aligned} \quad (13)$$

where

$$A(\mu) = A_1 + e^{j\mu}A_2 \quad (14)$$

and the last equality in (13) is due to the fact that

$$f(\mu) = \inf_{-\pi\le\omega\le\pi} \underline{\sigma}[e^{j\omega}I_n - A(\mu)] \quad (15)$$

is an even function for $\mu \in [-\pi, \ \pi]$. With the definition of $f(\mu)$ in (15), (13) becomes

$$\gamma = \inf_{0 \le \mu \le \pi} f(\mu). \tag{16}$$

To explore the quantitative relation of γ to the robust stability measure ν, we note that (1) is asymptotically stable if and only if all eigenvalues of $A(\mu)$ for each $\mu \in [0, \ \pi]$ are strictly inside the unit circle [9]. Hence $A+\Delta A$ represents a stable system matrix if and only if $\Delta A \begin{bmatrix} I \\ e^{j\mu} I \end{bmatrix}$ is a stable perturbation of $A(\mu)$ for each $\mu \in [0, \ \pi]$. Since the exact 2-norm bound of complex, stable perturbations of $A(\mu)$ is equal to $f(\mu) = \min_{-\pi \le \omega \le \pi} \underline{\sigma}[e^{j\omega} I - A(\mu)]$ [10]–[12], it follows that ΔA is a stable perturbation of A if

$$\left\| \Delta A \begin{bmatrix} I \\ e^{j\mu} I \end{bmatrix} \right\| < f(\mu) \quad \text{for all} \quad \mu \in [0, \ \pi]. \tag{17}$$

As $\left\| \Delta A \begin{bmatrix} I \\ e^{j\mu} I \end{bmatrix} \right\| \le \sqrt{2} \|\Delta A\|$ and $f(\mu) \ge \gamma$, (17) holds if

$$\|\Delta A\| < \frac{\gamma}{\sqrt{2}} \equiv \beta_0. \tag{18}$$

This proves the following proposition.

Proposition 1: Define γ by (12), $\beta_0 = \gamma/\sqrt{2}$ is a lower bound of the robust-stability measure ν defined by (6).

Two immediate consequences from above analysis are as follows.

Proposition 2: $\Delta A = [\Delta A_1 \quad \Delta A_2]$ is a stable perturbation for system (1) if

$$\|\Delta A_1\| + \|\Delta A_2\| < \gamma. \tag{19}$$

Proof: For any μ

$$\left\| \Delta A \begin{bmatrix} I \\ e^{j\mu} I \end{bmatrix} \right\| \le \|\Delta A_1\| + \|\Delta A_2\|. \tag{20}$$

So if (19) holds, then $f(\mu) \ge \gamma$ leads (20) to (17). □

Proposition 3: ΔA is a stable perturbation for system (1) if

$$\tilde{\gamma} < \gamma \tag{21}$$

where

$$\tilde{\gamma} = \sup_{0 \le \mu \le \pi} \|\Delta A_1 + e^{j\mu} \Delta A_2\|. \tag{22}$$

Proof: If (21) holds, then for any μ

$$\left\| \Delta A \begin{bmatrix} I \\ e^{j\mu} I \end{bmatrix} \right\| \le \tilde{\gamma} < \gamma \le f(\mu)$$

which leads to (17). □

B. Computational Issues

We see from (13) that computing γ is a minimization problem with the objective function given by $F(\omega, \mu) = \underline{\sigma}[e^{j\omega} I_n - A(\mu)]$. Because of the periodicity of $F(\omega, \mu)$ with respect to ω and μ, this minimization can be considered as an unconstrained problem, hence efficient optimization methods such as quasi-Newton algorithms can be used to find γ. As $F(\omega, \mu)$ represents the smallest singular value of a two-parameter matrix, there is in general no closed-form formula to compute the gradient of $F(\omega, \mu)$. For any $\delta > 0$, however, elementary properties of singular values [13, Chapter 6] imply that

$$|F(\omega + \delta, \mu) - F(\omega, \mu)| \le |e^{j(\omega+\delta)} - e^{j\omega}| \le \delta$$

and

$$|F(\omega, \mu + \delta) - F(\omega, \mu)| \le \|A(\mu + \delta) - A(\mu)\| \le \delta \|A_1\|$$

indicating that for any $\delta > 0$, ratio $[F(\omega + \delta, \mu) - F(\omega, \mu)]/\delta$ and $[F(\omega, \mu + \delta) - f(\omega, \mu)]/\delta$ are bounded by unity and $\|A_1\|$, respectively. Hence the gradient of $F(\omega, \mu)$ if it exists is bounded and reliable approximation of it can be obtained using simple numerical differentiations (i.e., the above difference ratio) with sufficiently small δ.

An alternative approach to the numerical evaluation of γ is to make use of (16) in conjunction with the fact that for a fixed μ, $f(\mu)$ defined by (15) is the exact bound for stable, complex perturbations of the system with $A(\mu)$ as its system matrix. Consequently, $f(\mu)$ for a fixed μ can be computed using the efficient bisection method proposed by Byers [14]. Again, due to the periodicity of $f(\mu)$, (15) is indeed a scalar minimization problem, which can be solved using, for example, the modified three-point-pattern algorithm [15, Section 7.3].

We shall touch upon these computational issues again in Section V through an illustrative example.

IV. Lower Bounds Derived Using the Generalized 2-D Lyapunov Equation

A. The Generalized 2-D Lyapunov Equation

In [7] the following proposition was given.

Proposition 4 [7]: Equation (1) is asymptotically stable if there are positive definite P, W_1, and W_2 such that

$$Q = \begin{bmatrix} P^{T/2} W_1 P^{1/2} & 0 \\ 0 & P^{T/2} W_2 P^{1/2} \end{bmatrix} - A^T P A > 0 \tag{23}$$

and that

$$I_n - W_1 - W_2 \ge 0. \tag{24}$$

This proposition is a generalization of the 2-D Lyapunov theorem proposed recently by Hinamoto [8]. It was shown in [7] that as compared to Hinamoto's result, the generalized Lyapunov equation (23) can be used to confirm stability of a broader class of 2-D discrete systems in the F-M state space setting. As [7] has not been available to the reader, for the sake of convenience a proof of Proposition 4 is given in Appendix A.

B. Two Lower Bounds of ν

Consider a stable system (1) that satisfies (23) and (24) with some positive definite P, W_1 and W_2, and let the perturbed system be given by

$$x(i+1, j+1) = (A_1 + \Delta A_1) x(i, j+1) + (A_2 + \Delta A_2) x(i+1, j). \tag{25}$$

The matrices P, W_1, and W_2 are employed to construct the Lyapunov function

$$\phi(i, j) = \phi_1(i, j) + \phi_2(i, j) \tag{26}$$

where

$$\phi_1(i, j) = x^T(i, j+1) P^{T/2} W_1 P^{1/2} x(i, j+1) \tag{27}$$

$$\phi_2(i, j) = x^T(i+1, j) P^{T/2} W_2 P^{1/2} x(i+1, j). \tag{28}$$

Define

$$\phi_{11}(i, j) = \phi_1(i+1, j) + \phi_2(i, j+1). \tag{29}$$

By (24)

$$\begin{aligned} \phi_{11}(i, j) &= x^T(i+1, j+1) P^{T/2} (W_1 + W_2) P^{1/2} x(i+1, j+1) \\ &\le x^T(i+1, j+1) P x(i+1, j+1). \end{aligned} \tag{30}$$

Now using (23) and (25)–(30) we compute

$$\begin{aligned} \Delta\phi(i, j) &= \phi_{11}(i, j) - \phi(i, j) \\ &\le \tilde{x}^T(i, j)[(A^T + \Delta A) P (A + \Delta A) - \tilde{P}] \tilde{x}(i, j) \\ &\le -[\underline{\sigma}(Q) - 2\bar{\sigma}^{1/2}(\tilde{P} - Q)\bar{\sigma}^{1/2}(P)\|\Delta A\| \\ &\quad - \bar{\sigma}(P)\|\Delta A\|^2] \, \|\tilde{x}(i, j)\|^2 \end{aligned}$$

where

$$\tilde{x}(i,j)=\begin{bmatrix}x(i,j+1)\\ x(i+1,j)\end{bmatrix} \text{ and } \tilde{P}=\begin{bmatrix}P^{T/2}W_1P^{T/2} & 0\\ 0 & P^{T/2}W_2P^{1/2}\end{bmatrix}. \tag{31}$$

Hence if

$$||\Delta A|| < \frac{[\bar{\sigma}(\tilde{P}-Q)+\underline{\sigma}(Q)]^{1/2}-\bar{\sigma}^{1/2}(\tilde{P}-Q)}{\bar{\sigma}^{1/2}(P)} \tag{32}$$

then $\Delta\phi(i,j) < 0$ for all i, j, and a typical argument (see, e.g., [16]) applies to show the stability of the perturbed system (25).

An alternative Lyapunov function may be defined as

$$\psi(i,j) = [\phi_1(i,j)+\phi_2(i,j)]^{1/2}. \tag{33}$$

Similarly we define

$$\psi_{11}(i,j) = [\phi_1(i+1,j)+\phi_2(i,j+1)]^{1/2} \tag{34}$$

and use (24) to write

$$\psi_{11}(i,j) \le [x^T(i+1,j+1)Px(i+1,j+1)]^{1/2}. \tag{35}$$

Now (23), (25), (33), and (35) imply that

$$\begin{aligned}\Delta\psi(i,j) &= \psi_{11}(i,j)-\psi(i,j)\\ &\le [\tilde{x}^T(i,j)(A+\Delta A)^TP(A+\Delta A)\tilde{x}(i,j)]^{1/2}\\ &\quad - [\tilde{x}^T(i,j)\tilde{P}\tilde{x}(i,j)]^{1/2}\\ &\le -\left[\frac{\underline{\sigma}(Q)}{\bar{\sigma}^{1/2}(\tilde{P}-Q)+\bar{\sigma}^{1/2}(\tilde{P})}-\bar{\sigma}^{1/2}(P)||\Delta A||\right]\\ &\quad ||\tilde{x}(i,j)||.\end{aligned}$$

Thus if

$$||\Delta A|| < \frac{\underline{\sigma}(Q)}{\bar{\sigma}^{\frac{1}{2}}(P)[\bar{\sigma}^{\frac{1}{2}}(\tilde{P}-Q)+\bar{\sigma}^{\frac{1}{2}}(\tilde{P})]} \tag{36}$$

then $\Delta\psi(i,j) < 0$ for all i,j, and therefore, the perturbed system (25) will remain stable.

Although our numerical experience indicates that the lower bounds of ν given in (32) and (36) usually stay quite close each other, the bound given in (32) is always better than that in (36) as is shown in the next proposition.

Proposition 5:

$$\beta_{L_2} \le \beta_{L_1} \tag{37}$$

where

$$\beta_{L_1} = \frac{[\bar{\sigma}(\tilde{P}-Q)+\underline{\sigma}(Q)]^{1/2}-\bar{\sigma}^{1/2}(\tilde{P}-Q)}{\bar{\sigma}^{1/2}(P)} \tag{38}$$

$$\beta_{L_2} = \frac{\underline{\sigma}(Q)}{\bar{\sigma}^{1/2}(P)[\bar{\sigma}^{1/2}(\tilde{P}-Q)+\bar{\sigma}^{1/2}(\tilde{P})]} \tag{39}$$

with $\tilde{P}$ given by (31).

The proof of the above proposition can be found in Appendix B.

C. Remarks on Numerical Solutions of the Generalized 2-D Lyapunov Equation

To compute lower bound β_{L_1} or β_{L_2}, one must find positive definite P, W_1, and W_2 that satisfy (23) and (24). Although constraints (24), $P > 0$, $W_1 > 0$ and $W_2 > 0$ can be put together as a set of linear matrix inequalities (LMI's), it is difficult to formulate the problem of solving (23) as a convex programming problem such as those investigated in [23] and [24], since solving (23) cannot be formulated as a problem of minimizing λ with $\lambda B(x) - A(x) > 0$ where x is the vector collecting all the parameters in P, W_1, and W_2, and $B(x)$ and $A(x)$ are matrices depending on x affinely. The approach taken here is to reformulate the problem as conventional constrained minimization problem as follows. Note that (23) and (24) are equivalent to

$$\hat{Q} \equiv I_{2n} - \begin{bmatrix}V_1^T & 0\\ 0 & V_2^T\end{bmatrix}\tilde{A}^T\tilde{A}\begin{bmatrix}V_1 & 0\\ 0 & V_2\end{bmatrix} > 0 \tag{40}$$

and

$$I - V_1^{-T}V_1^{-1} - V_2^{-T}V_2^{-1} \ge 0 \tag{41}$$

respectively, where $T = P^{-1/2}$, $\tilde{A} = [\tilde{A}_1 \;\; \tilde{A}_2]$, $\tilde{A}_i = T^{-1}A_iT$, $W_i = V_i^{-T}V_i^{-1}$ $(i = 1, 2)$, which are in turn equivalent to

$$\underset{\substack{T \text{ nonsingular}\\ I-V_1^{-T}V_1^{-1}-V_2^{-T}V_2^{-1}\ge 0}}{\text{minimize}} \left\|\tilde{A}\begin{bmatrix}V_1 & 0\\ 0 & V_2\end{bmatrix}\right\| < 1. \tag{42}$$

Solving the constrained minimization problem (42) is in general numerically intensive in the sense that it usually requires a fairly large amount of computations to obtain a local minimum, due primarily to the iterative nature of the algorithm. When the norm of system matrix A or the norm of a weighted version of A is small in a sense specified below, however, solutions to the generalized 2-D Lyapunov equation (23) and (24) can be found easily. For example, if

$$||A|| < \frac{1}{\sqrt{2}} \tag{43}$$

then $W_1 = W_2 = \frac{1}{2}I_n$ and $P = I_n$ satisfy (24) and lead Q in (23) to

$$Q = \frac{1}{2}I_{2n} - A^TA$$

which is positive definite. The above choice of W_1, W_2, and P leads to $\tilde{P} = \frac{1}{2}I_{2n}$, $\bar{\sigma}(\tilde{P}-Q) = ||A||^2$, $\underline{\sigma}(Q) = \frac{1}{2}-||A||^2$, and $\bar{\sigma}(P) = 1$. The two lower bounds in this case are found equal, being

$$\beta_{L_1} = \beta_{L_2} = \frac{1}{\sqrt{2}} - ||A||. \tag{44}$$

Another simple case is when weights $\alpha > 0$ and $\beta > 0$ can be found such that

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2} = 1 \tag{45}$$

and

$$||[\alpha A_1 \;\; \beta A_2]|| < 1. \tag{46}$$

Obviously, $W_1 = \frac{1}{\alpha^2}I_n$, $W_2 = \frac{1}{\beta^2}I_n$, and $P = I_n$ satisfy (24) and lead Q in (23) to

$$Q = \begin{bmatrix}\frac{1}{\alpha^2}I & 0\\ 0 & \frac{1}{\beta^2}I\end{bmatrix} - A^TA \tag{47}$$

which is positive definite if (46) holds. The two lower bounds in this case are given by

$$\beta_{L_1} = [||A||^2 + \underline{\sigma}(Q)]^{1/2} - ||A|| \tag{48}$$

and

$$\beta_{L_2} = \frac{\underline{\sigma}(Q)}{||A|| + \max(\alpha^{-1},\beta^{-1})}. \tag{49}$$

V. An Example

Consider a fourth-order F-M model with

$$A_1 = \begin{bmatrix}0.0058 & 0.2853 & -0.2432 & 0.1246\\ -0.0084 & 0.2377 & 0.1606 & -0.1404\\ -0.0084 & -0.0217 & 0.1785 & 0.1662\\ -0.0245 & 0.0884 & -0.0321 & 0.2428\end{bmatrix}$$

$$A_2 = \begin{bmatrix}0.0495 & 0.0687 & 0.2030 & -0.3013\\ 0.1630 & 0.4817 & -0.0993 & 0.1814\\ -0.2842 & 0.1790 & 0.5551 & 0.0799\\ 0.2112 & -0.1887 & -0.0895 & 0.5604\end{bmatrix}.$$

As $||A|| = 0.7791$ exceeds $1/\sqrt{2} \approx 0.7071$, we attempt to find positive scalars α and β that satisfy (45) and (46). Using (45) we write $\beta = \alpha/\sqrt{\alpha^2 - 1}$ and define the objective function

$$e(\alpha) = ||[\alpha A_1 \quad \frac{\alpha}{\sqrt{\alpha^2 - 1}} A_2]|| \quad \text{for} \quad 1 < \alpha < \infty. \tag{50}$$

Applying the modified three-point-pattern minimization algorithm [15, Chapter 7] to the bracket (i.e., an interval of α) [1.1, 3], it took 12 iterations with 17.796 Kflops (10^3 floating point operations) to find the minimum of $e(\alpha)$, $e(\alpha^*) = 0.9001 < 1$ at $\alpha^* = 2$. This leads to a solution to (23)–(24) with $W_1 = 0.25I_4$, $W_2 = 0.75I_4$, and $P = I_4$. It follows from Proposition 4 that the 2-D system is stable. From (47)–(49), lower bounds β_{L_1} and β_{L_2} were found to be $\beta_{L_1} = 0.0310$ and $\beta_{L_2} = 0.0300$. The total number of flops used in computing β_{L_1} or β_{L_2} is about 21 Kflops. To evaluate the lower bound β_0 in (18), a quasi-Newton optimization method was used to evaluate ν, where the gradient of $F(\omega, \mu)$ was performed numerically with $\delta = 10^{-6}$. It took nine iterations (with 241.6 Kflops) to obtain $\gamma = 0.1775$ at $[\omega^*, \mu^*] = [0.0252 \ \ 0.3856]$, which leads to $\beta_0 = 0.1255$. It is noted that bound β_0 is considerably less conservative as compared to β_{L_1} and β_{L_2} although computing β_0 is far more expensive in this case.

VI. Concluding Remarks

A robust stability measure has been proposed and three lower bounds of ν for F-M 2-D discrete systems have been derived by two distinct approaches. It appears from the numerical example discussed in Section V that the bounds β_{L_1} and β_{L_2} derived using the Lyapunov approach are quite conservative. This could be improved by carrying out the optimization problem (42) rather than using (50). As a matter of fact, the norm in (42) becomes $e(\alpha)$ in (50) if $T = I_4$, $V_1 = \alpha I_4$ and $V_2 = \frac{\alpha}{\sqrt{\alpha^2-1}} I_4$ with $\alpha > 1$ are chosen. With more parameters in (42), better lower bounds β_{L_i} $(i = 1, 2)$ can be found at the expense of higher computation intensity.

Appendix A

Proof of Proposition 4: We prove the proposition by contradiction. Suppose the conditions of the proposition are satisfied but (1) is unstable. Then there is $(z_1, z_2) \in \bar{U}$ such that

$$\det(I_n - z_1 A_1 - z_2 A_2) = 0. \tag{51}$$

Hence there exists vector $v \neq 0$ such that

$$v = A \begin{bmatrix} z_1 I_n \\ z_2 I_n \end{bmatrix} v. \tag{52}$$

Using (23) and (52), we compute

$$v^* P v = \\ v^*[\bar{z}_1 I_n \quad \bar{z}_2 I_n] \begin{bmatrix} P^{T/2} W_1 P^{1/2} & 0 \\ 0 & P^{T/2} W_2 P^{1/2} \end{bmatrix} \begin{bmatrix} z_1 I_n \\ z_2 I_n \end{bmatrix} v - v_z^* Q v_z$$

where v^* denotes the complex conjugate transposition of v, $\bar{z}_i$ is the complex conjugate of z_i $(i = 1, 2)$, and $v_z^* = v^*[\bar{z}_1 I_n \quad \bar{z}_2 I_n]$. It follows that

$$v^* P^{T/2}(I_n - |z_1|^2 W_1 - |z_2|^2 W_2) P^{1/2} v = -v_z^* Q v_z. \tag{53}$$

By (51) we note that $(z_1, z_2) \neq (0, 0)$, hence $v_z \neq 0$. So $Q > 0$ means that the right-hand side of (53) is strictly negative. On the other side of equation, $|z_1| \leq 1$ and $|z_2| \leq 1$ and the positive semidefiniteness of $I_n - W_1 - W_2$ [see (24)] imply that $I - |z_1|^2 W_1 - |z_2|^2 W_2 \geq 0$. Therefore the left-hand side of (53) is nonnegative, leading to a contradiction. This completes the proof. $\square$

Appendix B

Proof of Proposition 5: Let $a = \underline{\sigma}(Q)$, $b = \bar{\sigma}(P)$, $c = \bar{\sigma}(\tilde{P} - Q)$, and $d = \bar{\sigma}(\tilde{P})$. Note that $a > 0$, $b > 0$, $c \geq 0$, $d > 0$, and that

$$\bar{\sigma}(\tilde{P} - Q) \leq \bar{\sigma}(\tilde{P}) - \underline{\sigma}(Q)$$

which implies that

$$(a + c)^{1/2} + c^{1/2} \leq c^{1/2} + d^{1/2}. \tag{54}$$

Since $b > 0$ and $d > 0$, it follows from (54) that

$$\frac{a}{b^{1/2}(c^{1/2} + d^{1/2})} \leq \frac{(a + c)^{1/2} - c^{1/2}}{b^{1/2}}.$$

$\square$

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