Two-Dimensional State-Space Digital Filters with Minimum Frequency-Weighted L_2-Sensitivity under L_2-Scaling Constraints

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Outline

- Early and Recent Work
- System Model
- Sensitivity Measure and Scaling Constraints
- Problem Formulation
- A Solution Method
- Experimental Results

Early and Recent Work

- Kawamata, Lin, and Higuchi, 1987. (L_2/L_1 mixed sensitivity)
- Hinamoto, Hamanaka, and Maekawa, 1990. (L_2/L_1 mixed sensitivity)
- Hinamoto, Takao, and Muneyasu, 1992. (L_2/L_1 mixed sensitivity)
- Hinamoto and Takao, 1992. (L_2/L_1 mixed sensitivity)
- Hinamoto, Zempo, Nishino, and Lu 1999. (L_2/L_1 mixed sensitivity)
- Li, 1997 and 1998. (L_2 sensitivity)
- Hinamoto, Yokoyama, Inoue, Zeng, and Lu, 2002. (L_2 sensitivity)
- Hinamoto and Sugie, 2002. (L_2 sensitivity)

Some of the above work have considered *frequency weighted sensitivity*, but they do not impose scaling constraints on the design variables. This paper presents a study concerning a frequency weighted L_2 measure subject to L_2 scaling constraints.

System Model

- We consider stable and locally controllable and observable 2-D state-space digital filters that are modeled by Roesser's local state space model as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}}_{A} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{b} u(i,j)$$

$$y(i,j) = \underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_{c} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + d\,u(i,j)$$

- Transfer function:

$$H(z_1,z_2) = c\left(Z - A\right)^{-1} b, \quad Z = z_1 I_m \oplus z_2 I_n$$

Sensitivity Measure and Scaling Constraints

- Frequency-weighted L_2-sensitivity

$$S = \left\| W_A(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial A} \right\|_2^2 + \left\| W_B(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial b} \right\|_2^2 + \left\| W_C(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial c} \right\|_2^2$$

- Evaluation of S

$$S = \left\| W_A(z_1, z_2) G^T(z_1, z_2) F^T(z_1, z_2) \right\|_2^2$$
$$+ \left\| W_B(z_1, z_2) G^T(z_1, z_2) \right\|_2^2 + \left\| W_C(z_1, z_2) F(z_1, z_2) \right\|_2^2$$

where

$$F(z_1, z_2) = (Z - A)^{-1} b$$
$$G(z_1, z_2) = c(Z - A)^{-1}$$

and the squared L_2-norm $\left\| Y(z_1, z_2) \right\|_2^2$ can be computed as

trace of a certain matrix:

$$\left\| Y(z_1, z_2) \right\|_2^2 = \text{trace}\left[\frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} Y(z_1, z_2) Y^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \right]$$

which leads to

- An alternative expression of S:

$$S = \text{trace}\left[M_A \right] + \text{trace}\left[W_B \right] + \text{trace}\left[K_C \right]$$

- L_2 signal scaling constraints:

$$\left(K_1 \right)_{i,i} = 1, \quad \left(K_4 \right)_{k,k} = 1$$

where

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \left\| X(z_1, z_2) \right\|_2^2 = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} F(z_1, z_2) F^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

Problem Formulation

• Minimization of the frequency-weighted L_2-sensitivity subject to L_2 scaling constraints is achieved by using an optimized state-space coordinate transformation

$$\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} = \begin{bmatrix} T_1 \hat{x}^h(i,j) \\ T_4 \hat{x}^v(i,j) \end{bmatrix} = \underbrace{\begin{bmatrix} T_1 & 0 \\ 0 & T_4 \end{bmatrix}}_{T} \begin{bmatrix} \hat{x}^h(i,j) \\ \hat{x}^v(i,j) \end{bmatrix}$$

• The transfer function is *invariant* under a state-space transformation T, but system realization $\{A, b, c, d\}$ is changed to $\{\hat{A}, \hat{b}, \hat{c}, \hat{d}\}$ with

$$\hat{A} = T^{-1}AT,\ \hat{b} = T^{-1}b,\ \hat{c} = cT,\ \hat{d} = d$$

• Sensitivity measure S under transformation T is changed

accordingly to

$$S(P) = \text{trace}\left[M_A(P)P\right] + \text{trace}\left[W_B P\right] + \text{trace}\left[K_C P^{-1}\right]$$

where $P = TT^T$ and

$$M_A(P) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} Y(z_1, z_2) P^{-1} Y^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$

with $Y(z_1, z_2) = W_A(z_1, z_2) G^T(z_1, z_2) F^T(z_1, z_2)$.

• And here is the point: one can select a state space transformation T to minimize the sensitivity $S(P)$ subject to L_2 scaling constraints:

$$\underset{T,\, P=TT^T}{\text{minimize}} \quad S(P)$$

$$\text{subject to: } \left(T_1^{-1} K_1 T_1^{-T}\right)_{i,i} = 1, \quad \left(T_4^{-1} K_4 T_4^{-T}\right)_{k,k} = 1$$

A Solution Method

- The solution method proposed here eliminates the L_2 scaling conditions to convert the problem at hand into an unconstrained problem which is then solved using a quasi-Newton method.
- Let

$$\hat{T}_1 = T_1^T K_1^{-1/2},\ \hat{T}_4 = T_4^T K_4^{-1/2}$$

then constraints

$$\left(T_1^{-1} K_1 T_1^{-T}\right)_{i,i} = 1, \quad \left(T_4^{-1} K_4 T_4^{-T}\right)_{k,k} = 1$$

become

$$\left(\hat{T}_1^{-T} \hat{T}_1^{-1}\right)_{i,i} = 1, \quad \left(\hat{T}_4^{-T} \hat{T}_4^{-1}\right)_{k,k} = 1$$

which are automatically satisfied if we set

$$\hat{T}_1^{-1} = \begin{bmatrix} \frac{t_{11}}{\|t_{11}\|} & \frac{t_{12}}{\|t_{12}\|} & \cdots & \frac{t_{1m}}{\|t_{1m}\|} \end{bmatrix}, \quad \hat{T}_4^{-1} = \begin{bmatrix} \frac{t_{41}}{\|t_{41}\|} & \frac{t_{42}}{\|t_{42}\|} & \cdots & \frac{t_{4n}}{\|t_{4n}\|} \end{bmatrix}$$

- The L_2-sensitivity in terms of $\hat{T}_1$ and $\hat{T}_4$ are given by

$$S(x) = \text{trace}\left[\hat{T} M_A(\hat{P})\hat{T}^T\right] + \text{trace}\left[\hat{T}\hat{W}_B\hat{T}^T\right] + \text{trace}\left[\hat{T}^{-T}\hat{K}_C\hat{T}^{-1}\right]$$

where $\hat{P} = \hat{T}^T\hat{T}$ and

$$x = \begin{bmatrix} t_{11}^T & t_{12}^T & \cdots & t_{1m}^T & t_{41}^T & t_{41}^T & \cdots & t_{4n}^T \end{bmatrix}^T$$

- Minimizing $S(x)$ is an unconstrained problem that can be carried out using an efficient iterative algorithm such as a quasi-Newton algorithm as follows:

(1) Start with an initial point x_0 corresponding to $\hat{T}_1 = I_m$, $\hat{T}_4 = I_n$. Set $k = 0$ and $S_0 = I$.

(2) Update x_k to $x_{k+1} = x_k + \alpha_k d_k$ where

$$d_k = -S_k \nabla S(x_k), \quad \alpha_k = \arg\min_{\alpha} S(x_k + \alpha d_k)$$

(3) Update S_k to

$$S_{k+1} = S_k + \left(1 + \frac{\gamma_k^T S_k \gamma_k}{\gamma_k^T \delta_k}\right)\frac{\delta_k \delta_k^T}{\gamma_k^T \delta_k} - \frac{\delta_k \gamma_k^T S_k + S_k \gamma_k \delta_k^T}{\gamma_k^T \delta_k}$$

$$\delta_k = x_{k+1} - x_k, \quad \gamma_k = \nabla S(x_{k+1}) - \nabla S(x_k)$$

(4) If $\left|S(x_{k+1}) - S(x_k)\right| < \varepsilon$, terminate the iteration, otherwise set $k := k + 1$ and repeat from step (2).

Experimental Results

- An Example: Consider a stable recursive digital filter realization $\left(A^o, b^o, c^o, d\right)_{4,4}$ where

$$A^o = \begin{bmatrix} A_1^o & A_2^o \\ A_1^o & A_2^o \end{bmatrix},\ b^o = \begin{bmatrix} b_1^o \\ b_2^o \end{bmatrix},\ c^o = \begin{bmatrix} c_1^o & c_2^o \end{bmatrix} \text{ with}$$

$$A_1^o = \begin{bmatrix} 0 & 0.481228 & 0 & 0 \\ 0 & 0 & 0.510378 & 0 \\ 0 & 0 & 0 & 0.525287 \\ -0.031857 & 0.298663 & -0.808282 & 1.044600 \end{bmatrix}$$

$$A_2^o = \begin{bmatrix} -0.226080 & 0.776837 & 0.024693 & -0.000933 \\ -0.843550 & 1.610400 & -0.309366 & 0.065898 \\ -1.260339 & 2.005100 & -0.453220 & 0.203118 \\ -1.121498 & 1.636435 & -0.590516 & 0.562890 \end{bmatrix}$$

$$b_1^o = b_2^o = \begin{bmatrix} 0 & 0 & 0 & 0.198473 \end{bmatrix}^T$$

$$c_1^o = \begin{bmatrix} -0.567054 & 0.231913 & 0.197016 & 0.239932 \end{bmatrix}$$

$$c_2^o = \begin{bmatrix} 0.464344 & 0.441837 & -0.061100 & 0.105505 \end{bmatrix}$$

$$d = 0.009430$$

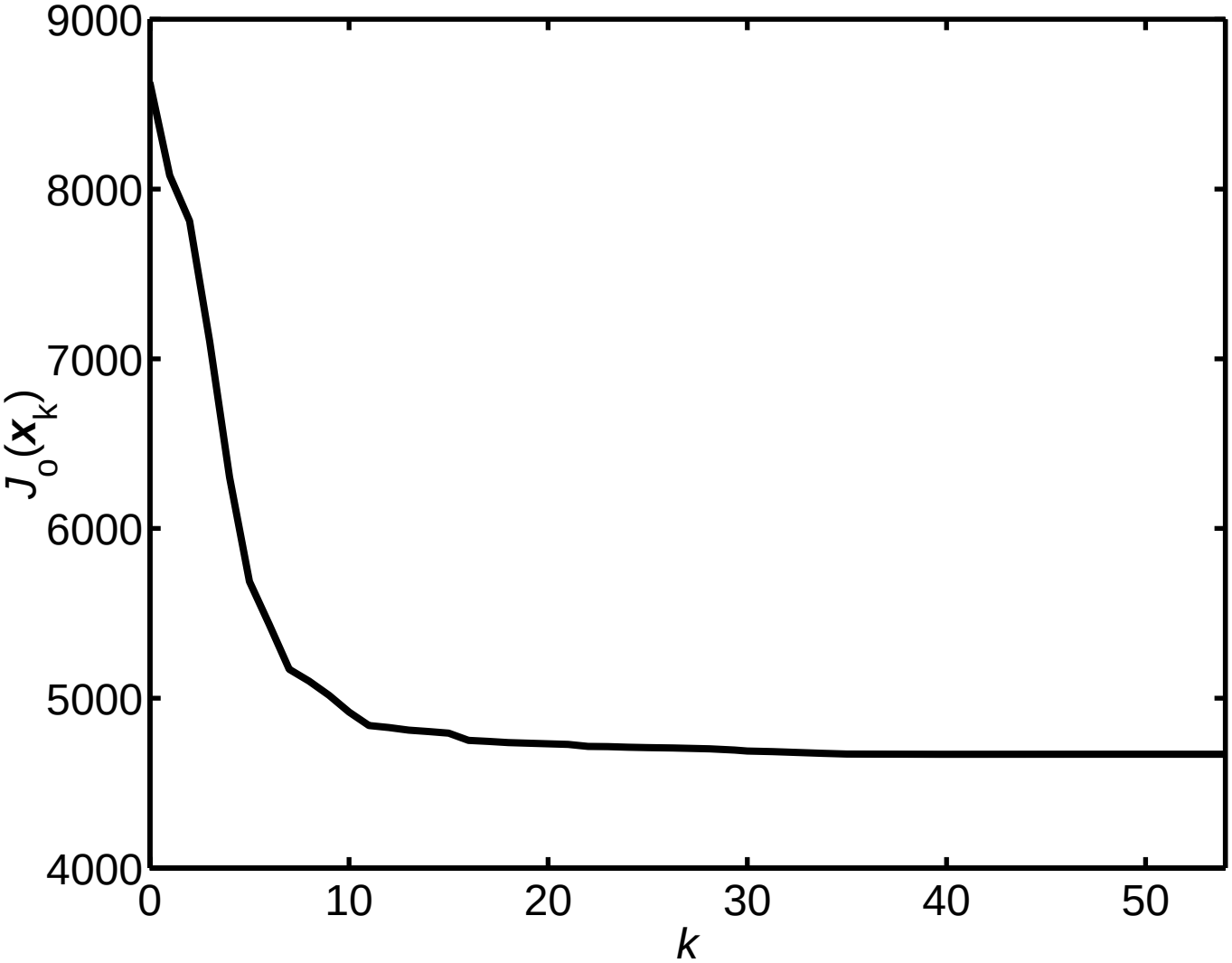
- Frequency-weighted functions were z-transforms of

$$w_A(i,j) = w_B(i,j) = w_C(i,j) = 0.256322 \cdot e^{-0.103203\left[(i-4)^2+(j-4)^2\right]}$$

for $(0,0) \le (i,j) \le (20,20)$ and zero elsewhere.

- The frequency weighted L_2-sensitivity of the filter was found to be $J_0(\hat{T}_0) \approx 8.60 \times 10^3$ (with $\hat{T}_0 = I_4 \oplus I_4$).

- With an initial $\hat{T} = I_4 \oplus I_4$ and tolerance $\varepsilon = 10^{-8}$, it took the algorithm 54 iterations to converge to a solution.


9000
8000
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4000
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30
40
50
k
J_o(x_k)

- The optimized $\hat{T}$ was found to be

$$\hat{T}^{opt} = \begin{bmatrix} 3.056671 & -2.673365 & 0.575882 & -0.429287 \\ -0.331629 & 2.142411 & -0.401503 & -0.192081 \\ -2.530651 & 0.932586 & 0.553002 & -0.136935 \\ 1.754363 & -0.312582 & 0.624509 & 0.515370 \end{bmatrix}$$

$$\oplus \begin{bmatrix} 1.307170 & -0.419919 & 0.045538 & -0.194118 \\ 0.762443 & 0.830435 & -0.297531 & 0.062104 \\ -0.405202 & 0.189220 & 0.976564 & -0.250656 \\ 1.071478 & -0.069804 & 0.315533 & 0.828727 \end{bmatrix}$$

- The minimized frequency weighted L_2-sensitivity was found to be $J_0(\hat{T}^{opt}) \approx 4.67 \times 10^3$.