Two-Dimensional State-Space Digital Filters with Minimum Frequency-Weighted $L_2$-Sensitivity under $L_2$-Scaling Constraints

T. Hinamoto, T. Oumi, O. I. Omoifo
Graduate School of Engineering
Hiroshima University, Japan

W.-S. Lu
Electrical & Computer Engineering
University of Victoria, Canada
Outline

• Early and Recent Work
• System Model
• Sensitivity Measure and Scaling Constraints
• Problem Formulation
• A Solution Method
• Experimental Results
Early and Recent Work

- Kawamata, Lin, and Higuchi, 1987. ($L_2/L_1$ mixed sensitivity)
- Hinamoto, Hamanaka, and Maekawa, 1990. ($L_2/L_1$ mixed sensitivity)
- Hinamoto, Takao, and Muneyasu, 1992. ($L_2/L_1$ mixed sensitivity)
- Hinamoto and Takao, 1992. ($L_2/L_1$ mixed sensitivity)
- Hinamoto, Zempo, Nishino, and Lu 1999. ($L_2/L_1$ mixed sensitivity)
- Li, 1997 and 1998. ($L_2$ sensitivity)
- Hinamoto, Yokoyama, Inoue, Zeng, and Lu, 2002. ($L_2$ sensitivity)
- Hinamoto and Sugie, 2002. ($L_2$ sensitivity)

Some of the above work have considered frequency weighted sensitivity, but they do not impose scaling constraints on the design variables. This paper presents a study concerning a frequency weighted $L_2$ measure subject to $L_2$ scaling constraints.
System Model

- We consider stable and locally controllable and observable 2-D state-space digital filters that are modeled by Roesser’s local state space model as

\[
\begin{bmatrix}
  x^h(i+1, j) \\
  x^v(i, j+1)
\end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\
  x^v(i, j) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(i, j)
\]

\[
y(i, j) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x^h(i, j) \\
  x^v(i, j) \end{bmatrix} + d u(i, j)
\]

- Transfer function:

\[
H(z_1, z_2) = c (Z - A)^{-1} b, \quad Z = z_1 I_m \oplus z_2 I_n
\]
Sensitivity Measure and Scaling Constraints

• Frequency-weighted $L_2$-sensitivity

$$S = \left\| W_A(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial A} \right\|_2^2 + \left\| W_B(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial b} \right\|_2^2 + \left\| W_C(z_1, z_2) \frac{\partial H(z_1, z_2)}{\partial c} \right\|_2^2$$

• Evaluation of $S$

$$S = \left\| W_A(z_1, z_2) G^T(z_1, z_2) F^T(z_1, z_2) \right\|_2^2 + \left\| W_B(z_1, z_2) G^T(z_1, z_2) \right\|_2^2$$

where

$$F(z_1, z_2) = (Z - A)^{-1} b$$

$$G(z_1, z_2) = c(Z - A)^{-1}$$

and the squared $L_2$-norm $\|Y(z_1, z_2)\|_2^2$ can be computed as
trace of a certain matrix:

$$\|Y(z_1, z_2)\|_2^2 = \text{trace}\left[\frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} Y(z_1, z_2)Y^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}\right]$$

which leads to

- An alternative expression of $S$:

$$S = \text{trace}[M_A] + \text{trace}[W_B] + \text{trace}[K_C]$$

- $L_2$ signal scaling constraints:

$$(K_1)_{i,i} = 1, \quad (K_4)_{k,k} = 1$$

where

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} = \|X(z_1, z_2)\|_2^2 = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} F(z_1, z_2)F^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2}$$
Problem Formulation

• Minimization of the frequency-weighted $L_2$-sensitivity subject to $L_2$ scaling constraints is achieved by using an optimized state-space coordinate transformation

$$
\begin{bmatrix}
  x^h(i, j) \\
  x^v(i, j)
\end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ T_4 & 0 \end{bmatrix} \begin{bmatrix}
  \hat{x}^h(i, j) \\
  \hat{x}^v(i, j)
\end{bmatrix}
$$

• The transfer function is *invariant* under a state-space transformation $T$, but system realization $\{A, b, c, d\}$ is changed to $\{\hat{A}, \hat{b}, \hat{c}, \hat{d}\}$ with

$$
\hat{A} = T^{-1}AT, \quad \hat{b} = T^{-1}b, \quad \hat{c} = cT, \quad \hat{d} = d
$$

• Sensitivity measure $S$ under transformation $T$ is changed
accordingly to

\[ S(P) = \text{trace}[M_A(P)P] + \text{trace}[W_B P] + \text{trace}[K_C P^{-1}] \]

where \( P = TT^T \) and

\[ M_A(P) = \frac{1}{(2\pi j)^2} \oint_{\Gamma_1} \oint_{\Gamma_2} Y(z_1, z_2) P^{-1} Y^*(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} \]

with \( Y(z_1, z_2) = W_A(z_1, z_2) G^T(z_1, z_2) F^T(z_1, z_2) \).

• And here is the point: one can select a state space transformation \( T \) to minimize the sensitivity \( S(P) \) subject to \( L_2 \) scaling constraints:

\[
\begin{align*}
\text{minimize} & \quad S(P) \\
\text{subject to:} & \quad (T_1^{-1} K_1 T_1^{-T})_{i,i} = 1, \quad (T_4^{-1} K_4 T_4^{-T})_{k,k} = 1
\end{align*}
\]
A Solution Method

• The solution method proposed here eliminates the $L_2$ scaling conditions to convert the problem at hand into an unconstrained problem which is then solved using a quasi-Newton method.

• Let

$$\hat{T}_1 = T_1^T K_1^{-1/2}, \quad \hat{T}_4 = T_4^T K_4^{-1/2}$$

then constraints

$$\left(T_1^{-1} K_1 T_1^{-T}\right)_{i,i} = 1, \quad \left(T_4^{-1} K_4 T_4^{-T}\right)_{k,k} = 1$$

become

$$\left(\hat{T}_1^{-T} \hat{T}_1^{-1}\right)_{i,i} = 1, \quad \left(\hat{T}_4^{-T} \hat{T}_4^{-1}\right)_{k,k} = 1$$
which are automatically satisfied if we set

\[ \hat{T}_1^{-1} = \begin{bmatrix} \frac{t_{11}}{\left\| t_{11} \right\|} & \frac{t_{12}}{\left\| t_{12} \right\|} & \ldots & \frac{t_{1m}}{\left\| t_{1m} \right\|} \end{bmatrix}, \quad \hat{T}_4^{-1} = \begin{bmatrix} \frac{t_{41}}{\left\| t_{41} \right\|} & \frac{t_{42}}{\left\| t_{42} \right\|} & \ldots & \frac{t_{4n}}{\left\| t_{4n} \right\|} \end{bmatrix} \]

- The $L_2$-sensitivity in terms of $\hat{T}_1$ and $\hat{T}_4$ are given by

\[ S(x) = \text{trace}\left[ \hat{T} M_A(\hat{P})\hat{T}^T \right] + \text{trace}\left[ \hat{T} W_B \hat{T}^T \right] + \text{trace}\left[ \hat{T}^{-T} K_C \hat{T}^{-1} \right] \]

where $\hat{P} = \hat{T}^T \hat{T}$ and

\[ x = \begin{bmatrix} t_{11}^T & t_{12}^T & \ldots & t_{1m}^T & t_{41}^T & t_{41}^T & \ldots & t_{4n}^T \end{bmatrix}^T \]

- Minimizing $S(x)$ is an unconstrained problem that can be carried out using an efficient iterative algorithm such as a quasi-Newton algorithm as follows:
(1) Start with an initial point \( x_0 \) corresponding to \( \hat{T}_1 = I_m, \hat{T}_4 = I_n \). Set \( k = 0 \) and \( S_0 = I \).

(2) Update \( x_k \) to \( x_{k+1} = x_k + \alpha_k d_k \) where
\[
d_k = -S_k \nabla S(x_k), \quad \alpha_k = \arg\min_{\alpha} S(x_k + \alpha d_k)
\]

(3) Update \( S_k \) to
\[
S_{k+1} = S_k + \left(1 + \frac{\gamma_k^T S_k \gamma_k}{\gamma_k^T \delta_k}ight) \frac{\delta_k \delta_k^T}{\gamma_k^T \delta_k} - \frac{\delta_k \gamma_k^T S_k + S_k \gamma_k \delta_k^T}{\gamma_k^T \delta_k}
\]
\[
\delta_k = x_{k+1} - x_k, \quad \gamma_k = \nabla S(x_{k+1}) - \nabla S(x_k)
\]

(4) If \( |S(x_{k+1}) - S(x_k)| < \varepsilon \), terminate the iteration, otherwise set \( k := k + 1 \) and repeat from step (2).
Experimental Results

• An Example: Consider a stable recursive digital filter realization \((A^o, b^o, c^o, d)^{4,4}\) where

\[
A^o = \begin{bmatrix}
A_1^o & A_2^o \\
A_1^o & A_2^o
\end{bmatrix}, \quad b^o = \begin{bmatrix} b_1^o \\
b_2^o \end{bmatrix}, \quad c^o = \begin{bmatrix} c_1^o & c_2^o \end{bmatrix}
\]

with

\[
A_1^o = \begin{bmatrix}
0 & 0.481228 & 0 & 0 \\
0 & 0 & 0.510378 & 0 \\
0 & 0 & 0 & 0.525287 \\
-0.031857 & 0.298663 & -0.808282 & 1.044600
\end{bmatrix}
\]

\[
A_2^o = \begin{bmatrix}
-0.226080 & 0.776837 & 0.024693 & -0.000933 \\
-0.843550 & 1.610400 & -0.309366 & 0.065898 \\
-1.260339 & 2.005100 & -0.453220 & 0.203118 \\
-1.121498 & 1.636435 & -0.590516 & 0.562890
\end{bmatrix}
\]
\[ b_1^o = b_2^o = [0 \ 0 \ 0 \ 0.198473]^T \]

\[ c_1^o = [-0.567054 \ 0.231913 \ 0.197016 \ 0.239932] \]

\[ c_2^o = [0.464344 \ 0.441837 \ -0.061100 \ 0.105505] \]

\[ d = 0.009430 \]

- Frequency-weighted functions were z-transforms of

\[ \omega_A(i,j) = \omega_B(i,j) = \omega_C(i,j) = 0.256322 \cdot e^{-0.103203[(i-4)^2+(j-4)^2]} \]

for \((0,0) \leq (i,j) \leq (20,20)\) and zero elsewhere.

- The frequency weighted \(L_2\)-sensitivity of the filter was found to be \( J_0(\hat{T}_0) \approx 8.60 \times 10^3 \) (with \( \hat{T}_0 = I_4 \oplus I_4 \)).

- With an initial \( \hat{T} = I_4 \oplus I_4 \) and tolerance \( \varepsilon = 10^{-8} \), it took the algorithm 54 iterations to converge to a solution.
• The optimized $\hat{T}$ was found to be

$$\hat{T}^{opt} = \begin{bmatrix}
3.056671 & -2.673365 & 0.575882 & -0.429287 \\
-0.331629 & 2.142411 & -0.401503 & -0.192081 \\
-2.530651 & 0.932586 & 0.553002 & -0.136935 \\
1.754363 & -0.312582 & 0.624509 & 0.515370 \\
1.307170 & -0.419919 & 0.045538 & -0.194118 \\
0.762443 & 0.830435 & -0.297531 & 0.062104 \\
-0.405202 & 0.189220 & 0.976564 & -0.250656 \\
1.071478 & -0.069804 & 0.315533 & 0.828727 \\
\end{bmatrix}$$

• The minimized frequency weighted $L_2$-sensitivity was found to be $J_0(\hat{T}^{opt}) \approx 4.67 \times 10^3$. 