Design of Frequency-Response-Masking FIR Filters
Using SOCP with Coefficient Sensitivity Constraint

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Outline

• Early and Recent Work on FRM Filters
• Coefficient Sensitivity (CS) Performance of an Second-Order Cone Programming (SOCP) Algorithm for FRM Filters
• An Enhanced SOCP Algorithm with a CS Constraint
• Experimental Results
Early and Recent Work on FRM Filters

• Lim, 1986.
• Lim and Lian, 1993.
• Lee and Chen, 1993.
• Lim and Lian, 1994.
• Bellanger, 1996.
• Barcellos, Netto, and Diniz, 2003.
• Saramaki and Lim, 2003.
• Lu and Hinamoto, 2003. (SDP and SOCP techniques)
• Lian and Yang, 2003.
• Lian, 2003.
• Saramaki, Yli-Kaakinen, and Johansson, 2003.
• Lee, Rehbock, and Teo, 2003.
• Gustafsson, Johansson, and Wanhammar, 2003.
• Yu, Teo, Lim, and Zhao, 2005.
• Rodrigues and Pai, 2005.
• Cen and Lian, 2005.
• Lim, Yu, Teo, and Saramaki, 2007. (Coefficient Sensitivity)
CS Performance of an SOCP Algorithm for FRM Filters


\[
H_a(z^M) \quad H_a(z) \quad H_{ma}(z) \\
\quad + \quad - \quad +
\]

\[
H_{-0.5(N-1)M} \quad H_c(z) \quad H_{me}(z) \\
\quad + \quad +
\]

delay = \((N-1)M/2\)  \quad \text{delay} = d

\[
H(\omega, x) = \left[ a^T c(\omega) \right] \left[ a_a^T c_a(\omega) \right] + \left[ 1 - a^T c(\omega) \right] \left[ a_c^T c_c(\omega) \right], \quad x = \begin{bmatrix} a \\ a_a \\ a_c \end{bmatrix}
\]
\[ x_{k+1} = x_k + \delta_k \quad \text{with} \quad \delta_k \, \text{"small"} \]

\[ H(\omega, x_{k+1}) \approx H(\omega, x_k) + g_k^T(\omega) \delta_k \quad \text{with} \quad g_k(\omega) = \nabla H(\omega, x_k) \]

- **SOCP Formulation for an optimal \( \delta_k \):**

  minimize \[ \eta \]

  subject to: \[ W(\omega)\left| H(\omega, x_k) + g_k^T(\omega) \delta - H_d(\omega) \right| \leq \eta \]

  \[ \|\delta\| \leq \beta \]

- The coefficient vector of the optimal FRM filter is given by

\[ x^* = x_0 + \sum_{k=0}^{K-1} \delta_k \]

Hence

\[ \|x^*\| \leq \|x_0\| + \sum_{k=0}^{K-1} \|\delta_k\| \leq \|x_0\| + K\beta \]
and
\[ \left\| x^* - x_0 \right\| \leq \sum_{k=0}^{K-1} \left\| \delta_k \right\| \leq K \beta \]

Typically $K$ is in the range of 8 to 15 (regardless of filter length).

$\Rightarrow$ ♦ $\|x^*\|$ remains small as long as $\|x_0\|$ is small.

♦ Because a small $\|x^*\|$ implies a low CS, the CS of an SOCP-based solution is low as long as the CS of the initial FRM filter is low.

♦ Using SOCP, one can always find an optimal FRM filter in a small vicinity of a reasonable initial design (obtained e.g. using the method of Lim 1986).
• **Example** $N = 45$, $N_a = 27$, $N_c = 19$, $M = 9$, $\omega_p = 0.3$, and $\omega_a = 0.305$

  ♦ CS of the initial FRM filter (Lim 1986): $S_1^2 = 24.4532$

  ♦ $\beta$ was set to 0.1533 (problem size: 44)

  ♦ the SOCP algorithm converges in 8 iterations

  ♦ step size for coefficient quantization was set to $2^{-14}$

  ♦ CS of the optimal FRM filter: $S_1^2 = 38.9035$

  ♦ peak ripple magnitude in passband: 0.009586

  ♦ minimum stopband attenuation: 40.4179 dB

  ♦ coefficient differences in 2-norm:

    $$\|h^* - h^{(0)}\| = 0.5394, \quad \|h_a^* - h_a^{(0)}\| = 0.0816, \quad \|h_c^* - h_c^{(0)}\| = 0.1052$$
◆ peak ripple magnitude in passband: 0.009586
◆ minimum stopband attenuation: 40.4179 dB
◆ CS of the FRM filter: $S_l^2 = 38.9035$
\[ \| h^* - h^{(0)} \| = 0.5394, \quad \| h_a^* - h_a^{(0)} \| = 0.0816, \quad \| h_c^* - h_c^{(0)} \| = 0.1052 \]
An Enhanced SOCP Algorithm with a CS Constraint

- The CS measure defined in Lim et al 2007 is given by

\[
S_1^2 = \left\| \begin{bmatrix} \sqrt{N_c} h \\ \sqrt{N_a} h \\ \sqrt{N} (h_a - \hat{h}_c) \end{bmatrix} - \hat{e} \right\|^2
\]

with

\[
h = [J \ 0 \ 0] x, \quad h_a = [0 \ J_a \ 0] x, \quad h_c = [0 \ 0 \ \hat{J}_c] x
\]

Hence we can write

\[
\begin{bmatrix} \sqrt{N_c} h \\ \sqrt{N_a} h \\ \sqrt{N} (h_a - \hat{h}_c) \end{bmatrix} = A^T x \quad \text{with} \quad A^T = \begin{bmatrix} \sqrt{N_c} J & 0 & 0 \\ \sqrt{N_a} J & 0 & 0 \\ 0 & \sqrt{N} J_a & -\sqrt{N} \hat{J}_c \end{bmatrix}
\]
If $d_{FWL}$ is the desired upper bound for $S_1$, then the constraint that $S_1$ be bound above by $d_{FWL}$ can be cast as

$$\left\| A^T x - \hat{e} \right\| \leq d_{FWL}$$

In the $k$th iteration, $x = x_k + \delta$, the CS constraint becomes

$$\left\| A^T \delta + b_k \right\| \leq d_{FWL} \quad \text{with} \quad b_k = Ax_k - \hat{e}$$

• Incorporating above constraint on CS into our early SOCP formulation leads an enhanced SOCP problem:

minimize $\eta$

subject to: $W(\omega) \left| H(\omega, x_k) + g_k^T (\omega) \delta - H_d (\omega) \right| \leq \eta$

$\left\| \delta \right\| \leq \beta$

$\left\| A^T \delta + b_k \right\| \leq d_{FWL}$
Experimental Results

• Design of an FRM filter with $N = 45$, $N_a = 27$, $N_c = 19$, $M = 9$, $\omega_p = 0.3$, and $\omega_a = 0.305$ (same as that in Lim et al, 2007).

• Other design parameters: $\beta = 0.2168$, $d_{FWL} = 5.4 \iff S_1^2 \leq 29.16$

• quantization step-size: $2^{-14}$

• number of freq. grids in passband and stopband: 2000

• Results: ♦ 8 iterations to converge
  ♦ peak ripple magnitude in passband: 0.009874
  ♦ minimum stopband attenuation: 40.6479 dB
  ♦ CS of the FRM filter: $S_1^2 = 28.2468$
  ♦ CPU time (3.4 GHz, Pentium 4): 26.91 seconds
<table>
<thead>
<tr>
<th>FRM Filter</th>
<th>Peak passband ripple</th>
<th>Minimum stopband attenuation (dB)</th>
<th>$S_1^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Filter in Table I of Lim et al, 2007</td>
<td>0.009949</td>
<td>40.0674</td>
<td>6.7797 × 10⁹</td>
</tr>
<tr>
<td>Filter in Table II of Lim et al, 2007</td>
<td>0.010041</td>
<td>39.9628</td>
<td>26.4288</td>
</tr>
<tr>
<td>SOCP-based design without CS constraint</td>
<td>0.009586</td>
<td>40.4187</td>
<td>38.9035</td>
</tr>
<tr>
<td>SOCP-based design with CS constraint</td>
<td>0.009874</td>
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