Realization of 3-D Separable-Denominator Digital Filters with Very Low l_2-Sensitivity

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1. Motivations and Approach

- When a transfer function with infinite accuracy coefficients is designed and realized by a state-space model, the coefficients in the model must be truncated or rounded to fit the finite-word-length constraints.
- This coefficient quantization usually alters the characteristics of the filter. For instance, it may change a stable filter to an unstable one.
- This motivates the study of coefficient sensitivity minimization problem.
- Here we investigate the problem of reducing the l_2-sensitivity for 3-D separable-denominator digital filters.

- First, a 3-D transfer function with separable denominator is represented with cascade connection of three 1-D transfer functions by applying a minimal realization technique.
- Next, the MIMO 1-D transfer function in the middle of the cascade connection is realized by a minimal state-space model.
- Third, a l_2-norm coefficient sensitivity of the model is analyzed.
- A technique for the optimal synthesis of the minimal state-space model is developed so as to minimize the l_2-sensitivity subject to l_2-scaling constraints.

2. Problem Statement

A 3-D separable-denominator digital filter is given by

$$H(z_1, z_2, z_3) = \frac{N(z_1, z_2, z_3)}{D_1(z_1) D_2(z_2) D_3(z_3)}$$

with

$$N(z_1, z_2, z_3) = \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} \sum_{k=0}^{N_3} a_{i,j,k} z_1^{-i} z_2^{-j} z_3^{-k}$$

$$D_1(z_1) = 1 + b_{11} z_1^{-1} + \cdots + b_{1N_1} z_1^{-N_1}$$

$$D_2(z_2) = 1 + b_{21} z_2^{-1} + \cdots + b_{2N_2} z_2^{-N_2}$$

$$D_3(z_3) = 1 + b_{31} z_3^{-1} + \cdots + b_{3N_3} z_3^{-N_3}$$

The 3-D transfer function is decomposed as

$$H(z_1, z_2, z_3) = \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \boldsymbol{H}_2(z_2) \frac{\boldsymbol{Z}_3}{D_3(z_3)}$$

with

$$\boldsymbol{Z}_1 = \begin{bmatrix} 1 & z_1^{-1} & \cdots & z_1^{-N_1} \end{bmatrix}^T$$

$$\boldsymbol{Z}_3 = \begin{bmatrix} 1 & z_3^{-1} & \cdots & z_3^{-N_3} \end{bmatrix}^T$$

$$\boldsymbol{H}_2 = \frac{\boldsymbol{\Delta}_0 + \boldsymbol{\Delta}_1 z_2^{-1} + \cdots + \boldsymbol{\Delta}_{N_2} z_2^{-N_2}}{D_2(z_2)}$$

$$\boldsymbol{\Delta}_m = \begin{bmatrix} a_{0m0} & a_{0m1} & \cdots & a_{0mN_3} \\ a_{1m0} & a_{1m1} & \cdots & a_{1mN_3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N_1m0} & a_{N_1m1} & \cdots & a_{N_1mN_3} \end{bmatrix} \text{ for } m = 0, 1, \ldots, N_2$$

Let a minimal realization of $\boldsymbol{H}_2(z_2)$ be given by

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_2\boldsymbol{x}(k) + \boldsymbol{B}_2\boldsymbol{u}(k)$$
$$\boldsymbol{y}(k) = \boldsymbol{C}_2\boldsymbol{x}(k) + \Delta_0\boldsymbol{u}(k)$$

with

$\boldsymbol{x}(k)$: a $p \times 1$ state-variable vector

$\boldsymbol{u}(k)$: an $(N_3+1) \times 1$ input vector

$\boldsymbol{y}(k)$: an $(N_1+1) \times 1$ output vector

The transfer function becomes

$$\boldsymbol{H}_2(z_2) = \boldsymbol{C}_2(z_2\boldsymbol{I}_p - \boldsymbol{A}_2)^{-1}\boldsymbol{B}_2 + \Delta_0$$

The l_2-sensitivity of $H(z_1, z_2, z_3)$ with respect to $\boldsymbol{A}_2, \boldsymbol{B}_2$, and $\boldsymbol{C}_2$:

$$S = \left\| \frac{\partial H(z_1, z_2, z_3)}{\partial \boldsymbol{A}_2} \right\|_2^2 + \left\| \frac{\partial H(z_1, z_2, z_3)}{\partial \boldsymbol{B}_2} \right\|_2^2 + \left\| \frac{\partial H(z_1, z_2, z_3)}{\partial \boldsymbol{C}_2} \right\|_2^2$$

$$= \left\| \boldsymbol{g}(z_1, z_2)^T \boldsymbol{f}(z_2, z_3)^T \right\|_2^2$$

$$+ \left\| \boldsymbol{g}(z_1, z_2)^T \frac{\boldsymbol{Z}_3^T}{D_3(z_3)} \right\|_2^2 + \left\| \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \boldsymbol{f}(z_2, z_3)^T \right\|_2^2$$

with

$$\boldsymbol{f}(z_2, z_3) = (z_2 \boldsymbol{I}_p - \boldsymbol{A}_2)^{-1} \boldsymbol{B}_2 \frac{\boldsymbol{Z}_3}{D_3(z_3)}$$

$$\boldsymbol{g}(z_1, z_2) = \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \boldsymbol{C}_2 (z_2 \boldsymbol{I}_p - \boldsymbol{A}_2)^{-1}$$

$$\frac{\boldsymbol{Z}_1^T}{D_1(z_1)} = \boldsymbol{c}_1 (z_1 \boldsymbol{I}_{N_1} - \boldsymbol{A}_1)^{-1} \boldsymbol{B}_1 + \boldsymbol{d}_1$$

$$\frac{\boldsymbol{Z}_3}{D_3(z_3)} = \boldsymbol{C}_3 (z_3 \boldsymbol{I}_{N_3} - \boldsymbol{A}_3)^{-1} \boldsymbol{b}_3 + \boldsymbol{d}_3$$

It turns out that the sensitivity measure S can be expressed as

$$S = \mathrm{tr}\left[\boldsymbol{M}_A(\boldsymbol{I}_p)\right] + \mathrm{tr}\left[\boldsymbol{W}_B\right] + \mathrm{tr}\left[\boldsymbol{K}_C\right]$$

where $\boldsymbol{M}_A(\boldsymbol{I}_p), \boldsymbol{W}_B,$ and $\boldsymbol{K}_C$ are obtained using

$$\boldsymbol{X} = \frac{1}{(2\pi j)^3} \oint_{|z_1|=1} \oint_{|z_2|=1} \oint_{|z_3|=1} Y(z_1, z_2, z_3) \cdot Y^*(z_1, z_2, z_3) \frac{dz_1}{z_1} \frac{dz_2}{z_2} \frac{dz_3}{z_3}$$

with

$$\boldsymbol{Y}(z_1, z_2, z_3) = \boldsymbol{g}(z_1, z_2)^T \boldsymbol{f}(z_2, z_3)^T \quad \text{for } \boldsymbol{X} = \boldsymbol{M}_A(\boldsymbol{I}_p)$$

$$\boldsymbol{Y}(z_1, z_2, z_3) = \left[\frac{\boldsymbol{Z}_3}{D_3(z_3)} \boldsymbol{g}(z_1, z_2)\right]^* \quad \text{for } \boldsymbol{X} = \boldsymbol{W}_B$$

$$\boldsymbol{Y}(z_1, z_2, z_3) = \boldsymbol{f}(z_2, z_3) \frac{\boldsymbol{Z}_1^T}{D_1(z_1)} \quad \text{for } \boldsymbol{X} = \boldsymbol{K}_C$$

Applying a coordinate transformation $\bar{\boldsymbol{x}}(k) = \boldsymbol{T}^{-1}\boldsymbol{x}(k)$ to the 1-D system $(\boldsymbol{A}_2, \boldsymbol{B}_2, \boldsymbol{C}_2, \Delta_0)_p$ yields an equivalent realization

$$\left(\bar{\boldsymbol{A}}_2, \bar{\boldsymbol{B}}_2, \bar{\boldsymbol{C}}_2, \Delta_0\right)_p = \left(\boldsymbol{T}^{-1}\boldsymbol{A}_2\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{B}_2, \boldsymbol{C}_2\boldsymbol{T}, \Delta_0\right)_p$$

whose l_2-sensitivity measure S is given by

$$S = \mathrm{tr}\left[\boldsymbol{M}_A(\boldsymbol{P})\boldsymbol{P}\right] + \mathrm{tr}\left[\boldsymbol{W}_B\boldsymbol{P}\right] + \mathrm{tr}\left[\boldsymbol{K}_C\boldsymbol{P}^{-1}\right]$$

with $\boldsymbol{P} = \boldsymbol{T}\boldsymbol{T}^T$.

To impose a l_2-scaling condition, define controllability Gramian

$$\boldsymbol{K} = \frac{1}{(2\pi j)^2} \oint_{|z_2|=1} \oint_{|z_3|=1} \boldsymbol{f}(z_2, z_3) \cdot \boldsymbol{f}^*(z_2, z_3) \frac{dz_2}{z_2} \frac{dz_3}{z_3}$$

the l_2-scaling constraints are given by

$$\left(\overline{\boldsymbol{K}}\right)_{ii} = \left(\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-T}\right)_{ii} = 1 \quad \text{for } i = 1, 2, \ldots, p$$

Now the coefficient sensitivity minimization problem can be

formulated as

To obtain a coordinate transformation matrix $\boldsymbol{T}$ *that minimizes sensitivity measure* $S(\boldsymbol{P})$ *subject to the* l_2*-scaling constraints.*

3. l_2-Sensitivity Minimization

The l_2-scaling constraints are relaxed as

$$\mathrm{tr}\left[\boldsymbol{T}^{-1}\boldsymbol{K}\boldsymbol{T}^{-T}\right] = \mathrm{tr}\left[\boldsymbol{K}\boldsymbol{P}^{-1}\right] = p$$

leading to a relaxed optimization problem

$$\begin{aligned} &\text{minimize} \;\; S(\boldsymbol{P}) \\ &\text{subject to: } \mathrm{tr}\left[\boldsymbol{K}\boldsymbol{P}^{-1}\right] = p \end{aligned}$$

To solve it, we introduce the *Lagranging* of the problem

$$J(\boldsymbol{P},\lambda) = \mathrm{tr}\left[\boldsymbol{M}_A(\boldsymbol{P})\boldsymbol{P}\right] + \mathrm{tr}\left[\boldsymbol{W}_B\boldsymbol{P}\right] + \mathrm{tr}\left[\boldsymbol{K}_C\boldsymbol{P}^{-1}\right] + \lambda\left(\mathrm{tr}\left[\boldsymbol{K}\boldsymbol{P}^{-1}\right] - p\right)$$

Setting $\dfrac{\partial J(\boldsymbol{P},\lambda)}{\partial \boldsymbol{P}} = \mathbf{0}$ yields

$$\boldsymbol{P}\boldsymbol{F}(\boldsymbol{P})\boldsymbol{P} = \boldsymbol{G}(\boldsymbol{P},\lambda)$$

where

$$\boldsymbol{F}(\boldsymbol{P}) = \boldsymbol{M}_A(\boldsymbol{P}) + \boldsymbol{W}_B$$
$$\boldsymbol{G}(\boldsymbol{P},\lambda) = \boldsymbol{N}_A(\boldsymbol{P}) + \boldsymbol{K}_C + \lambda\boldsymbol{K}$$

with

$$\boldsymbol{N}_A(\boldsymbol{P}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\begin{bmatrix} I_p & 0 \end{bmatrix}\begin{bmatrix} A_2 & B_2 R_{ij} C_2 \\ 0 & A_2 \end{bmatrix}^k \begin{bmatrix} 0 & 0 \\ 0 & P \end{bmatrix}\begin{bmatrix} A_2^T & 0 \\ C_2^T R_{ij}^T B_2^T & A_2^T \end{bmatrix}^k \begin{bmatrix} I_p \\ 0 \end{bmatrix}$$

The above equation can be solved iteratively using the scheme

$$\boldsymbol{P}^{(k+1)}\boldsymbol{F}(\boldsymbol{P}^{(k)})\boldsymbol{P}^{(k+1)} = \boldsymbol{G}(\boldsymbol{P}^{(k)}, \lambda^{(k+1)})$$

whose solution is given by

$$\boldsymbol{P}^{(k+1)} = \boldsymbol{F}^{-1/2}(\boldsymbol{P}^{(k)}) \cdot \left[\boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)})\boldsymbol{G}(\boldsymbol{P}^{(k)}, \lambda^{(k+1)})\boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)}) \right]^{1/2} \cdot \boldsymbol{F}^{-1/2}(\boldsymbol{P}^{(k)})$$

The Lagrange multiplier is updated by solving

$$\mathrm{tr}\left[\tilde{\boldsymbol{K}}^{(k)}\tilde{\boldsymbol{G}}^{(k)}(\lambda^{(k+1)}) \right] = p$$

using a bisection method, where

$$\boldsymbol{K}^{(k)} = \boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)})\boldsymbol{K}\boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)})$$

$$\tilde{\boldsymbol{G}}^{(k)}(\lambda^{(k+1)}) = \left[\boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)})\boldsymbol{G}(\boldsymbol{P}^{(k)}, \lambda^{(k+1)})\boldsymbol{F}^{1/2}(\boldsymbol{P}^{(k)}) \right]^{-1/2}$$

4. A Numerical Example

Here we consider a stable 3-D state-space digital filter with

$$\Delta_0 = 10^{-2}\begin{bmatrix} 0.00730 & 0.34297 & -0.09594 & 0.20541 \\ 3.33408 & -5.73707 & 3.94939 & -1.61598 \\ -1.46081 & 2.66051 & -1.68094 & 0.68022 \\ 1.12651 & -1.62192 & 1.24735 & -0.55781 \end{bmatrix}$$

$$\Delta_1 = 10^{-2}\begin{bmatrix} 2.81318 & -5.00467 & 3.46926 & -0.84798 \\ -5.29980 & 9.24831 & -6.29206 & 2.80791 \\ 4.95232 & -8.39641 & 5.73329 & -1.62170 \\ 0.72029 & -1.34272 & 0.95941 & 0.54827 \end{bmatrix}$$

$$\Delta_2 = 10^{-2}\begin{bmatrix} -0.69409 & 1.54874 & -0.94779 & 0.39116 \\ 3.93785 & -6.79910 & 4.66564 & -1.96344 \\ -2.37995 & 4.20737 & -2.75482 & 0.95329 \\ 0.70545 & -0.90615 & 0.73168 & -0.55633 \end{bmatrix}$$

$$\Delta_3 = 10^{-2}\begin{bmatrix} 1.67681 & -2.69078 & 1.98218 & -0.33567 \\ -0.59937 & 1.11289 & -0.71981 & 0.43504 \\ 1.87472 & -2.93685 & 2.11591 & -0.43417 \\ 1.28875 & -2.01749 & 1.51782 & -0.09016 \end{bmatrix}$$

and

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix} = \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} -1.81600 & 1.23756 & -0.31382 \end{bmatrix}$$

$$\begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} -1.81611 & 1.23775 & -0.31391 \end{bmatrix}$$

The above filter admits a minimal state-space model with

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & -0.19089 & 0.29060 \\ 0.74393 & -86.40470 & 133.71075 \\ -0.27211 & -57.01643 & 88.22081 \end{bmatrix}$$

$$\boldsymbol{B}_2 = 10^3 \begin{bmatrix} 0.00602 & -0.00921 & 0.00699 & -0.00095 \\ -1.10247 & 1.68622 & -1.27902 & 0.17267 \\ -0.71455 & 1.09291 & -0.82977 & 0.11192 \end{bmatrix}$$

$$\boldsymbol{C}_2 = \begin{bmatrix} 0.07236 & 0.06711 & -0.10298 \\ 0.01930 & 0.01789 & -0.02745 \\ 0.05887 & 0.05460 & -0.08378 \\ 0.07079 & 0.06565 & -0.10073 \end{bmatrix}$$

After l_2-scaling, the l_2-sensitivity of the filter was found to be

$$S = 9.8732 \times 10^{8}$$

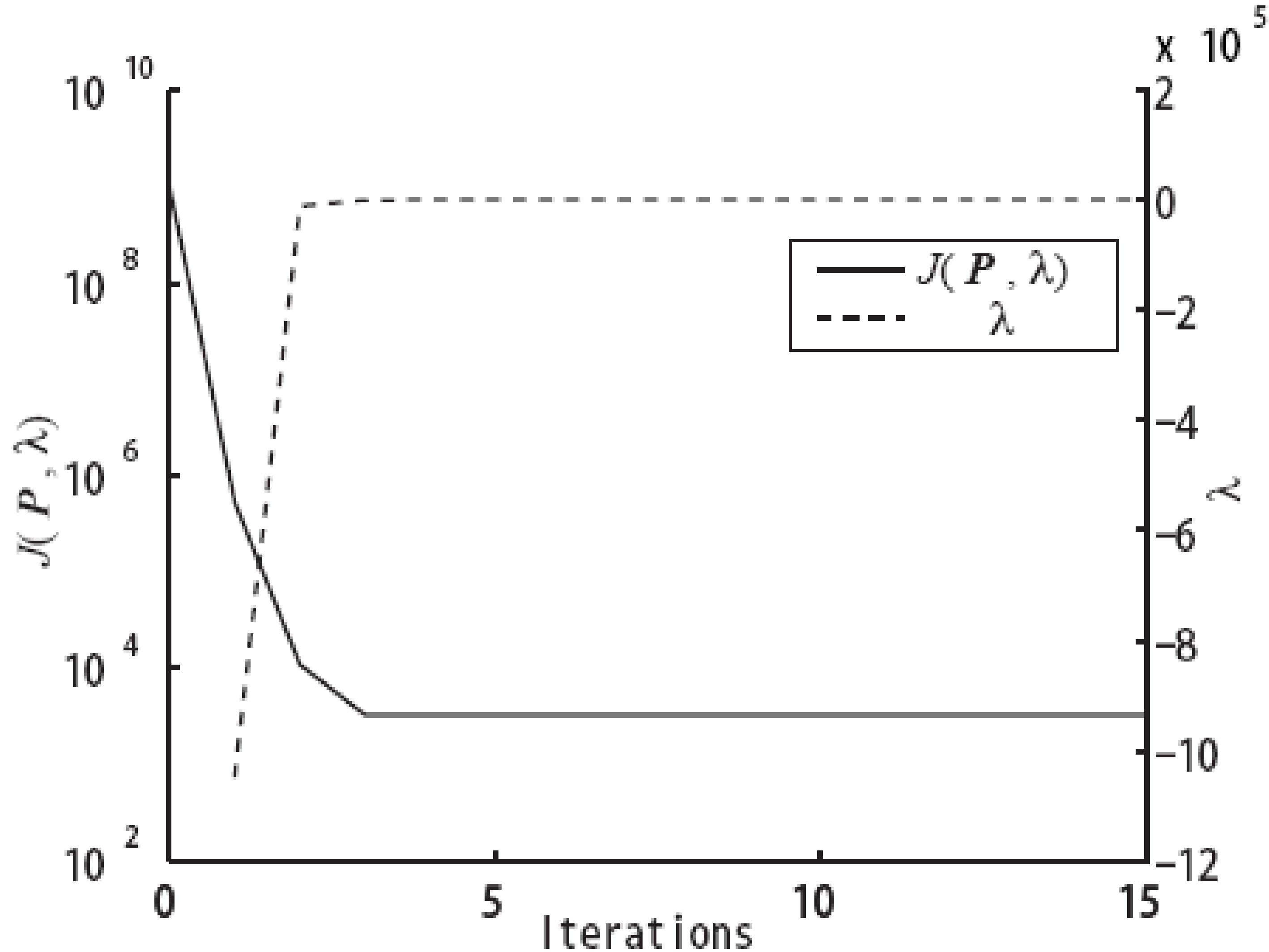
By applying the proposed algorithm with initial $\boldsymbol{P}^{(0)} = \boldsymbol{I}_3$ and $\varepsilon = 10^{-8}$, it took the algorithm 15 iterations to converge to a solution

$$\boldsymbol{P}^{(opt)} = \begin{bmatrix} 2.267890 & -2.297027 & 2.289396 \\ -2.297027 & 3.274871 & 3.268974 \\ 2.289396 & 3.268974 & 3.263110 \end{bmatrix}$$

$$\boldsymbol{T}^{(opt)} = \begin{bmatrix} 0.180276 & 1.360946 & -0.619043 \\ -1.018340 & -1.083999 & 1.030932 \\ -1.020420 & -1.079492 & 1.027884 \end{bmatrix}$$

and $\lambda^{(opt)} = 8.4292 \times 10^{3}$. The minimized l_2-sensitivity was found to be

$$J(P^{(opt)}, \lambda^{(opt)}) = 3.2436 \times 10^{3}$$



10^10
10^8
10^6
10^4
10^2
J(P , λ)
x 10^5
2
0
−2
−4
−6
−8
−10
−12
λ
J(P , λ)
λ
0
5
10
15
Iterations

4. Conclusion

- The problem of minimizing a l_2-coefficient sensitivity measure for separable-denominator 3-D state-space digital filters has been investigated.
- To this end, the problem is formulated as a constrained optimization problem and is solved using an iterative technique.
- Our computer simulation has indicated that the algorithm proposed here works well with satisfactory efficiency.


Thank you for your attention !