# Penalty Convex-Concave Procedure
## *for Source Localization Problem*


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**Abstract**

In this paper, we focus on the least-squares (LS) formulation for the localization problem, where the $l_2$-norm of the residual errors is minimized in a setting known as difference-of-convex-functions programming. The problem at hand is then solved by applying a penalty convex-concave procedure (PCCP) in a successive manner. Algorithmic details that are tailored to the localization problem, such as imposing additional constraints to enforce iteration path towards the LS solution and strategies to secure a good initial point, are also provided. Simulation results demonstrate promising localization performance when compared with some best known results from the literature.


## Introduction

- Least squares (LS) algorithms for range-based localization:
  - geometrically meaningful
  - provide low complexity solutions with competitive accuracy
- However the error measure is non-convex which excludes many *local* methods, that are iterative
- Solutions obtained using *global* localization techniques such as semidefinite programming (SDP) are not optimal in LS sense.
- Methods by A.Beck, P.Stoica, J.Li [BSL2008] for *squared* range LS
  - obtain exact global solutions
  - remain suboptimal in the maximum likelihood (ML) sense
- Proposed formulation:
  - based on a penalty convex-concave procedure (PCCP)
  - accepts infeasible initial points
  - additional constraints that enforce the algorithms iteration path towards the LS solution
  - strategies to secure good initial points

## Problem Statement

### *Measurement Model*

Throughout it is assumed that *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

$\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of $m$ sensors;
$\boldsymbol{a}_i \in R^n$ - $n$ coordinates of the $i$th sensor in space $R^n$, n = 2 or 3;
$r_i$ - received noisy distance reading from sensor $i$;
$\varepsilon_i$ - unknown noise associated with measurement from the $i$th sensor.
**Problem statement**: estimate the exact source location $\boldsymbol{x} \in R^n$ from noisy range measurements $\boldsymbol{r} = [r_1\, r_2 \ldots r_m]^T$.

### *LS Formulation*

The range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}}\; F(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \qquad \text{(R)}$$

If $\boldsymbol{\varepsilon} \sim N(0, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} \propto \boldsymbol{I}$, then the ***R-LS solution of problem* (R) *is identical to the ML location estimator***. Unfortunately, the objective in (R) is highly non-convex with many local minimizers even for small-scale systems.

## CCP Framework for Localization

### *Basic Convex-Concave Procedure (CCP)*

It is a descent algorithm that requires a *feasible* initial point $\boldsymbol{x}_0$, i.e. $f_i(\boldsymbol{x}) - g_i(\boldsymbol{x}) \le 0$ for $i = 1, 2 \ldots, m$. The CCP finds local optima of *nonconvex* problems of the form

$$\begin{aligned} &\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x}) - g(\boldsymbol{x}) \\ &\text{subject to:} \quad f_i(\boldsymbol{x}) \le g_i(\boldsymbol{x}) \quad \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

where $f(\boldsymbol{x}), g(\boldsymbol{x}), f_i(\boldsymbol{x}), g_i(\boldsymbol{x})$ for $i = 1, 2 \ldots, m$ are convex.

The basic CCP algorithm is an iterative procedure including two key steps (in the $k$-th iteration):

1. ***Convexify***: form $\hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) = g(\boldsymbol{x}_k) + \bigtriangledown g(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$ and $\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = g_i(\boldsymbol{x}_k) + \bigtriangledown g_i(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k)$ for $i = 1, 2 \ldots, m$
2. ***Solve the convex problem***:

$$\begin{aligned} &\underset{\boldsymbol{x}}{\text{minimize}} \quad f(\boldsymbol{x}) - \hat{g}(\boldsymbol{x}, \boldsymbol{x}_k) \\ &\text{subject to:} \quad f_i(\boldsymbol{x}) - \hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) \le 0 \\ &\qquad\qquad \text{for: } i = 1, 2, \ldots, m \end{aligned}$$

### *Problem Reformulation*

We begin by re-writing the objective $F(\boldsymbol{x})$ up to a constant as:

$$\sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T \sum_{i=1}^{m} \boldsymbol{a}_i - 2\sum_{i=1}^{m} r_i \|\boldsymbol{x} - \boldsymbol{a}_i\|$$

which allows to formulate it in a basic CCP form $F(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$ with

$$f(\boldsymbol{x}) = m\boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T \sum_{i=1}^{m} \boldsymbol{a}_i \quad \text{- convex}$$

$$g(\boldsymbol{x}) = 2\sum_{i=1}^{m} r_i \|\boldsymbol{x} - \boldsymbol{a}_i\| \qquad \text{- convex.}$$

Since $g(\boldsymbol{x})$ is not differentiable at the point where $\boldsymbol{x} = \boldsymbol{a}_i$ for some $1 \le i \le m$, we replace $\bigtriangledown g(\boldsymbol{x}_k)$ by a subgradient of $g(\boldsymbol{x})$ at $\boldsymbol{x}_k$ as

$$\partial g(\boldsymbol{x}_k) = 2\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

where

$$\partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\| = \begin{cases} \dfrac{\boldsymbol{x}_k - \boldsymbol{a}_i}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|}, & \text{if } \boldsymbol{x}_k \ne \boldsymbol{a}_i \\ \boldsymbol{0}, & \text{otherwise} \end{cases}$$

Up to a multiplicative factor $1/m$ and an additive constant term the objective in (R) can be written as

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \hat{F}(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k$$

where

$$\boldsymbol{v}_k = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|, \quad \bar{\boldsymbol{a}} = \frac{1}{m}\sum_{i=1}^{m} \boldsymbol{a}_i$$

Given $\boldsymbol{x}_k$ (in the $k$-th iteration) the solution of the quadratic problem can be obtained as

$$\boldsymbol{x}_{k+1} = \bar{\boldsymbol{a}} + \frac{1}{m}\sum_{i=1}^{m} r_i \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|$$

### *Imposing Error Bounds*

The algorithm can be enhanced by imposing a bound on each squared measurement error

$$(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2 \le \delta_i^2$$

which leads to

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le 0 & \qquad (C1) \\ r_i - \delta_i \le \|\boldsymbol{x} - \boldsymbol{a}_i\|, \text{ for } 1 \le i \le m. & \qquad (C2) \end{aligned}$$

Both sets of constraints can be written in a form $f_i(\boldsymbol{x}) \le g_i(\boldsymbol{x})$.

Constraints in (C1) are convex, with $f_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i$, and $g_i(\boldsymbol{x}) = 0$. In case of (C2): define $f_i(\boldsymbol{x}) = r_i - \delta_i$ and $g_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}_i\|$. Then replace $g_i(\boldsymbol{x})$ with its approximation

$$\hat{g}_i(\boldsymbol{x}, \boldsymbol{x}_k) = \|\boldsymbol{x}_k - \boldsymbol{a}_i\| + \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k)$$

This allows to convexify constraints $r_i - \delta_i \le \|\boldsymbol{x} - \boldsymbol{a}_i\|$ as

$$-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le 0$$

Summarizing, the problem in the $k$-th iteration can be stated as

$$\begin{aligned} &\underset{\boldsymbol{x}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k \\ &\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le 0 \\ &-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \partial \|\boldsymbol{x}_k - \boldsymbol{a}_i\|^T (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le 0 \end{aligned}$$

### Penalty CCP (PCCP)

***Technical problem***: the formulation requires a *feasible* initial point.
***Solution approach***: allow *infeasible* initial points by introducing slack variables $s_i \ge 0, \hat{s}_i \ge 0, 1 \le i \le m$ into constraints (C1) and (C2) and penalizing the sum of violations.
This leads to a ***penalty* CCP based formulation**:

$$\begin{aligned} &\underset{\boldsymbol{x}, \boldsymbol{s}, \hat{\boldsymbol{s}}}{\text{minimize}} \quad \boldsymbol{x}^T\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{v}_k + \tau_k \sum_{i=1}^{m} (s_i + \hat{s}_i) \\ &\text{subject to:} \quad \|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i - \delta_i \le s_i \\ &-\|\boldsymbol{x}_k - \boldsymbol{a}_i\| - \frac{(\boldsymbol{x}_k - \boldsymbol{a}_i)^T}{\|\boldsymbol{x}_k - \boldsymbol{a}_i\|} (\boldsymbol{x} - \boldsymbol{x}_k) + r_i - \delta_i \le \hat{s}_i \\ &\qquad s_i \ge 0, \hat{s}_i \ge 0, \text{ for: } i = 1, 2, \ldots, m \end{aligned}$$

where $0 \le \tau_k \le \tau_{max}$.

### *Input Parameters*

***Bound $\delta_i$ on the measurement error***:

- Lower $\delta_i$ leads to a "tighter" solution;
- Larger $\delta_i$ makes the algorithm less sensitive to outliers;
- If $\boldsymbol{\varepsilon}$ obeys a Gaussian distribution with zero mean and covariance $\boldsymbol{\Sigma} = \text{diag}(\sigma_1^2, \ldots, \sigma_m^2)$, then $\delta_i = \gamma\sigma_i$, where $\gamma$ determines the width of confidence interval. For example, for $\gamma = 3$ we have the probability $Pr\{|\varepsilon_i| \le 3\sigma_i\} \approx 0.99$.

***Techniques to select good initial point $\boldsymbol{x}_0$***:

- Uniformly randomly over the same region as the unknown source;
- Set the initial point to the origin;
- Run the algorithm from a set of candidate initial points and identify the solution as the one with lowest LS error;
- Apply a *global* localization algorithm to generate an approximate LS solution, then take it as the initial point.

## Numerical Results

### *System Setup*

- Sensors: $\{\boldsymbol{a}_i, i = 1, 2, \ldots, 5\}$ randomly placed in the planar region in $[-15; 15] \times [-15; 15]$
- Source: $\boldsymbol{x}_s$, located randomly in $\{\boldsymbol{x} = [x_1; x_2], -10 \le x_1, x_2 \le 10\}$
- Noise: $\{\varepsilon_i, i = 1, \ldots, m\}$ was modelled as i.i.d random variables with zero mean and variance $\sigma^2$, $\sigma \in \{10^{-3}, 10^{-2}, 10^{-1}, 1\}$
- $\gamma = 3$, $K_{max} = 20$.

Averaged MSE for SR-LS and PCCP methods

| $\sigma$ | MLE | SR - LS | PCCP | R elative.I. |
|---|---|---|---|---|
| 1e-03 | 6.0159e-01 | 1.3394e-06 | **9.5243e-07** | 29% |
| 1e-02 | 3.5077e-01 | 1.4516e-04 | **9.5831e-05** | 34% |
| 1e-01 | 3.7866e-01 | 1.2058e-02 | **8.7107e-03** | 28% |
| 1e+00 | 1.4470e+00 | 1.3662e+00 | **1.2346e+00** | 10% |

## Conclusions

- New iterative method for locating a radiating source based on noisy range measurements that transforms original least-squares problem to a DC programming problem
- This in turn is relaxed to a sequential convex minimization based on PCCP that can be efficiently solved with an infeasible initial point
- CCP allows a natural embedding of the LS formulation for localization into a sequential convex formulation

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