Improved Least-Squares Methods for Source Localization: An Iterative Re-Weighting Approach

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Outline

Introduction

- Locating a radiating source from noisy range or range-difference measurements in a passive sensor network has a wide range of applications in teleconferencing, wireless communications, surveillance, navigation, and geophysics.

- Least squares (LS) based algorithms are an important class of solution techniques for localization problems. Under a certain probabilistic model of the noise an LS solution is identical to the maximum likelihood (ML) estimator.

- The error measure in an LS formulation is highly non-convex, possessing multiple local solutions with degraded performance.

Introduction

- Methods developed by A. Beck, P. Stoica, J. Li [BSL2008] for *squared* range LS (SR-LS) and *squared* range difference LS (SDR-LS) problems allow to obtain exact and *global* solutions.
- The results produced are merely approximations of the original LS problems because SR-LS and SRD-LS are no longer an ML solutions.
- Proposed iterative procedure where the SR-LS (or SRD-LS) algorithm is applied to a *weighted* sum of squared terms and special weights construction allow to obtain a solution which is conciderably closer to the original range-based (or range-difference-based) LS solution.

Source Localization From Range Measurements

Measurement Model

- Throughout it is assumed that *range measurements* obey the model

$$r_i = \|\boldsymbol{x} - \boldsymbol{a}_i\| + \varepsilon_i, \quad i = 1, \ldots, m.$$

  where $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ - given array of m sensors;
  $\boldsymbol{a}_i \in R^n$ contains n coordinates of the ith sensor in space R^n;
  r_i - received noisy distance reading from the ith sensor;
  ε_i - unknown noise associated with measurement from the ith sensor.

- The problem can be stated as to estimate the exact source location $\boldsymbol{x} \in R^n$ from noisy range measurements $\boldsymbol{r} = [r_1 \; r_2 \ldots r_m]^T$.

Source Localization From Range Measurements

LS Formulations

- The range-based least squares (R-LS) estimate refers to the solution of the problem

$$\underset{\boldsymbol{x}}{\text{minimize}}\, f(\boldsymbol{x}) = \sum_{i=1}^{m} (r_i - \|\boldsymbol{x} - \boldsymbol{a}_i\|)^2 \qquad \text{(R)}$$

- If $\varepsilon \sim N(0, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma} \propto \boldsymbol{I}$, then the R-LS solution of problem (R) is identical to the ML location estimator.

- Unfortunately, the objective in (R) is highly non-convex, posessing many local minimizers even for small-scale systems.

LS Formulations

- Alternatively, location estimate can be obtained by solving the *squared range based LS* (SR-LS) problem [BSL2008]

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} (\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2)^2 \qquad \text{(SR)}$$

- The SR-LS estimate is no longer an ML solution, hence, only an approximation of the original R-LS problem.
- To reduce the gap between the two solutions we propose a weighted SR-LS (WSR-LS) problem:

$$\underset{\boldsymbol{x}}{\text{minimize}} \sum_{i=1}^{m} w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2\right)^2 \qquad \text{(WSR)}$$

An Iterative Re-Weighting Strategy

- WSR-LS with properly chosen weights facilitates an excellent approximation of the R-LS estime.

- The main idea is to use the weigths $w_i, i = 1, \ldots, m$ to tune the objective in (WSR) toward the objective in (R). We compare the ith term of the objective in (WSR) with its counterpart in (R) as:

$$\underbrace{w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2\right)^2}_{\text{in (WSR)}} \leftrightarrow \underbrace{(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i)^2}_{\text{in (R)}}$$

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- By writing the ith term in (WSR) as

$$w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 = w_i \left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2 \underbrace{\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| - r_i \right)^2}_{\text{same as in (R)}}$$

  we note that the objective in (WSR) would be the same as in (R) if the weight w_i was assigned to $1/\left(\|\boldsymbol{x} - \boldsymbol{a}_i\| + r_i \right)^2$.

- Evidently, such weight assignments cannot be realized.

Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- In the proposed iterative procedure we solve a weighted SR-LS sub-problem,where at each iteration the weights are fixed:

$$\underset{x}{\text{minimize}} \sum_{i=1}^{m} w_i^{(k)} \left(\|\boldsymbol{x} - \boldsymbol{a}_i\|^2 - r_i^2 \right)^2 \qquad \text{(IRWSR)}$$

- for $k = 1$ all weights $\{w_i^{(1)}, i = 1, \ldots, m\}$ are set to unity;
- for $k \geq 2$ the weights $\{w_i^{(k)}, i = 1, \ldots, m\}$ are assigned using the previous iterate $\boldsymbol{x}_{k-1}$ as

$$w_i^{(k)} = \frac{1}{\left(\|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\| + r_i\right)^2}.$$

Source Localization From Range-Difference Measurements

SRD-LS and WSRD-LS formulations

- An approximation of the RD-LS solution can be obtained by solving the *squared range difference based LS* (SRD-LS) problem.
- By re-writing the measurements model as $d_i + \|\boldsymbol{x}\| = \|\boldsymbol{x} - \boldsymbol{a}_i\|$ and squaring both sides, we obtain

$$-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T\boldsymbol{x} = g_i, \quad i = 1, \ldots, m$$

where $g_i = d_i^2 - \|\boldsymbol{a}_i\|^2$. The SRD-LS solution can be obtained by minimizing following criterion:

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} \left(-2\boldsymbol{a}_i^T\boldsymbol{x} - 2d_i\|\boldsymbol{x}\| - g_i\right)^2$$

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We now present a method for improved solutions over SRD-LS solutions.

- We consider the weighted SRD-LS problem

$$\underset{\boldsymbol{x} \in R^n}{\text{minimize}} \sum_{i=1}^{m} w_i \left(-2\boldsymbol{a}_i^T \boldsymbol{x} - 2d_i \|\boldsymbol{x}\| - g_i \right)^2 \qquad \text{(WSRD)}$$

where weights w_i for $i = 1, \ldots, m$ are *fixed* nonnegative constants.

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- The ith term of the objective function in (WSRD) can be written as:

$$
\begin{aligned}
& w_i \left(-2d_i\|\boldsymbol{x}\| - 2\boldsymbol{a}_i^T \boldsymbol{x} - g_i \right)^2 \\
= & w_i \left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\| \right) \underbrace{\left(d_i + \|\boldsymbol{x}\| - \|\boldsymbol{x} - \boldsymbol{a}_i\| \right)}_{\text{same as in RD}}
\end{aligned}
$$

- If weights w_i were set to $1/\left(d_i + \|\boldsymbol{x}\| + \|\boldsymbol{x} - \boldsymbol{a}_i\|\right)^2$ the objective in (WSRD) would be the same as in (RD).

Source Localization From Range-Difference Measurements

Improved Solution Using Iterative Re-weighting

- We employ an iterative procedure where the weights in the kth iteration are assigned to

$$w_i^{(k)} = \frac{1}{(d_i + \|\boldsymbol{x}_{k-1}\| + \|\boldsymbol{x}_{k-1} - \boldsymbol{a}_i\|)^2}, i = 1, \ldots, m$$

with $\{w_i^{(1)} = 1, i = 1, \ldots, m\}$.

- We will refer to the derived problem as the iterative re-weighted SRD-LS (WSRD-LS) problem and the solution obtained as IRWSRD-LS solution.

Performance Evaluation for SR-LS and IRWSR-LS

- We can see that IRWSR-LS solutions offer considerable improvement over SR-LS solutions.

Table: Averaged MSE for SR-LS and IRWSR-LS methods by noise level

σ	**SR - LS**	**IRWSR-LS**	**Improvement (%)**
1e-03	1.897294e-06	1.123411e-06	40.8
1e-02	1.779870e-04	1.081470e-04	39.2
1e-01	1.831870e-02	1.128165e-02	38.4
1e+0	2.415438e+00	1.877930e+00	22.3

Performance Evaluation for SRD-LS and IRWSRD-LS

Table: Averaged MSE for SRD-LS and IRWSRD-LS methods by noise level

σ	**SRD - LS**	**IRWSRD-LS**	**Improvement (%)**
1e-04	8.4918e-09	4.1050e-09	51.7
1e-03	5.8553e-06	3.5105e-06	40.0
1e-02	6.3508e-05	5.0378e-05	20.7
1e-01	1.6057e-02	1.0055e-02	37.3
1e+0	1.2773e+00	6.2221e-01	51.2

Conclusions

- New global methods for locating a radiating source based on noisy range or range difference measurements have been proposed.
- These methods are developed by transforming the SR-LS and SRD-LS algorithms [BSL2008] into an iterative procedure so that a weighted SR-LS (SRD-LS) objective assymptotically approaches the original R-LS objective.
- Proposed algorithms are found to outperform the best known results from the literature.