Improved Least-Squares Methods for Source Localization: An Iterative Re-Weighting Approach

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Introduction

- Locating a radiating source from noisy range or range-difference measurements in a passive sensor network has a wide range of applications in teleconferencing, wireless communications, surveillance, navigation, and geophysics.

- Least squares (LS) based algorithms are an important class of solution techniques for localization problems. Under a certain probabilistic model of the noise an LS solution is identical to the maximum likelihood (ML) estimator.

- The error measure in an LS formulation is highly non-convex, possessing multiple local solutions with degraded performance.
Introduction

- Methods developed by A. Beck, P. Stoica, J. Li [BSL2008] for squared range LS (SR-LS) and squared range difference LS (SDR-LS) problems allow to obtain exact and *global* solutions.

- The results produced are merely approximations of the original LS problems because SR-LS and SRD-LS are no longer an ML solutions.

- Proposed iterative procedure where the SR-LS (or SRD-LS) algorithm is applied to a *weighted* sum of squared terms and special weights construction allow to obtain a solution which is considerably closer to the original range-based (or range-difference-based) LS solution.
Measurement Model

Throughout it is assumed that range measurements obey the model

\[ r_i = \|x - a_i\| + \varepsilon_i, \quad i = 1, \ldots, m. \]

where \( \{a_1, \ldots, a_m\} \) - given array of \( m \) sensors;
\( a_i \in \mathbb{R}^n \) contains \( n \) coordinates of the \( i \)th sensor in space \( \mathbb{R}^n \);
\( r_i \) - received noisy distance reading from the \( i \)th sensor;
\( \varepsilon_i \) - unknown noise associated with measurement from the \( i \)th sensor.

The problem can be stated as to estimate the exact source location \( x \in \mathbb{R}^n \) from noisy range measurements \( r = [r_1 \ r_2 \ldots r_m]^T \).
Source Localization From Range Measurements

**LS Formulations**

- The range-based least squares (R-LS) estimate refers to the solution of the problem

\[
\text{minimize } f(x) = \sum_{i=1}^{m} (r_i - \|x - a_i\|)^2
\]  

(R)

- If \( \varepsilon \sim N(0, \Sigma) \) and \( \Sigma \propto I \), then the R-LS solution of problem (R) is identical to the ML location estimator.

- Unfortunately, the objective in (R) is highly non-convex, possessing many local minimizers even for small-scale systems.
**LS Formulations**

- Alternatively, location estimate can be obtained by solving the *squared range based LS* (SR-LS) problem [BSL2008]

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{m} (\|x - a_i\|^2 - r_i^2)^2 \\
\text{SR} \quad & (SR)
\end{align*}
\]

- The SR-LS estimate is no longer an ML solution, hence, only an approximation of the original R-LS problem.

- To reduce the gap between the two solutions we propose a weighted SR-LS (WSR-LS) problem:

\[
\begin{align*}
\text{minimize} \quad & \sum_{i=1}^{m} w_i (\|x - a_i\|^2 - r_i^2)^2 \\
\text{WSR} \quad & (WSR)
\end{align*}
\]
Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- WSR-LS with properly chosen weights facilitates an excellent approximation of the R-LS estimate.

- The main idea is to use the weights $w_i, i = 1, \ldots, m$ to tune the objective in (WSR) toward the objective in (R). We compare the $i$th term of the objective in (WSR) with its counterpart in (R) as:

$$w_i \left( \|x - a_i\|^2 - r_i^2 \right)^2 \leftrightarrow \left( \|x - a_i\| - r_i \right)^2$$

in (WSR) \leftrightarrow \text{in (R)}
An Iterative Re-Weighting Strategy

By writing the $i$th term in (WSR) as

$$w_i \left( \|x - a_i\|^2 - r_i^2 \right)^2 = w_i \left( \|x - a_i\| + r_i \right)^2 \left( \|x - a_i\| - r_i \right)^2$$

we note that the objective in (WSR) would be the same as in (R) if the weight $w_i$ was assigned to $1/ \left( \|x - a_i\| + r_i \right)^2$.

Evidently, such weight assignments cannot be realized.
Source Localization From Range Measurements

An Iterative Re-Weighting Strategy

- In the proposed iterative procedure we solve a weighted SR-LS sub-problem, where at each iteration the weights are fixed:

\[
\minimize_x \sum_{i=1}^{m} w_i^{(k)} \left( \|x - a_i\|^2 - r_i^2 \right)^2 \tag{IRWSR}
\]

- for \( k = 1 \) all weights \( \{w_i^{(1)}, i = 1, \ldots, m\} \) are set to unity;

- for \( k \geq 2 \) the weights \( \{w_i^{(k)}, i = 1, \ldots, m\} \) are assigned using the previous iterate \( x_{k-1} \) as

\[
w_i^{(k)} = \frac{1}{\left( \|x_{k-1} - a_i\| + r_i \right)^2}.
\]
SRD-LS and WSRD-LS formulations

- An approximation of the RD-LS solution can be obtained by solving the \textit{squared range difference based LS} (SRD-LS) problem.
- By re-writing the measurements model as $d_i + \|x\| = \|x - a_i\|$ and squaring both sides, we obtain

$$-2d_i\|x\| - 2a_i^T x = g_i, \quad i = 1, \ldots, m$$

where $g_i = d_i^2 - \|a_i\|^2$. The SRD-LS solution can be obtained by minimizing following criterion:

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \left(-2a_i^T x - 2d_i\|x\| - g_i\right)^2$$
Source Localization From Range-Difference Measurements

**Improved Solution Using Iterative Re-weighting**

- We now present a method for improved solutions over SRD-LS solutions.

- We consider the weighted SRD-LS problem

\[
\minimize_{\mathbf{x} \in \mathbb{R}^n} \sum_{i=1}^{m} w_i \left( -2 a_i^T \mathbf{x} - 2 d_i \| \mathbf{x} \| - g_i \right)^2 \tag{WSRD}
\]

where weights \( w_i \) for \( i = 1, \ldots, m \) are fixed nonnegative constants.
Improved Solution Using Iterative Re-weighting

- The $i$th term of the objective function in (WSRD) can be written as:
  \[ w_i \left( -2d_i \|x\| - 2a_i^T x - g_i \right)^2 \]
  \[ = w_i \left( d_i + \|x\| + \|x - a_i\| \right) \left( d_i + \|x\| - \|x - a_i\| \right) \]
  same as in RD

- If weights $w_i$ were set to $1/ (d_i + \|x\| + \|x - a_i\|)^2$ the objective in (WSRD) would be the same as in (RD).
Improved Solution Using Iterative Re-weighting

- We employ an iterative procedure where the weights in the $k$th iteration are assigned to

$$w_i^{(k)} = \frac{1}{(d_i + \|x_{k-1}\| + \|x_{k-1} - a_i\|)^2}, \quad i = 1, \ldots, m$$

with \(\{w_i^{(1)} = 1, \ i = 1, \ldots, m\}\).

- We will refer to the derived problem as the iterative re-weighted SRD-LS (WSRD-LS) problem and the solution obtained as IRWSRD-LS solution.
We can see that IRWSR-LS solutions offer considerable improvement over SR-LS solutions.

### Table: Averaged MSE for SR-LS and IRWSR-LS methods by noise level

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>SR - LS</th>
<th>IRWSR-LS</th>
<th>Improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1e-03</td>
<td>1.897294e-06</td>
<td>1.123411e-06</td>
<td>40.8</td>
</tr>
<tr>
<td>1e-02</td>
<td>1.779870e-04</td>
<td>1.081470e-04</td>
<td>39.2</td>
</tr>
<tr>
<td>1e-01</td>
<td>1.831870e-02</td>
<td>1.128165e-02</td>
<td>38.4</td>
</tr>
<tr>
<td>1e+0</td>
<td>2.415438e+00</td>
<td>1.877930e+00</td>
<td>22.3</td>
</tr>
</tbody>
</table>
**Table: Averaged MSE for SRD-LS and IRWSRD-LS methods by noise level**

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>SRD - LS</th>
<th>IRWSRD-LS</th>
<th>Improvement (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1e-04</td>
<td>8.4918e-09</td>
<td>4.1050e-09</td>
<td>51.7</td>
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<tr>
<td>1e-03</td>
<td>5.8553e-06</td>
<td>3.5105e-06</td>
<td>40.0</td>
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<tr>
<td>1e-02</td>
<td>6.3508e-05</td>
<td>5.0378e-05</td>
<td>20.7</td>
</tr>
<tr>
<td>1e-01</td>
<td>1.6057e-02</td>
<td>1.0055e-02</td>
<td>37.3</td>
</tr>
<tr>
<td>1e+0</td>
<td>1.2773e+00</td>
<td>6.2221e-01</td>
<td>51.2</td>
</tr>
</tbody>
</table>
Conclusions

- New global methods for locating a radiating source based on noisy range or range difference measurements have been proposed.

- These methods are developed by transforming the SR-LS and SRD-LS algorithms [BSL2008] into an iterative procedure so that a weighted SR-LS (SRD-LS) objective asymptotically approaches the original R-LS objective.

- Proposed algorithms are found to outperform the best known results from the literature.