Minimax Design of Stable IIR Filters with Sparse Coefficients

W.-S. Lu
Dept. of Electrical & Computer Engineering
University of Victoria
Victoria, Canada

T. Hinamoto
Graduate School of Engineering
Hiroshima University
Hiroshima, Japan

Outline

- Design Problem and Related Work
- Significance
- Design Method at a Glance
- A Design Example
- Concluding Remarks

1. Design Problem and Related Work

Design Problem

Find a stable IIR digital filter of order $(n, 2r)$

$$H(z) = \frac{a(z)}{d(z)} = \frac{\sum_{i=0}^{n} a_i z^{-i}}{\prod_{i=1}^{r} (d_{i2} z^{-2} + d_{i1} z^{-1} + 1)}$$

with sparsity of $\{a_i\} = K$ that optimally approximates a desired frequency response $H_d(\omega)$ in a least-squares or minimax sense.

Related Work

- Frequency-response masking filters (Y.C. Lim, 1986)
  -- Special filter structure.

- Shaped window functions for narroband 2D fan filters (L. Khademi and L.T. Bruton, 2003) – Special filter type.

- Sparse half-band like FIR filters (O. Gustafsson, L.S. DeBrunner, V. DeBrunner, and H. Johnnsson, 2007)

- Linear programming algorithms for sparse FIR filters (T. Baran, D. Wei, and A.V. Oppenheim, March 2010)

- l_1-minimization algorithms for sparse 1-D and 2-D FIR filters (W.-S. Lu and T. Hinamoto, May 2010, Jan. 2011)

2. Significance

- Digital filters with sparse coefficients are of interest because the sparsity implies real-time application potential and reduction in implementation complexity (hence cost).
- IIR filters are *not* sparse in general.

3. Design Method at a Glance

Design Phase 1

- To identify an index set of the most appropriate locations for numerator polynomial $a(z)$ to be zero in order to satisfy a target sparsity.
- This is done subject to:

  (i) Keeping closeness of $H(e^{j\omega})$ to $H_d(\omega)$.

  (ii) stability of $H(z)$.
- The target sparsity is achieved by the l_1-norm of coefficient vector $\boldsymbol{a}$ into an objective function so as to promote its sparsity. This yields

$$\begin{aligned} &\underset{\boldsymbol{x}}{\text{minimize}} \;\; \left[\left\| H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega) \right\|_\infty + \mu \left\| \boldsymbol{a} \right\|_1 \right] \\ &\text{subject to:} \quad H(z) \;\; \text{stable} \end{aligned} \qquad (1)$$

where

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{d} \end{bmatrix}, \quad \left\| \boldsymbol{a} \right\|_1 = \sum_{i=0}^{n} |a_i|$$

- (1) is *not* a convex problem because $\left\| H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega) \right\|_\infty$ is not convex, however,
- the feasibility region is convex, and
- the second term of the objective function, $\mu \left\| \boldsymbol{a} \right\|_1$, is convex.
- under these circumstances we decide to use a sequential design approach to an optimal solution of (1), where in

each step in the sequential technique the term $\left\| H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega) \right\|_\infty$ is approximated by a convex term, hence we deal with a sequential convex problem to achieve the design.

- The convex subproblem in *k*th iteration looks like this:

$$
\begin{aligned}
&\text{minimize} && \boldsymbol{f}^T \boldsymbol{y} \\
&\text{subject to:} && \left\| \boldsymbol{G}_k(\omega)\boldsymbol{y} + \boldsymbol{e}_k(\omega) \right\|_2 \le \boldsymbol{b}^T \boldsymbol{y}, \quad \omega \in \Omega \\
& && \left\| a_k(i) + \delta_a(i) \right\| \le u_i \quad \text{for } \ 0 \le i \le n \\
& && \boldsymbol{C}\boldsymbol{y} + \boldsymbol{s}_k \ge \boldsymbol{0} \\
& && \left\| \hat{\boldsymbol{I}}\boldsymbol{y} \right\|_2 \le \beta
\end{aligned}
\qquad (2)
$$

where $\boldsymbol{y} = \begin{bmatrix} \underbrace{\eta}_{\text{minimax error}} & \underbrace{\boldsymbol{\delta}^T}_{\text{for filter coeff.}} & \underbrace{\boldsymbol{u}^T}_{\text{upper bound of } \|\boldsymbol{a}\|_1} \end{bmatrix}^T$

- Once the unique global solution δ^* of (2) is obtained, $\boldsymbol{x}_k$ is updated to $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{\delta}^*$.
- The process continues until certain termination condition is met and a solution point $\boldsymbol{x}^* = \begin{bmatrix} \boldsymbol{a}^* \\ \boldsymbol{d}^* \end{bmatrix}$ is found.
- An index set is generated by hard thresholding as

$$I^* = \left\{ i : \left| a_i^* \right| < \varepsilon_t \right\} \tag{3}$$

where ε_t is a threshold so tuned that the set I^* contains $n - k + 1$ indices.

- It is the index set in (3) that is the goal of the first phase of the design.

Design Phase 2

The goal is to design an IIR digital filter that optimally approximates a desired frequency response $H_d(\omega)$ in minimax sense subject to sparsity and stability.

- The problem at hand can be formulated as

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \left\| H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega) \right\|_\infty \qquad (4) \\ \text{subject to:} \quad & H(z) \text{ is stable} \\ & a_i = 0 \ \text{ for } i \in I^* \end{aligned}$$

- The constraints in (4) define a convex region. However, the objective function is *not* convex.

- We apply a sequential design approach similar to that used in phase 1, where in each step $\left\| H(e^{j\omega}, \boldsymbol{x}) - H_d(\omega) \right\|_\infty$ is approximated by a convex term, hence we deal with a sequence of convex problems to achieve the design. The convex subproblem if the *k*th iteration looks like this:

$$\begin{aligned} \min_{\delta} \max_{\omega} \quad & \left\| \tilde{\boldsymbol{G}}_k(\omega)\boldsymbol{\delta} + \boldsymbol{e}_k(\omega) \right\|_2 \\ \text{subject to:} \quad & a_i = 0 \quad \text{for} \ \ i \in I^* \\ & \tilde{\boldsymbol{C}}\boldsymbol{\delta}_{\boldsymbol{d}} + \boldsymbol{s}_k \geq \boldsymbol{0} \\ & \left\| \boldsymbol{\delta} \right\|_2 \leq \beta \end{aligned} \qquad (5)$$

4. A Design Example

The Design Problem

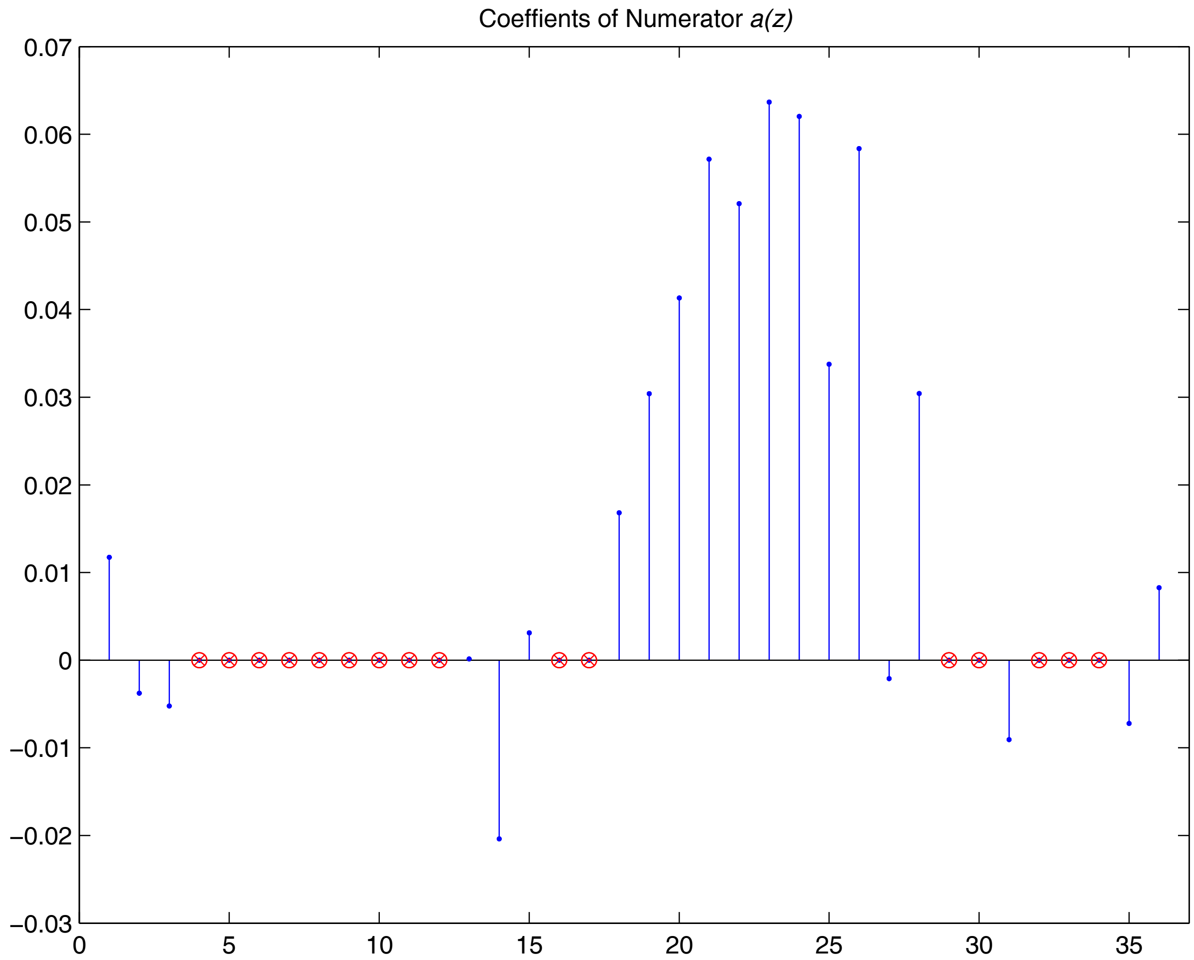
- Design a stable lowpass IIR filter of order (n = 35, r = 1) with $\omega_p = 0.2\pi$ and $\omega_a = 0.23\pi$, where the numerator $a(z)$ possesses at least 16 zeros i.e. k = 20. Passband group delay = 21.

- A total of 200 grid points are evenly placed over the passband and stopband.

- Other parameters were set to $\beta = 0.1, \mu = 0.005, \varepsilon_t = 0.004$.

- It took 20 iterations for the algorithm in design phase 1 to yield an index set I^* = {4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17, 29, 30, 32, 33, 34}

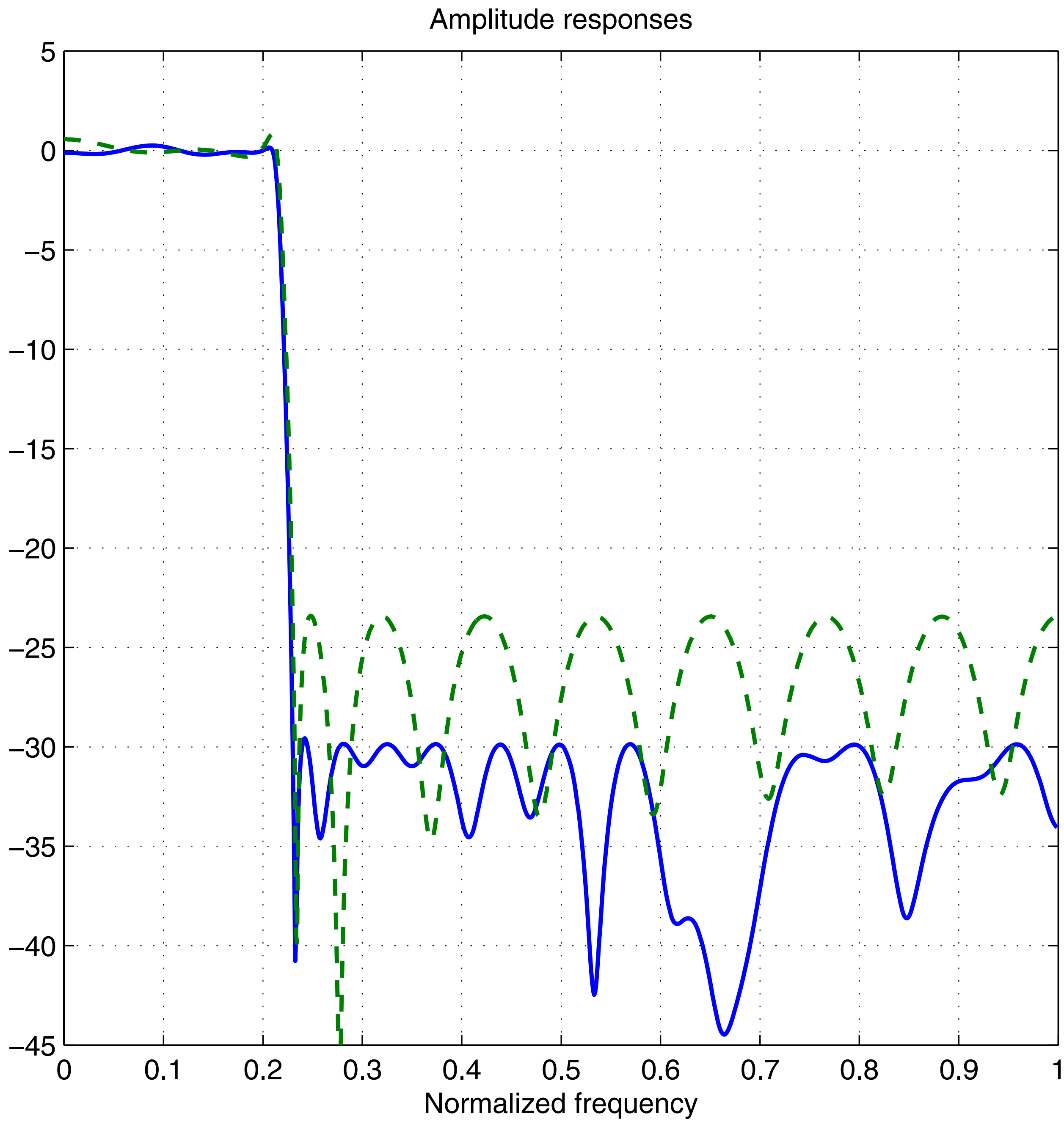
- It took another 20 iterations for the algorithm in phase 2 to converge. The denominator coefficients were found to be $d_{11} = -1.5230137262$, $d_{12} = 0.9500000026$. The coefficients of numerator $a(z)$ are shown in the figure below.



- maximum passband ripple: 0.0325
- minimum stopband attenuation: 29.7270 dB
- passband group delay 21 with maximum ripple 0.1818
- magnitude of the two poles: 0.9747
- number of zeros in numerator: 16

========= Comparison 1 ========

- Compare with an equivalent nonsparse IIR filter of order (n = 19, r = 1) with the same design specifications:
- maximum passband ripple: 0.0676
- minimum stopband attenuation: 23.4019 dB
- passband group delay 12 with maximum ripple 0.4579


Amplitude responses
5
0
−5
−10
−15
−20
−25
−30
−35
−40
−45
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized frequency

========= Comparison 2 ========

- Compared with an equiripple linear-phase FIR filter of order of length 88 that was designed using the Parks-McClellan algorithm with the same design specifications
- and comparable maximum passband ripple and
- comparable minimum stopband attenuation
- passband group delay = 43.5 (versus 21)
- 44 multiplications versus 18 multiplications per output sample required by the sparse IIR filter.

5. Concluding Remarks

- The proposed technique works for minimax design of stable IIR filters with sparse coefficients. The performance of these filters appears to be satisfactory compared with their non-sparse counterparts.
- A drawback of these filters is the longer group delay relative to their non-sparse counterparts, thus a topic of future research: sparse filters with low group delay.
- Implementation techniques for sparse filters with irregularly located zero coefficients seems worth investigation.