Variable Fractional Delay FIR Filters with Sparse Coefficients

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Outline

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1. Design Problem and Related Work

Design Problem

Find a VFD FIR digital filter of order (N, K)

$$H(z,p)=\sum_{n=0}^{N} a_n(p)z^{-n}, \quad a_n(p)=\sum_{k=0}^{K} a_{nk}p^k$$

with $0 \le p \le 1$ and sparse $\boldsymbol{A}$ = $\{a_{ij}\}$ $\in R^{(N+1)\times(K+1)}$ that optimally approximates a desired frequency response

$$H_d(\omega,p)=e^{-j\omega(D+p)}$$

in a weighted least-squares sense.

$$J(\boldsymbol{A})=\tfrac{1}{2}\int_0^{\pi}\int_0^{1} W(\omega,p)\left|H(\omega,p)-H_d(\omega,p)\right|^2 dp\,d\omega$$

$$\text{subject to:} \quad \text{sparsity}(\boldsymbol{A})=N_z$$

Related Work

- A. Tarczynski, G. D. Cain, E. Hermanowicz and M. Rojewki, 1997
- W.-S. Lu and T.-B. Deng, 1999.
- T.-B. Deng, 2001.
- C.-C. Tseng, 2004.
- T.-B. Deng and Y. Lian, 2006.
- Linear programming algorithms for sparse FIR filters (T. Baran, D. Wei, and A.V. Oppenheim, March 2010)
- l_1-minimization algorithms for sparse 1-D and 2-D FIR filters (W.-S. Lu and T. Hinamoto, 2010, 2011)

2. Significance

- Implementing a VFD filter in Farrow model is costly as each coefficient of a VFD filter is a polynomial rather than a scalar. Hence VFD digital filters with sparse coefficients are of interest because the sparsity implies reduced implementation complexity (cost), hence real-time application potential.

- VFD filters are *not* sparse in general.

3. Design Method at a Glance

Define

$$\boldsymbol{\omega} = \begin{bmatrix} 1 & e^{-j\omega} & \cdots & e^{-jN\omega} \end{bmatrix}^T \text{ and } \boldsymbol{p} = \begin{bmatrix} 1 & p & \cdots & p^K \end{bmatrix}^T$$

and assume a separable weighting function $W(\omega, p) = W_1(\omega)W_2(p)$,

then up to a constant one can write

$$J(\boldsymbol{A}) = \tfrac{1}{2}\int_0^{\pi}\int_0^{1} W(\omega, p)\left|H(\omega, p) - H_d(\omega, p)\right|^2 dp\,d\omega$$
$$= \tfrac{1}{2}\,\mathrm{tr}(\boldsymbol{P}\boldsymbol{A}^T\boldsymbol{\Omega}\boldsymbol{A}) - \mathrm{tr}(\boldsymbol{S}\boldsymbol{A})$$

where

$$\boldsymbol{P} = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{p}^T dp, \quad \boldsymbol{\Omega} = \mathrm{Re}\left[\int_0^{\pi} W_1(\omega)\overline{\boldsymbol{\omega}}\boldsymbol{\omega}^T d\omega\right]$$

$$\boldsymbol{S} = \int_0^1 W_2(p)\boldsymbol{p}\boldsymbol{\omega}_p^T dp, \quad \boldsymbol{\omega}_p^T = \mathrm{Re}\left[\int_0^{\pi} W_1(\omega)\boldsymbol{\omega}^T e^{j\omega(D+p)} d\omega\right]$$

To deal with large condition numbers of $\boldsymbol{P}$ and $\boldsymbol{\Omega}$, Cholesky decompositions $\boldsymbol{P} = \boldsymbol{P}_1^T \boldsymbol{P}_1$, $\boldsymbol{\Omega} = \boldsymbol{\Omega}_1^T \boldsymbol{\Omega}_1$ are used, and up to a constant $J(\boldsymbol{A})$ can be written as

$$J(\boldsymbol{a}) = \tfrac{1}{2} \left\| \boldsymbol{\Gamma} \boldsymbol{a} - \boldsymbol{y} \right\|_2^2, \; \boldsymbol{\Gamma} = \boldsymbol{P}_1 \otimes \boldsymbol{\Omega}_1, \; \boldsymbol{y} = \boldsymbol{\Gamma}^{-T} \boldsymbol{s}$$

where $\boldsymbol{a}$ and $\boldsymbol{s}$ are the vectors generated by concatenating the columns of $\boldsymbol{A}$ and $\boldsymbol{S}$, and $\otimes$ is the Kronecker product.

Design Phase 1

- To identify an index set of the most appropriate locations in $\boldsymbol{a}$ to be set to zero in order to satisfy a target sparsity.
- This is done subject to maintaining closeness of $H(e^{j\omega}, p)$ to $H_d(\omega, p)$.

- The target sparsity is achieved by introducing sparsity promoting l_1-norm of $\boldsymbol{a}$ into an objective function:

$$\underset{\boldsymbol{a}}{\text{minimize}} \ \mu\|\boldsymbol{a}\|_1 + \tfrac{1}{2}\|\Gamma\boldsymbol{a} - \boldsymbol{y}\|_2^2 \qquad (1)$$

where $\|\boldsymbol{a}\|_1 = \sum_{i=0}^{n}|a_i|$. Problem (1) is a convex, for which many fast algorithms are available. E.g. using FISTA, (1) can be solved with a small number of iterations.

Input: Data $\boldsymbol{\Gamma}$ and $\boldsymbol{y}$, parameter μ and iteration number M.

Step 1. Compute $\boldsymbol{A}_0 = \boldsymbol{\Gamma}^{-1}\boldsymbol{y}$, $\boldsymbol{a}_0 = \boldsymbol{A}_0(:)$. Set $\boldsymbol{b}_1 = \boldsymbol{a}_0$, $t_1 = 1$, and $m = 1$.

Step 2. Compute $\boldsymbol{a}_m = S_{\mu/L}\{\boldsymbol{b}_m - (1/L)\boldsymbol{\Gamma}^T(\boldsymbol{\Gamma}\boldsymbol{b}_m - \boldsymbol{y})\}$,
where $S_\alpha(u) = \operatorname{sgn}(u)\cdot\max\{|u|-\alpha, 0\}$

Step 3. Update $t_{m+1} = (1+\sqrt{1+4t_m^2})/2$

Step 4. Update $\boldsymbol{b}_{m+1} = \boldsymbol{a}_m + ((t_m - 1)/t_{m+1})(\boldsymbol{a}_m - \boldsymbol{a}_{m-1})$.

Step 5. If $m < M$, set $m = m + 1$ and repeat from Step 2; otherwise stop and output $\boldsymbol{a}_m$ as solution $\hat{\boldsymbol{a}}$.

• Hard thresholding is applied to vector $\hat{\boldsymbol{a}}$ with an appropriate value of threshold ε^* so that the length of the index set $I^* = \{i, |\hat{a}(i)| < \varepsilon^*\}$ equals to target sparsity N_z. The index set I^* is a key ingredient of design phase 2.

Design Phase 2

• To find a coefficient matrix $\boldsymbol{A}$ that minimizes the WLS error $J(\boldsymbol{A})$ subject to the sparsity constraint. This part of the design is carried out by solving the convex problem

$$\underset{\boldsymbol{a}}{\text{minimize}} \; J(\boldsymbol{a}) = \tfrac{1}{2}\left\|\Gamma \boldsymbol{a} - \boldsymbol{y}\right\|_2^2$$
$$\text{subject to: } a(i) = 0 \text{ for } i \in I^*$$

• By simply substituting the constraints into the objective function, the above problem becomes an unconstrained least square problem whose solution is given by

$$\boldsymbol{a}^*(\bar{I}^*) = \boldsymbol{\Gamma}_s^{-1}\boldsymbol{y} \quad \text{and} \quad \boldsymbol{a}^*(I^*) = \boldsymbol{0}$$

where $\bar{I}^*$ denotes the set of index not in I^* and Γ_s is composed of those columns of Γ with indices in set $\bar{I}^*$.

4. A Design Example

The Design Problem

- Design a VFD FIR filter of order (N = 65 and K = 7) with cutoff $\omega_c = 0.9\pi$.
- Design performance is evaluated in terms of maximum error

$$e_{\max} = \max\left\{e(\omega, p), 0 \le \omega \le 0.9\pi, 0 \le p \le 1\right\}$$

with

$$e(\omega, p) = 20\log_{10}\left|H(\omega, p) - H_d(\omega, p)\right|$$

and L2-error

$$e_2 = \left[\int_0^{0.9\pi} \int_0^1 \left| H(\omega, p) - H_d(\omega, p) \right|^2 dp d\omega \right]^{1/2}$$

- The weighting function was set to $W(\omega, p) = W_1(\omega) W_2(p)$ with $W_2(p) = 1$ for p in $[0, 1]$ and

$$W_1(\omega) = \begin{cases} 1 & \text{for } \omega \in [0, 0.88\pi) \\ 3 & \text{for } \omega \in [0.88\pi, 0.8994\pi) \\ 0 & \text{for } \omega \in [0.88\pi, \pi] \end{cases}$$

- The target sparsity was set to N_z = 198 which means a 37.5% of coefficients were set to zero. To achieve this sparsity, the two key parameters in phase-1 were set to $\mu = 10^{-5}, \varepsilon^* = 10^{-3}$. It took 60 FISTA iterations for the algorithm in phase 1 to converge.

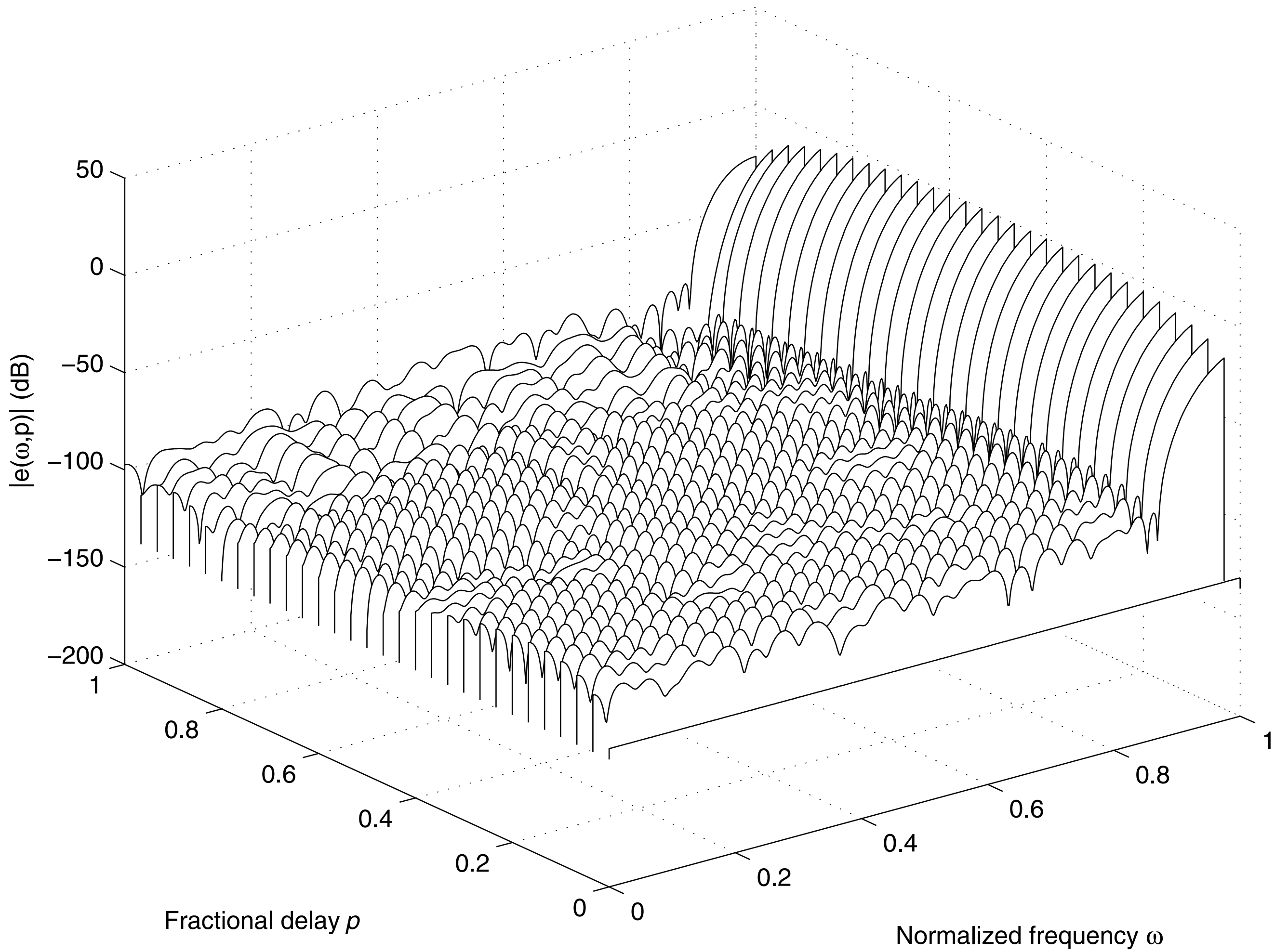
- Phase 2 of the design then produced an optimal $\boldsymbol{A^*}$ with sparsity($\boldsymbol{A^*}$) = 198. Below are the numerical evaluation results:
- maximum error $e_{max} = 0.0021$.
- L2-error $e_2 = -75.25$ dB.

========= Comparison 1 ========

- Compare with an equivalent nonsparse VFD filter of order (N = 65, K = 4) with the same specifications:
- maximum error $e_{max} = 0.0609$.
- L2-error $e_2 = -45.28$ dB.

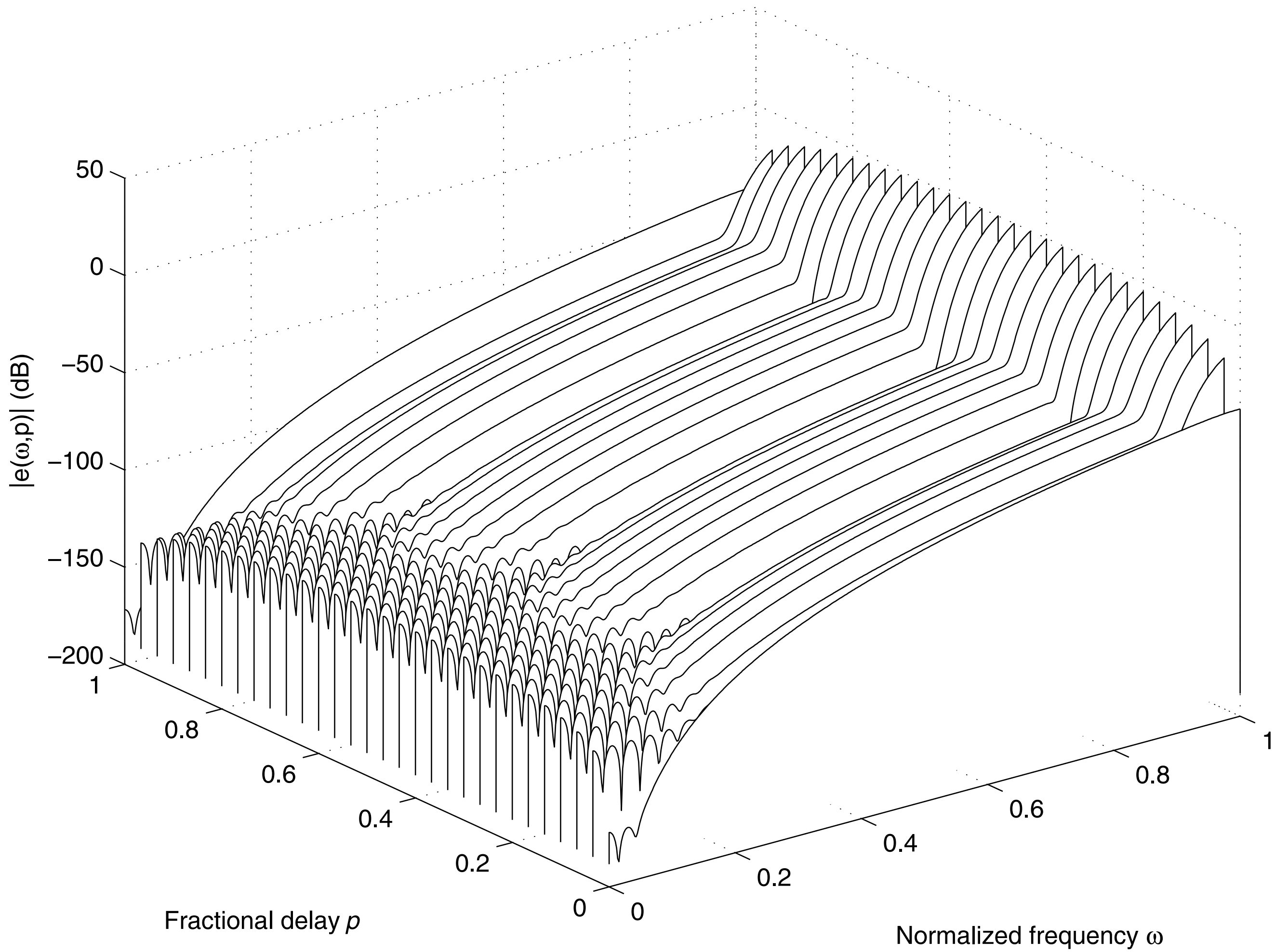
- Profile of frequency response error

  $|e(\omega, p)|$ over $0 \le \omega \le 0.9\pi, 0 \le p \le 1$: the sparse VFD filter:



- Profile of frequency response error

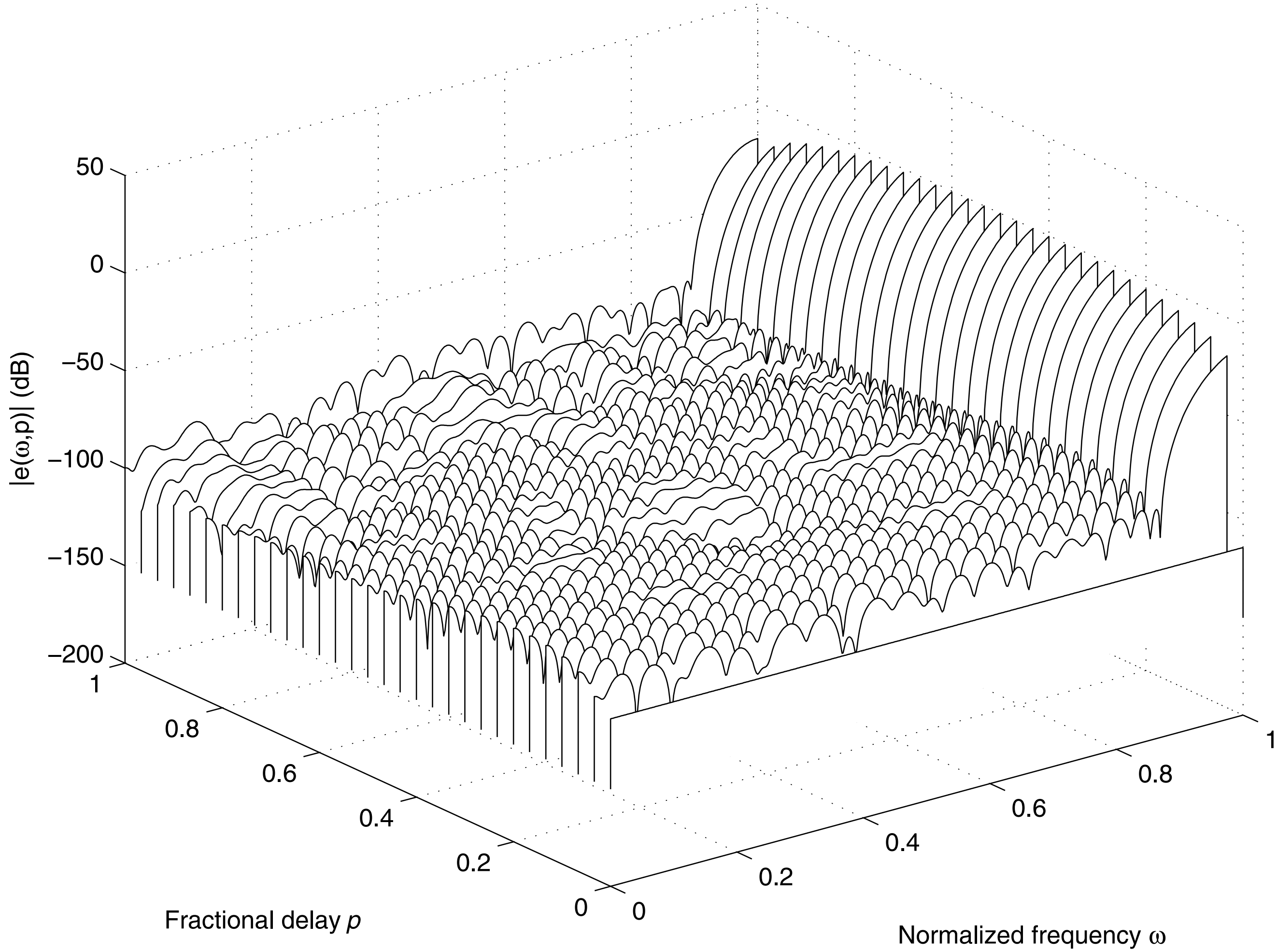
$|e(\omega, p)|$ over $0 \le \omega \le 0.9\pi, 0 \le p \le 1$: the nonsparse VFD filter:



========= Comparison 2 ========

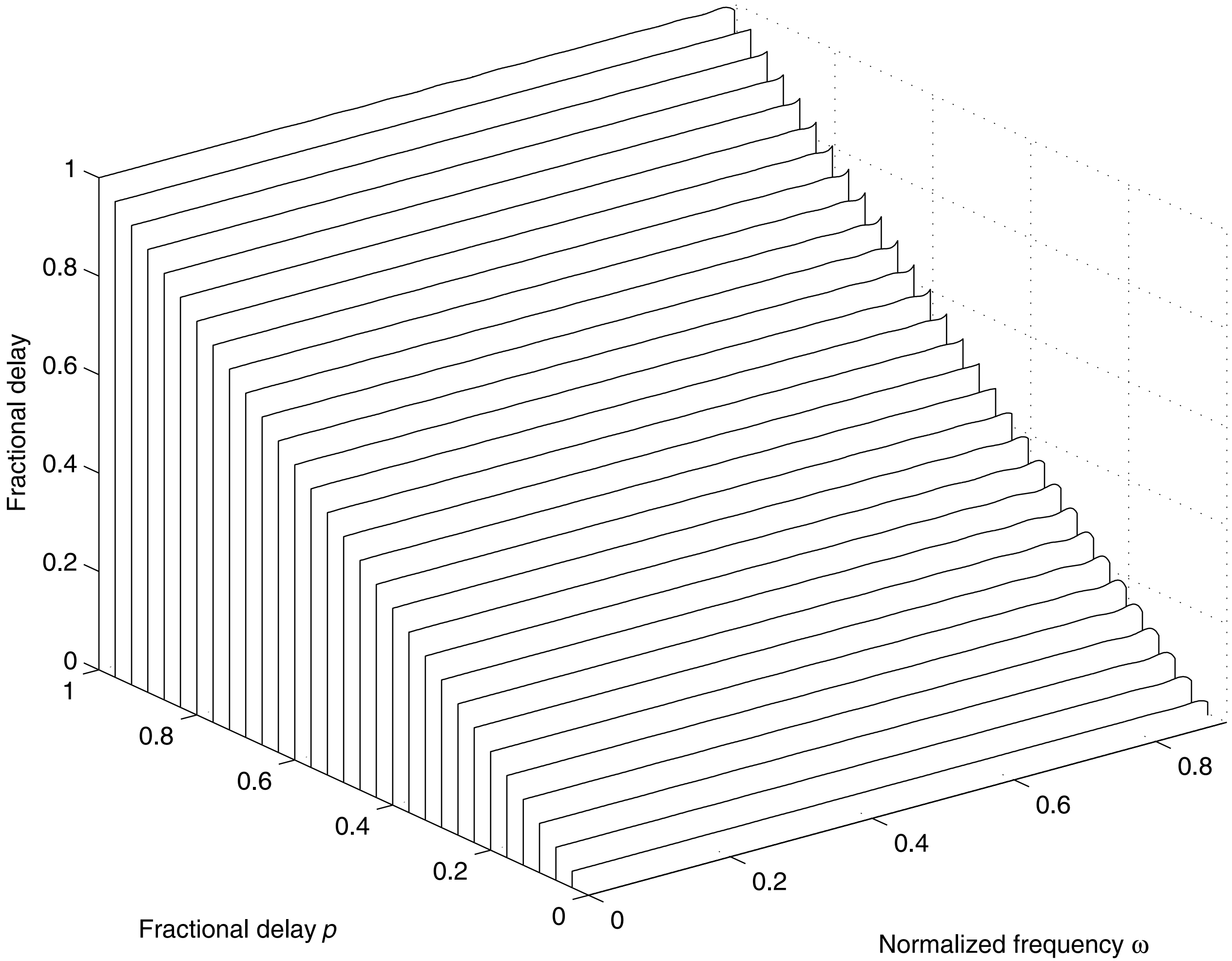
- To justify phase 1 of the design, we compare the above result with the following: We design a conversional (nonsparse) VFD filter of order (N = 65, K = 7), then applied hard thresholding to generate exactly 198 locations that may be considered appropriate to set to zero. We then went on the carried out phase 2 to yield a sparse VFD filter. It was found that
- maximum error $e_{max} = 0.0025$ (vs 0.0021 with phase 1)
- L2-error $e_2 = -73.41$ dB (vs -75.25 dB with phase 1).

- Profile of frequency response error $|e(\omega, p)|$ over $0 \le \omega \le 0.9\pi, 0 \le p \le 1$: the sparse VFD filter without phase 1:

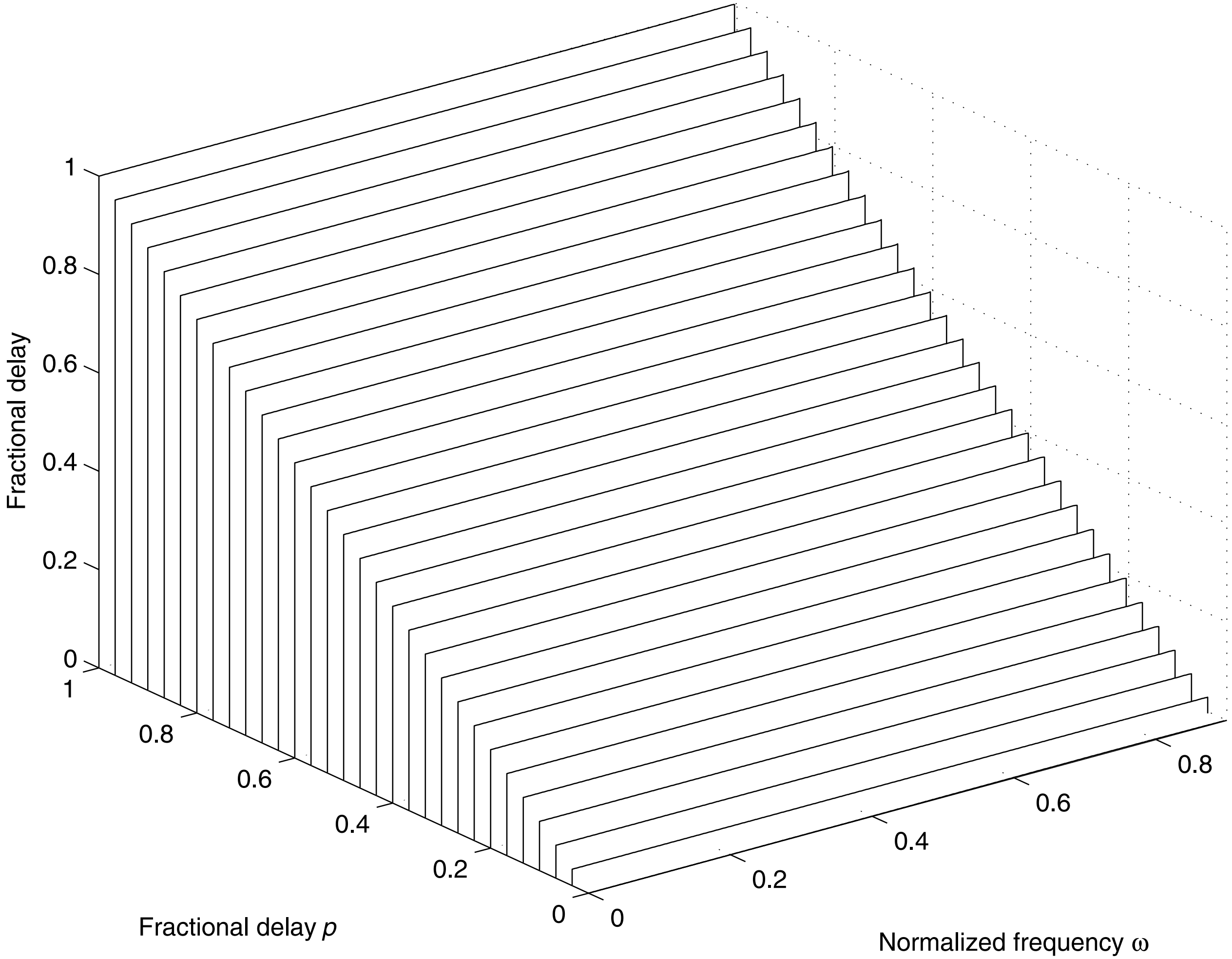


========= Comparison 3 ========

- Fractional delay over $0 \le \omega \le 0.9\pi, 0 \le p \le 1$. The sparse filter:



- Fractional delay over $0 \le \omega \le 0.9\pi, 0 \le p \le 1$. The equivalent nonsparse filter:



5. Concluding Remarks

- A two-phase technique for the WLS design of VFD FIR filters subject to a target coefficient sparsity constraint has been proposed.
- The design algorithm is easy to implement abd computationally efficient because it is based on $l_1 - l_2$ convex optimization.
- The performance of the filter appears to be satisfactory compared with its nonsparse counterpart.