New Algorithms for Minimax Design of Sparse IIR Filters

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Outline

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- Significance
- Design Method at a Glance
- A Design Example

1. Design Problem and Related Work

Design Problem

Find a stable IIR transfer function $H(z)$ of order (n, r)

$$H(z) = \frac{a(z)}{b(z)} = \frac{\sum_{k=0}^{n} a_k z^{-k}}{\sum_{i=0}^{r} b_i z^{-i}} \quad \text{with } b_0 = 1$$

that optimally approximates a desired frequency response $H_d(\omega)$ in a minimax sense subject to a coefficient sparsity constraint:

$$\begin{aligned} &\underset{\boldsymbol{a},\boldsymbol{b}}{\text{minimize}} \ \max_{\omega\in\Omega} |H(e^{j\omega}) - H_d(\omega)| \\ &\text{subject to: number of zeros in } \boldsymbol{a} \ge K \qquad \text{(1a-c)} \\ &\qquad H(z) \text{ is stable} \end{aligned}$$

where $\boldsymbol{a} = [a_0 \quad a_1 \quad \cdots \quad a_n]^T$ and $\boldsymbol{b} = [b_1 \quad b_2 \quad \cdots \quad b_r]^T$。

Related Work

- J.T. Kim, W. J. Oh, and Y.H. Lee, 1996.
- J. Webb and D.C. Munson, Jr., 1996.
- D. Mattera, F. Palmieri, and S. Haykin, 2002.
- R. NiemistÖ and B. Dumitrescu, 2004.
- O. Gustafsson, L.S.De Brunner, V.De Brunner, and H. Johansson, 2007.
- D. Wei, 2009.
- T. Baran, D. Wei, and A.V. Oppenheim, 2010.
- C.-C. Tseng and S.-L. Lee, 2011.
- C. Rusu and B. Dumitrescu, 2012.
- A. Jiang and H.K. Kwan, 2012.
- W.-S. Lu and T. Hinamoto, 2010, 2011, 2012.

2. Significance

- IIR digital filters are *not* sparse in general.
- Implementing an IIR filter can be costly, especially when the filter's order is high.
- Implementing an IIR filter of low order can be made more efficient if the filter in question is sparse.

3. Design Method at a Glance

A. Linear Representation of $H(e^{j\omega})$

We begin by designing a conventional stable IIR filter $H(z) = a(z)/b(z)$, then write $H(e^{j\omega})$ as

$$H(e^{j\omega}) = \sum_{k=0}^{n} a_k w_k(\omega) \tag{2}$$

where

$$w_k(\omega) = u_k(\omega) - jv_k(\omega)$$

$$u_k(\omega) = \frac{b_r(\omega)\cos k\omega + b_i(\omega)\sin k\omega}{p(\omega)}$$

$$v_k(\omega) = \frac{b_r(\omega)\sin k\omega - b_i(\omega)\cos k\omega}{p(\omega)}$$

$$p(\omega) = b_r^2(\omega) + b_i^2(\omega)$$

$$b_r(\omega) = \sum_{i=0}^{r} b_i \cos i\omega, \quad b_i(\omega) = \sum_{i=0}^{r} b_i \sin i\omega$$

This “linear” representation of frequency response $H(e^{j\omega})$ lies a foundation for the development of the proposed design method

- Our design practice with the new algorithm has indicated that the poles of the sparse filter designed (with both the coefficients of $a(z)$ and $b(z)$ being treated as design variables) practically do not move when compared with the poles obtained from a corresponding non-sparse design, especially when the order of $b(z)$ is not high. Note that this phenomenon coincides with an observation made in [16], although [16] does not deal with sparse filters.

The proposed design is accomplished in three phases that are described as follows.

- ***Phase 1 — Identifying Indices of Zero Coefficients***

Phase 1 identifies an index set I^* with length $|I^*| \geq K$ for coefficients $\{a_k\}$ that are most appropriate to be set to zero, namely the a_k's whose nullification leads to least performance degradation. To this end, we seek sparse $\{a_k\}$ subject to sufficient closeness between $H(e^{j\omega})$ and $H_d(\omega)$ across the entire region Ω.

- Note that stability is not an issue here because a stable and fixed $b(z)$ has been used in constructing the basis functions $\{w_k(\omega)\}$.

The index identification problem at hand is formulated as

$$\underset{\boldsymbol{a}}{\text{minimize}} \ \|\boldsymbol{a}\|_1 \tag{3a}$$

$$\text{subject to: } \left|\sum_{k=0}^{n} a_k w_k(\omega) - H_d(\omega)\right| \leq \delta \ \text{ for } \ \omega \in \Omega \tag{3b}$$

where $\|\boldsymbol{a}\|_1 = \sum_{i=0}^{n} |a_i|$ is the l_1-norm whose minimization promotes the sparsithy of coefficient vector $\boldsymbol{a}$, and δ is a constant to be specified.

- Problem (3) is convex, actually it is a second-order cone programming (SOCP) problem which can be solved efficiently.
- The solution of problem (3) is sparse in general, as long as the feasible region defined by (3b) is nonempty. The question is whether or not the sparsity constraint (1b) is met by the solution of (3). It is in this regard the error bound δ in (3b) plays a role: a greater δ defines a larger feasible region which includes an increasing number of sparser solutions. Therefore, the designer may use δ as a means to control the degree of sparsity in coefficients $\{a_k\}$ so as to satisfy (1b).

- In addition to using δ to control solution's sparsity, hard-thresholding with an appropriate threshold value may be applied to the solution of (3) for fine tuning the exact number of zero coefficients in $\{a_k\}$.
- In summary, design phase 1 produces an index set $I^* \subset \{0, 1, \ldots, n\}$ with $|I^*| \geq K$ for nullifying the coefficients $\{a_k, k \in I^*\}$.
- ***Phase 2 — Optimum Design of Sparse IIR Filters***

With index I^* identified, we now re-visit problem (1) which can now be expressed more specifically as

$$\begin{aligned} &\underset{\boldsymbol{a},\boldsymbol{b}}{\text{minimize}} \ \max_{\omega\in\Omega} |H(e^{j\omega}) - H_d(\omega)| \\ &\text{subject to:} \ \ a_k = 0 \ \text{ for } \ k \in I^* \\ &\qquad H(z) \text{ is stable} \end{aligned} \tag{4a-c}$$

By substituting (4b) into $H(e^{j\omega})$ in (4a), (4b) is eliminated and problem (4) is reduced to a standard formulation of *non-sparse* minimax design of stable IIR filters, for which many solution methods are available, see e.g. Ch. 16 of [17].

- ***Phase 3 — Performance Enhancement***

The design may be considered complete as long as both phases 1 and 2 are done. However there is a simple “follow-up” step that generates a filter having more sparse coefficients without considerably degrading approximation accuracy.

Let us take a look at the linear representation of $H(e^{j\omega})$ given by (2) in connection with the design steps in phases 1 and 2: Phase 1 essentially identifies a low-dimensional subspace spanned by basis functions $\{w_k(\omega), k \in I^*\}$ and claims that the design of a stable IIR filter with satisfactory performance can be carried out in that low-dimensional subspace; and Phase 2 actually carries out the design that yields a sparse IIR filter. From this perspective, the objective in this stage of the design is to identify a subspace of even lower dimension in which the final design shall be performed. A natural way to further reduce the dimension of the working subspace is to examine the design result from phase 2, namely the optimized coefficient vector $\boldsymbol{a}$, and approximate it with a sparser vector $\boldsymbol{a}^*$ so as to increase the sparsity of the IIR filter. Evidently, additional steps need to be done to ensure the filter's optimality within the reduced subspace. Specifically, design phase 3 consists of two steps as follows.

1) Apply hard-thresholding with an appropriate threshold to vector $\boldsymbol{a}$ obtained from phase 2 to generate more zero coefficients. Bear in mind there is always a tradeoff between the degree of sparsity and performance of the filter in terms of approximation accuracy in both magnitude and phase responses. The critical result of this step is an augmented index set I^{**} that contains set I^{*} (obtained from phase 1) as a subset.

2) Redo optimization (4) with I^{*} in (4b) replaced by I^{**}.

4. A Design Example

Below the proposed algorithm with an example is illustrated by designing a lowpass IIR filter of order $(n, r) = (26, 2)$ with a sparsity lower bound $K = 6$ for coefficients $\{a_k\}$. The desired frequency response is given in terms of normalized passband edge $\omega_p = 0.4\pi$, stopband edge $\omega_a = 0.45\pi$ and passband group delay 16. The largest magnitude of poles is required to be less than 0.95.

To prepare basis functions $\{w_k(\omega)\}$, a nonsparse IIR filter of order $(26, 2)$ with the same design specifications was designed using the method in Ch. 16 of [17]. The denominator obtained is given by

$$b(z) = 1 - 0.44115170\, z^{-1} + 0.9\, z^{-2}$$

which was used to construct $\{w_k(\omega)\}$ using (2). A total 200 grid points evenly placed over $[0, \omega_p] \cup [\omega_a, \pi]$ were used to form set Ω. With $\delta = 0.05$, SeDuMi was used to solve problem (3). The solution vector $\boldsymbol{a}$ contains six coefficients whose magnitudes are less than 10^{-8}. The indices of these practically zero coefficients are identified to form the index set $I^* = \{2, 4, 9, 11, 23, 25\}$. We stress that in case the number of (practically) zero coefficients does not meet the required sparsity bound K, constant δ in (3b) may be adjusted to expand the feasible region so as to include more sparse solutions. With I^* identified, phase 2 was performed to design a sparse IIR filter with $K = 6$. The coefficients of numerator $a(z)$ are shown in the left column in Table 1, the denominator was found to be

$$\hat{b}(z) = 1 - 0.43970895\, z^{-1} + 0.9\, z^{-2}$$

To carry out phase 3, a hard-thresholding with threshold 0.0035

was applied to the coefficients $\{a_k\}$ obtained from phase 2 that yielded two more zeros as seen in the augmented index set $I^{**} = \{2, 4, 6, 7, 9, 11, 23, 25\}$. This index set was used in the second step in phase 3 to solve problem (4) where I^* was replaced by I^{**}. The optimized $a^*(z)$ with 8 zero coefficients is shown in the right column in Table 1, and the optimized denominator is given by

$$b^*(z) = 1 - 0.43800965\, z^{-1} + 0.902\, z^{-2}$$

The magnitude of the two poles of the optimized $H^*(z) = a^*(z)/b^*(z)$ was found to be 0.9497, hence the filter is stable. The maximum passband ripple, minimum stopband attenuation, and relative maximum ripple in passband group delay were found to be 0.0255, 31.7625 dB, and 0.1416, respectively. The performance of the IIR filter obtained from phase 2 was quite comparable with that of

$H^*(z)$, hence $H^*(z)$ is considered a favorable design as it is more sparse. The amplitude response of sparse IIR filter $H^*(z)$ (solid line) is depicted in Fig. 1.

- For comparison, an equivalent nonspase lowpass IIR filter of order $(n = 18, r = 2)$ with the same design specifications (except the passband group delay which was set to 13 for best performance) was designed using a well established method described in Ch. 16 of [17]. The reason to compare $H^*(z)$ with a nonsparse IIR filter of order $(18, 2)$ is because they have same number of nonzero coefficients . The maximum passband ripple, minimum stopband attenuation, and relative maximum ripple in passband group delay of the non sparse IIR filter were found to be 0.0444, 27.0292 dB, and 0.2557. The largest magnitude of the poles was 0.9466. The amplitude response of the equivalent non-sparse IIR filter (dashed line) is shown in Fig. 1.

- It is interesting to note that the denominator $b(z)$ produced by a non-sparse design for constructing basis functions is practically the same as the denominators obtained in phases 2 and 3. As a matter of fact, the relative difference in poles between $b(z)$ and $\hat{b}(z)$ and between $b(z)$ and $b^*(z)$ was only 7.8×10^{-4} and 2.2×10^{-3}, respectively. This justifies the linear representation of $H(e^{j\omega})$ with a fixed $b(z)$ as the foundation of the new design algorithm.

- Comparison of the sparse IIR filter with a minimax linear phase FIR was also made. With $\omega_p = 0.4\pi$ and $\omega_a = 0.45\pi$, a 59-tap equiripple FIR filter was designed using the Parks-McClellan algorithm to achieve an amplitude response comparable to that of $H^*(z)$ designed above. The phase response of the FIR filter is perfectly linear, but its group delay is 29 compared with 16 offered by the sparse IIR filter. Moreover, with 59 taps the FIR filter requires more multiplications and additions per output relative to the sparse IIR filter which has only 18 nonzero coefficients in $a^*(z)$ and 2 non-unity coefficients in $b^*(z)$. From the numerical evidence given above, we see that sparse IIR filters have the potential to offer good filtering performance as well as implementation efficiency.

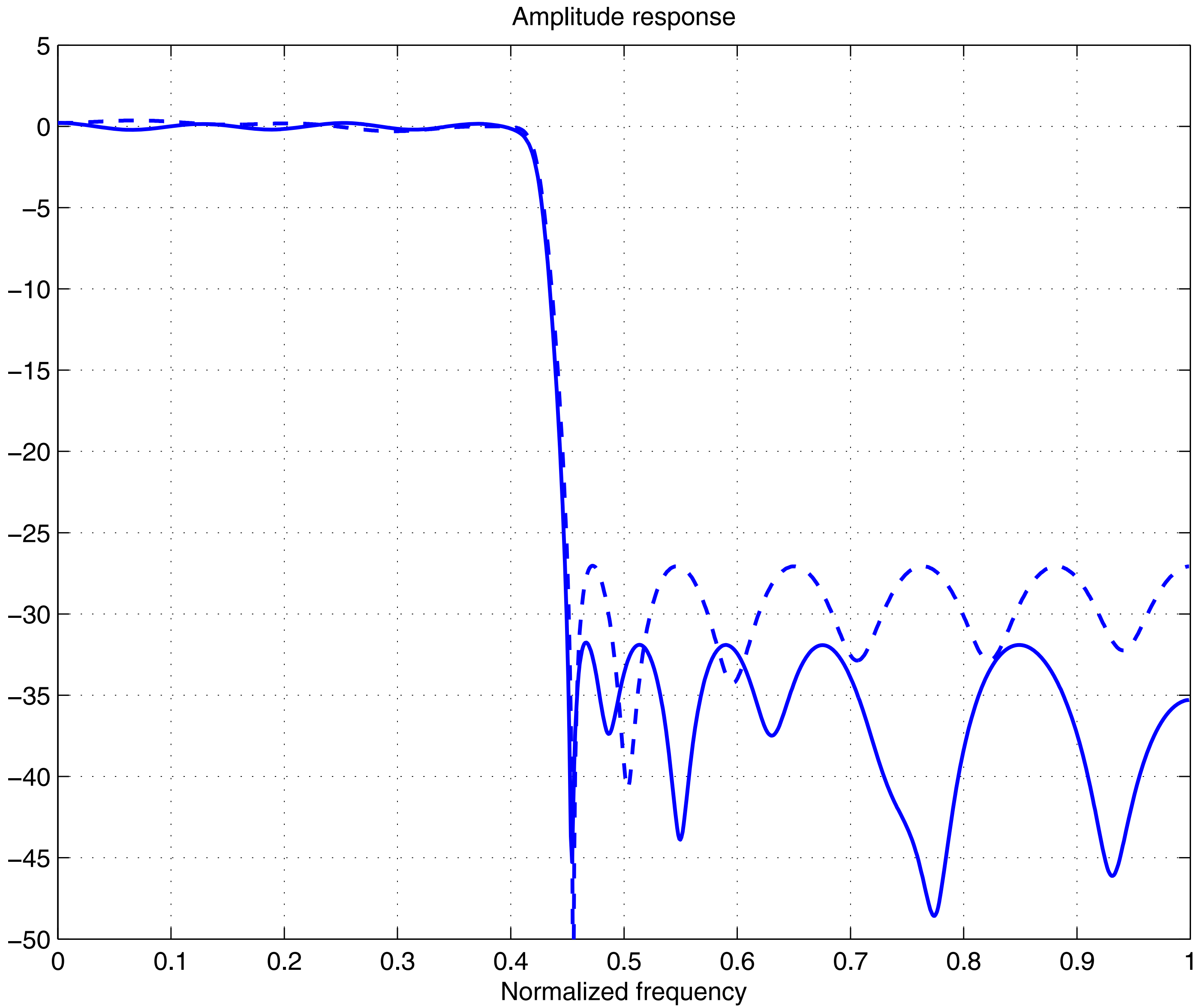

Fig. 1. Amplitude responses of the sparse IIR filter (solid line) and an equivalent nonsparse IIR filter (dashed line).

TABLE I

COEFFICIENTS of $a(z)$ from PHASE 2 (LEFT COLUMN) and PHASE 3 (RIGHT COLUMN)

0.010392851102970	0.010036395187765
0.017116960822615	0.015646826495216
0	0
0.005030800830541	0.003323342301283
0	0
0.007417943302547	0.008114566505141
-0.000001867894394	0
-0.003408484954507	0
-0.007738790632433	-0.006517428535895
0	0
0.020340709998230	0.023141230949229
0	0
-0.023613854872256	-0.027680623773149
-0.033053080229689	-0.027024000127553
0.052507098113062	0.042556154643593
0.200704762712400	0.207983936192666
0.357458937038486	0.349500928736363
0.400358031290358	0.407323106379040
0.316804799636574	0.315491540932951
0.171495341811609	0.171347025547514
0.027590628329020	0.032645254558257
-0.012835200844937	-0.016448691169804
-0.017963977929487	-0.014784668904078
0	0
0.010935099950532	0.010963543735549
0	0
-0.004853370866795	-0.005343831411609