Design of Composite Filters with Equiripple Passbands and Least-Squares Stopbands

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Outline

- Early Work on Composite Filters (C-Filters)
- Composite Filters and Problem Formulation
- Design Method
- Design Example and Comparisons

1. Early Work on Composite Filters

- Interpolated Filters (Neuvo, Dong, Mitra 1984; Saramäki, Neuvo, Mitra 1988).
- Frequency-Response-Masking Filters (Lim 1986).
- C-filters composed of a prototype filter and a shaping filter in cascade (Shiung, Yang, Yang 2016).

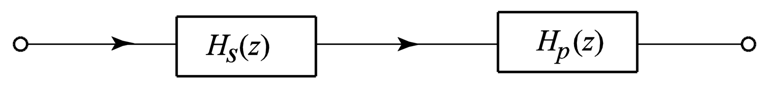
2. Composite Filters and Problem Formulation

- The linear-phase C-filter we study assumes the form

$$H(z) = H_p(z) \cdot H_s(z)$$

where

$$H_p(z) = \sum_{n=0}^{N-1} h_n z^{-n} \qquad H_s(z) = \prod_{l=1}^{L} (1+z^{-1})^{k_l}$$



- The frequency response of a C-filter is given by

$$H(\omega) = H_p(\omega) \cdot H_s(\omega)$$

$$H_p(\omega) = \sum_{n=0}^{N-1} h_n e^{-jn\omega} = e^{-j\tau_p\omega} \,|\, \boldsymbol{x}^T \boldsymbol{c}(\omega) \,|$$

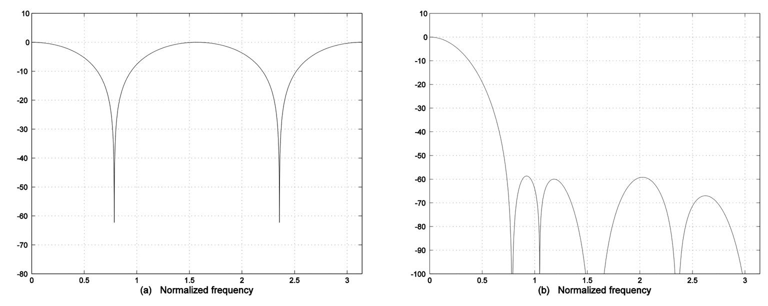
$$H_s(\omega) = \prod_{l=1}^{L} (1+e^{-jl\omega})^{k_l} = e^{-j\tau_s\omega} \prod_{l=1}^{L} \left(2\cos\frac{l\omega}{2}\right)^{k_l}$$

$$A(\omega) = A_p(\omega) \cdot A_s(\omega) = \boldsymbol{x}^T \boldsymbol{c}(\omega) \prod_{l=1}^{L} \left(2\cos\frac{l\omega}{2}\right)^{k_l}$$

- The shaping filters employed in this paper are complementary comb filters (CCFs) of the form

$$H_s(z) = \prod_{l=1}^{L} (1+z^{-l})^{k_l}$$

Cascading several CCFs with appropriate powers, a shaping filter can offer much reduced stopband energy. The figure on the right-hand side below illustrates the case $\{k_l\}$ = {4, 4, 1, 2}.



- Given a desired lowpass frequency response $H_d(\omega)$, prototype filter length N, and number of CCFs L, we seek to find a linear-phase FIR C-filter such that the peak-to-peak amplitude ripple in passband is minimized subject to constraints on filter's energy as well we peak gain in stopband and total group delay:

$$\underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} \quad \max_{\omega\in\Omega_p} |H(\omega) - H_d(\omega)| \qquad (7a)$$

$$\text{subject to:} \quad \int_{\omega_a}^{\pi} |H(\omega)|^2 \, d\omega \le e_a \qquad (7b)$$

$$\max_{\omega\in\Omega_a} |H(\omega)| \le \delta_a \qquad (7c)$$

$$\frac{1}{2}\sum_{l=1}^{L} l \cdot k_l \le D \qquad (7d)$$

3. Design Method

A. *Design Strategy*

Let $\boldsymbol{x}$ be the vector of the coefficients of the prototype filter, and $\boldsymbol{y} = [k_1 \;\; k_2 \;\; \cdots \;\; k_L]^T$. Note that

(a) In frequency response $A(\omega)$, design variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are separate from each other;

(b) $A(\omega)$ depends on $\boldsymbol{x}$ linearly, but on $\boldsymbol{y}$ nonlinearly.

It is therefore intuitively natural to optimize these design variables separately in an alternate fashion. This leads to a *sequential* design procedure where in each step one of the variables, say $\boldsymbol{x}$, is optimized while the other variable, $\boldsymbol{y}$, is held fixed, and the solution so produced, $\boldsymbol{x}_k$, is held fixed in the next step when variable $\boldsymbol{y}$ is updated to $\boldsymbol{y}_{k+1}$. The procedure continues until a stopping criterion is met.

B. *Solving (7) with y fixed to y = $\boldsymbol{y}_k$*

With a fixed $\boldsymbol{y}$ that satisfies (7d), (7d) can be neglected. By introducing an upper bound δ_p for the objective function, the problem becomes

$$\text{minimize} \quad \delta_p$$

$$\text{s.t.} \quad |H(\omega) - H_d(\omega)| \le \delta_p \quad \text{for} \quad \omega\in\Omega_p$$

$$\int_{\omega_a}^{\pi} |H(\omega)|^2 \, d\omega \le e_a$$

$$|H(\omega)| \le \delta_a \quad \text{for} \quad \omega\in\Omega_a$$

- If the desired frequency response assumes the form $H_d(\omega) = e^{-j\tau\omega} A_d(\omega)$ and

$$\tilde{\boldsymbol{c}}(\omega) = \boldsymbol{c}(\omega) \prod_{l=1}^{L} \left(2\cos\frac{l\omega}{2}\right)^{k_l}, \quad \boldsymbol{Q} = \int_{\omega_a}^{\pi} \tilde{\boldsymbol{c}}(\omega) \tilde{\boldsymbol{c}}^T(\omega) d\omega$$

then the problem can be formulated as

$$\text{minimize} \quad \delta_p \qquad (10a)$$

$$\text{s.t.} \quad |\boldsymbol{x}^T \tilde{\boldsymbol{c}}(\omega) - A_d(\omega)| \le \delta_p \quad \text{for} \quad \omega\in\tilde{\Omega}_p \qquad (10b)$$

$$\boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} \le e_a \qquad (10c)$$

$$|\boldsymbol{x}^T \tilde{\boldsymbol{c}}(\omega)| \le \delta_a \quad \text{for} \quad \omega\in\tilde{\Omega}_a \qquad (10d)$$

Note that

(a) the objective function is linear;

(b) constraints (10b) and (10d) are linear;

(c) constraint (10c) is convex because $\boldsymbol{Q}$ is P.D.

(d) as a result, (10) is a convex problem that admits a global solution which we denote by $(\delta_k, \boldsymbol{x}_k)$.

C. *Updating y with x fixed to x = $\boldsymbol{x}_k$*

Consider the stopband energy given by

$$J(y) = \int_{\omega_a}^{\pi} |H(\omega)|^2 \, d\omega = \int_{\omega_a}^{\pi} |\boldsymbol{x}^T \boldsymbol{c}(\omega)|^2 \prod_{l=1}^{L} \left(4\cos^2\frac{l\omega}{2}\right)^{k_l} d\omega$$

where $\boldsymbol{x} = \boldsymbol{x}_k$ is obtained by solving (10) and is fixed throughout this part of the algorithm.

- Our strategy to update y is via minimizing $J(\boldsymbol{y})$ with respect to $\boldsymbol{y}$ subject to several relevant constraints on $\boldsymbol{y}$.
- A technical difficulty to use continuous optimization to optimize $\boldsymbol{y}$ is that the components of $\boldsymbol{y}$ are constrained to be integers.
- This problem is handled by extending these components from nonnegative integers to nonnegative reals. Then an optimizing y with nonnegative integer components is obtained by rounding.

- The gradient and Hessian of $J(\boldsymbol{y})$ are evaluated in closed-form as

$$\frac{\partial J(y)}{\partial k_i} = \int_{\omega_a}^{\pi} |H(\omega)|^2 \cdot \log\left(4\cos^2\frac{i\omega}{2}\right) d\omega$$

$$\frac{\partial^2 J(y)}{\partial k_i \partial k_j} = \int_{\omega_a}^{\pi} |H(\omega)|^2 \cdot \log\left(4\cos^2\frac{i\omega}{2}\right) \log\left(4\cos^2\frac{j\omega}{2}\right) d\omega$$

From which the Hessian of $J(\boldsymbol{y})$ is shown to be P.S.D. because for an arbitrary vector $\boldsymbol{v}$ of length L we have

$$\boldsymbol{v}^T \nabla^2 J(y) \boldsymbol{v} = \int_{\omega_a}^{\pi} |H(\omega)|^2 \left(\sum_{i=1}^{L} v_i \log\left(4\cos^2\frac{i\omega}{2}\right)\right)^2 d\omega \ge 0$$

- Above analysis motivates a convex quadratic approximation of $J(\boldsymbol{y})$ as

$$\tilde{J}(y, y_k) = J(y_k) + \boldsymbol{d}_k^T \nabla J(y_k) + \frac{1}{2} \boldsymbol{d}_k^T \nabla^2 J(y_k) \boldsymbol{d}_k$$

where $\boldsymbol{d}_k = \boldsymbol{y} - \boldsymbol{y}_k$.

- We now update $\boldsymbol{y}$ by minimizing $\tilde{J}(\boldsymbol{y}, \boldsymbol{y}_k)$ subject to several constraints. These includes:

(a) An upper bound on group delay as seen in (7d);

(b) Nonnegativeness of the components of $\boldsymbol{y}$;

(c) An additional constraint is imposed to ensure the performance of the C-filter over the passband, especially at passband edge ω_p:

$$1 - d_p \le |H(\omega)|_{\omega=\omega_p} \le 1 + d_p$$

which is equivalent to

$$\log(1-d_p) \le c + \sum_{l=1}^{L} k_l \log\left|2\cos\frac{l\omega_p}{2}\right| \le \log(1+d_p) \qquad (12)$$

with $c = \log|\boldsymbol{x}_k^T \boldsymbol{c}(\omega_p)|$

Thus we update $\boldsymbol{y}$ by solving

$$\min \quad \tfrac{1}{2}(y-y_k)^T \nabla^2 J(y_k)(y-y_k) + (y-y_k)^T \nabla J(y_k) \qquad (13a)$$

$$\text{subject to:} \quad (7d),\; y \ge 0, \text{ and } (12) \qquad (13b)$$

D. *Summary of the Algorithm*

Step 1 Input $y_0, (N, L, D), \omega_p, \omega_a, \delta_a, e_a, d_p,$ and K.

Step 2 For k = 0, 1, ...,

(i) fix $\boldsymbol{y} = \boldsymbol{y}_k$ and solve (10) for $\boldsymbol{x}_k$;

(ii) fix $\boldsymbol{x} = \boldsymbol{x}_k$, perform K iterations of (13) for $\boldsymbol{y}_{k+1}$;

(iii) if $\boldsymbol{y}_{k+1}$ is unequal to $\boldsymbol{y}_k$, set $k = k + 1$ and repeat from step (i), otherwise go to step 3;

Step 3 Output $\boldsymbol{x}^* = \boldsymbol{x}_k$, $\boldsymbol{y}^* = \boldsymbol{y}_k$.

<u>Remarks</u>

- Because both (10) and (13) are convex problems, globally optimal iterates $\boldsymbol{x}_k$ and $\boldsymbol{y}_k$ can be calculated efficiently;
- The objective in (13a) represents the filter's stopband energy, minimizing it tends to increase some powers in the shaping filter until its group delay reaches upper bound D, see (7d). When this occurs, $\boldsymbol{y}_k$ remains unchanged and the algorithm terminates.

4. Design Example and Comparisons

- We illustrate the design method by applying it to design a narrow-band lowpass C-filter with linear-phase response and sharp transition band with $\omega_p = 0.1\pi$ and $\omega_a = 0.11\pi$. The length of prototype filter was set to N = 519 and the peak gain of the C-filter in the stopband was set to be no greater than -60 dB.
- The performance of th filter was evaluated in terms of peak-to-peak passband ripple A_p (in dB), minimum stopband attenuation A_a (in dB), stopband energy E_a, number of multiplications M per output sample, and group delay.
- With L = 7, D = 18, $\boldsymbol{y}_0$ = [1 1 1 1 1 1 1], d_p = 0.08, and e_a = 0.0005, problem (10) was solved and its solution $\boldsymbol{x}_0$ together with $\boldsymbol{y}_0$ defines a C-filter achieving A_p = 0.1681, A_a = 60, and E_a = 2.15×10^{-8}.

- With $\boldsymbol{x}_0$ help fixed, K = 5 iterations of (13) were performed to obtain an integer solution $\boldsymbol{y}_1$ = [1 1 1 1 1 1 2]. Since $\boldsymbol{y}_1$ differs $\boldsymbol{y}_0$, problem (10) was solved again with $\boldsymbol{y}$ fixed to $\boldsymbol{y}_1$, where ea was adjusted in order to obtain an $\boldsymbol{x}_1$ with practically the same peak gain in the stopband and stopband energy as $\boldsymbol{x}_0$ so that the two iterates $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ can be compared with each other in terms of peak passband ripple.
- With e_a = 0.000295, the C-filter defined by ($\boldsymbol{x}_1$, $\boldsymbol{y}_1$) achieved A_p = 0.1389, A_a = 60.04, and E_a = 2.12×10^{-8}.
- We then run (13) again with x fixed to $\boldsymbol{x}_1$ and K = 5, which yields $\boldsymbol{y}_2$ = [1 1 1 1 1 1 2]. Since $\boldsymbol{y}_2$ is equal to $\boldsymbol{y}_1$, the algorithm is terminated and ($\boldsymbol{x}_1$, $\boldsymbol{y}_1$) is claimed as the solution.
- The amplitude responses of the prototype and C-filter are shown in Fig. 3(a) and (b), respectively, and the amplitude response of the C-filter in passband is depicted in Fig. 3(c).

- The filters that are most relevant to the C-filters are linear-phase equiripple-passband-and-lease-squares-stopband (EPLSS) FIR filters with constrained peak gain in stopband, and linear-phase FIR filters with equiripple passbands and stopbands obtained using the Parks-McClellan (P-M) algorithm.
- For comparison purposes an EPLSS lowpass filter and a P-M filter with the same passband and stopband edges as those in the C-filter were designed, both EPLSS and P-M filters satisfy the same peak stopband gain of -60 dB as the C-filter. The evaluation results are summarized in Table I.

Fig. 3(a): Amplitude response of prototype filter

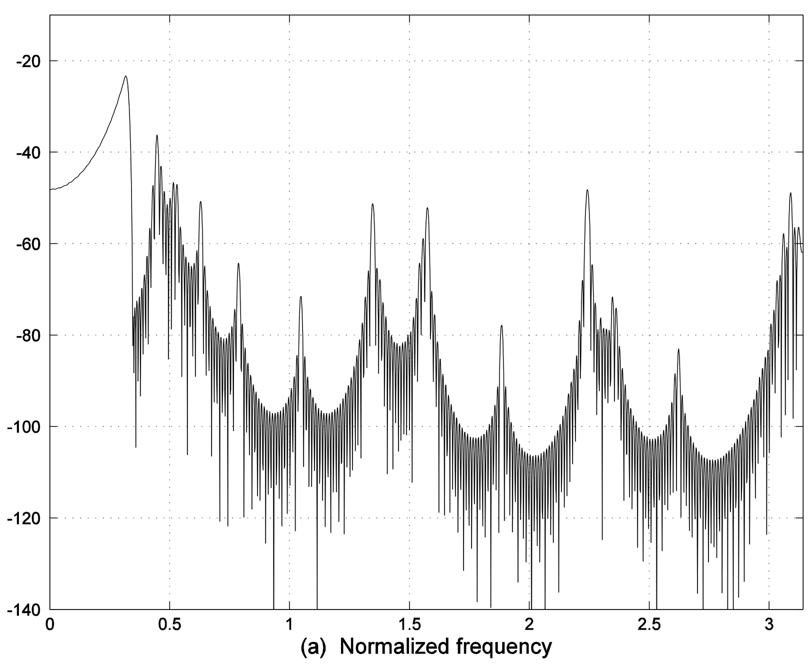


Fig. 3(b): Amplitude response of C-filter.

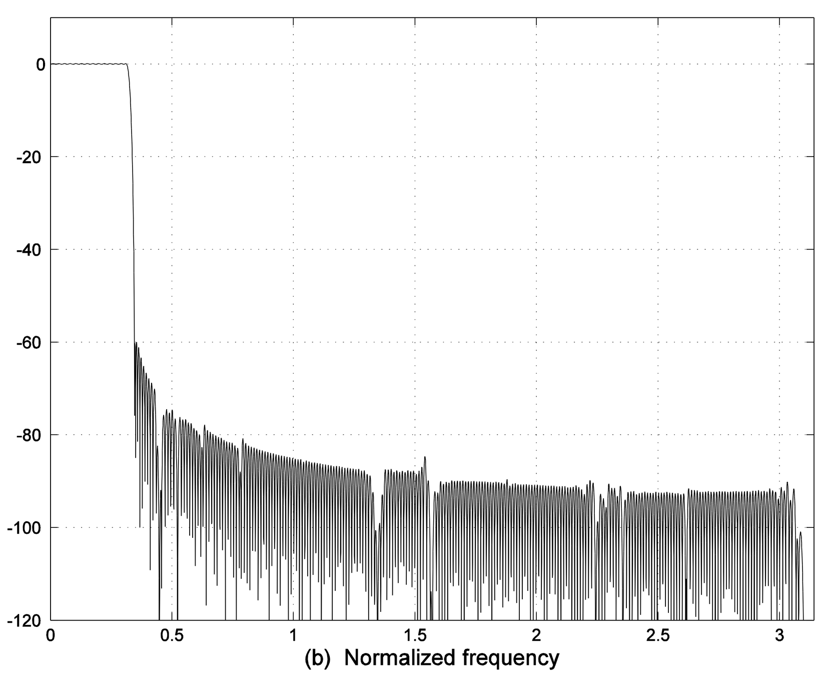


Fig. 3(c): Amplitude response of C-filter in passband.

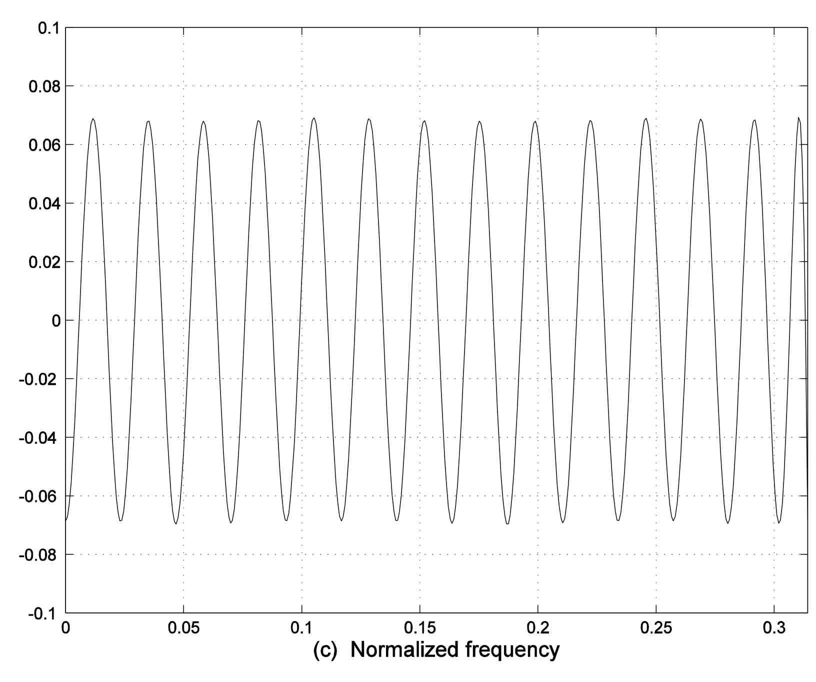


TABLE I
Comparisons of C-filter with EPLSS and P-M filters

Filters	N	A_p	A_a	E_a	M	Delay
C-Filter	**519**	**0.1389**	**60.04**	**2.12e-8**	**260**	277.5
EPLLS	551	0.1465	60.03	2.13e-8	276	275
P-M	531	0.1463	60.03	1.36e-6	266	**265**