Direct Design of Orthogonal Filter Banks and Wavelets

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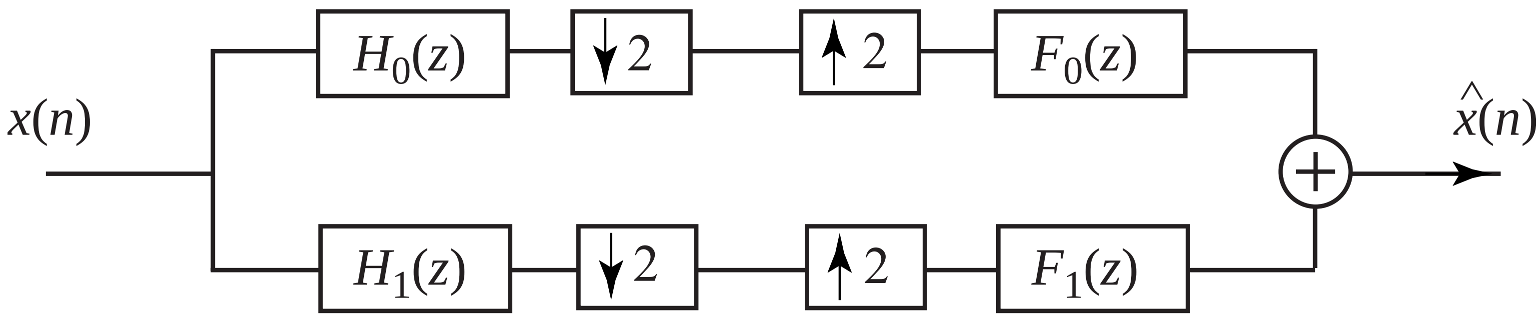
Outline

- Introduction
- Early and recent work
- Constrained linear updates and a convex QP formulation for least-squares design of conjugate quadrature (CQ) filters
- Constrained linear updates and an second-order cone programming formulation for minimax design of CQ filters
- Experimental results

1. Introduction

- Two-channel FIR filter bank



$$\hat{X}(z) = \frac{1}{2}[F_0(z)H_0(z) + F_1(z)H_1(z)]X(z) + \frac{1}{2}[F_0(z)H_0(-z) + F_1(z)H_1(-z)]X(-z)$$

- Perfect reconstruction (PR) conditions

$$F_0(z)H_0(z) + F_1(z)H_1(z) = 2z^{-l}$$

$$F_0(z)H_0(-z) + F_1(z)H_1(-z) = 0$$

- A conjugate quadrature (CQ) filter bank assumes

$$H_1(z) = -z^{-(N-1)}H_0(-z^{-1}),\ F_0(z) = z^{-(N-1)}H_0(z^{-1}),\ F_1(z) = z^{-(N-1)}H_1(z^{-1})$$

$\Rightarrow$ the 2nd PR condition is automatically satisfied

(no aliasing) and the 1st PR condition becomes

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \qquad \text{(PS)}$$

which is called the *power symmetric* (PS) *condition*

because it implies

$$\left|H_0\left(e^{j(\pi/2-\theta)}\right)\right|^2 + \left|H_0\left(e^{j(\pi/2+\theta)}\right)\right|^2 = 1 \qquad \text{for any } \theta$$

2. Early and Recent Work

- Representative early and recent work include
  - Smith and Barnwell (1984)
  - Mintzer (1985)
  - Vaidyanathan and Nguyen (1987)
  - Rioul and Duhamel (1994)
  - Lawton and Michelli (1997)
  - Tuqan and Vaidyanathan (1998)
  - Dumitrescu and Popeea (2000)
  - Tay (2005, 2006)

- The most common design technique:

♦ A half-band filter $P(z)$ is a zero-phase FIR filter satisfying

$$P(z) + P(-z) = 2$$

♦ Let $P(z) = H_0(z)H_0(z^{-1})$, then the PS condition

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$$

becomes

$$P(z) + P(-z) = 2$$

So $P(z)$ is a half-band filer and it is *nonnegative*:

$$P\left(e^{j\omega}\right) = H_0(e^{j\omega})H_0(e^{-j\omega}) = \left|H_0(e^{j\omega})\right|^2 \geq 0 \qquad \text{(P)}$$

♦ Design steps:

(a) Design a lowpass half-band FIR filter $P(z)$ with nonnegativity property $P(e^{j\omega}) \geq 0$

(b) Perform a spectral decomposition $P(z) = H_0(z)H_0(z^{-1})$

- Vanishing moment (VM): the number of VMs equals to the number of zeros of H_0 at $\omega = \pi$:

$$\left.\frac{d^l H_0(e^{j\omega})}{d\omega^l}\right|_{\omega=\pi} = (-j)^l \sum_{n=0}^{N-1} (-1)^n n^l h_n = 0, \quad \text{for } l = 0, 1, \ldots, L-1$$

3. Least-Squares Design of CQ Filters

Problem Formulation

- Let

$$H_0(z) = \sum_{n=0}^{N-1} h_n z^{-n} \text{ with } N \textit{ even}, \text{ and } h = [h_0 \;\; h_1 \; \ldots \; h_{N\text{-}1}]^T$$

- A direct approach: minimizing a least squares type

objective function subject to the PS constraint:

$$\underset{h}{\text{minimize}} \qquad \int_{\omega_a}^{\pi} \left| H_0(e^{j\omega}) \right|^2 d\omega$$

$$\text{subject to:} \quad H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$$

- The objective function is a positive definite quadratic form:

$$\int_{\omega_a}^{\pi} \left|H_0(e^{j\omega})\right|^2 d\omega = h^T Q h$$

with Q a symmetric positive definite Toeplitz matrix:

$$Q = \text{toeplitz}\left(\begin{bmatrix} \pi - \omega_a & -\sin\omega_a & \cdots & \dfrac{-1}{N-1}\sin[(N-1)\omega_a] \end{bmatrix}\right)$$

- The constraint is the PS condition:

$$H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2 \qquad \text{(PS)}$$

that is equivalent to $N/2$ second-order equality

constraints:

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for} \;\; m = 0, 1, \ldots, (N-2)/2$$

with $\delta(m) = 1$ for $m = 0$ and $\delta(m) = 0$ for $m \neq 0$.

- The design problem now becomes a *polynomial optimization problem* (POP):

$$\underset{h}{\text{minimize}} \quad h^T Q h = \left\| Q^{1/2} h \right\|^2$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for} \;\; 0 \le m \le (N-2)/2$$

- The POP can be modified to include VM requirement:

$$\underset{h}{\text{minimize}} \qquad h^T Q h = \left\| Q^{1/2} h \right\|^2$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } \ 0 \le m \le (N-2)/2$$

$$\sum_{n=0}^{N-1} (-1)^n n^l h_n = 0 \quad \text{for } \ l = 0, 1, \ldots, L-1$$

- Features of these problems:
  - ♦ All polynomials are of second-order.
  - ♦ The objective function is convex
  - ♦ *Nonconvex* problems because of the $N/2$

second-order equality constraints (the PS conditions).

- Examples of the PS constraints

1. $N = 4 \;\Rightarrow\; 2$ constraints:

$$h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1$$
$$h_0 \cdot h_2 + h_1 \cdot h_3 = 0$$

2. $N = 20 \Rightarrow 10$ constraints:

$$\begin{array}{ll}
h_0^2 + h_1^2 + \cdots + h_{19}^2 = 1 & (20\ \text{terms}) \\
h_0 \cdot h_2 + h_1 \cdot h_3 + \cdots + h_{17} \cdot h_{19} = 0 & (18\ \text{terms}) \\
h_0 \cdot h_4 + h_1 \cdot h_5 + \cdots + h_{15} \cdot h_{19} = 0 & (16\ \text{terms}) \\
h_0 \cdot h_6 + h_1 \cdot h_7 + \cdots + h_{13} \cdot h_{19} = 0 & (14\ \text{terms}) \\
\vdots & \vdots \\
h_0 \cdot h_{16} + h_1 \cdot h_{17} + h_2 \cdot h_{18} + h_3 \cdot h_{19} = 0 & (4\ \text{terms}) \\
h_0 \cdot h_{18} + h_1 \cdot h_{19} = 0 & (2\ \text{terms})
\end{array}$$

Constrained Linear Updates

- In the *k*th iteration of the algorithm we update filter coefficients from $h^{(k)}$ to $h^{(k+1)} = h^{(k)} + d$ to achieve two things:
  - ♦ to reduce the filter's stopband energy $h^T Qh$
  - ♦ to better approximate constraints

$$\sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m),\ 0 \le m \le N/2-1$$

- The stopband energy at $h^{(k+1)}$ is equal to $\left\| Q^{1/2}(d + h^{(k)}) \right\|^2$
- The constraints at $h^{(k+1)}$ becomes

$$\sum_n h_n^{(k)} h_{n+2m}^{(k)} + \sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \sum_n d_n d_{n+2m} = \delta(m)$$

- Imposing constraints on the smallness of increment vector d:

$$|d_i| \le \beta \qquad \text{for } i = 1, 2, \ldots, N$$

the 2nd-order constraints can be linearized:

$$\sum_n h_n^{(k)} h_{n+2m}^{(k)} + \sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} + \underbrace{\sum_n d_n d_{n+2m}}_{\text{very small, drop}}$$

$$\approx \underbrace{\sum_n h_n^{(k)} h_{n+2m}^{(k)}}_{\text{a known term, denoted by } s^{(k)}(m)} + \underbrace{\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)}}_{\text{linear updates}}$$

$$= \delta(m) \qquad \text{for} \qquad m = 0, 1, \ldots, (N-2)/2$$

- This leads to a set of $(N-2)/2$ linear equations:

$$\sum_n h_n^{(k)} d_{n+2m} + \sum_n d_n h_{n+2m}^{(k)} = \delta(m) - s^{(k)}(m) \equiv u^{(k)}(m)$$

which can be expressed as

$$C^{(k)} d = u^{(k)}$$

- The smallness constraint on d is given by

$$|d_i| \leq \beta \quad \text{for} \quad 1 \leq i \leq n \quad \Leftrightarrow \quad Ad \leq b$$

- The linear constraint on VMs is given by

$$\sum_{n=0}^{N-1} (-1)^n n^l (d_n + h_n^{(k)}) = 0 \quad \text{for} \quad 0 \leq l \leq L-1 \quad \Leftrightarrow \quad Dd = v^{(k)}$$

A Quadratic Programming (QP) Formulation

- Summarizing, the solution strategy is to *iteratively* update the filter coefficients from $h^{(k)}$ to $h^{(k+1)} = h^{(k)} + d^{(k)}$ with $d^{(k)}$ obtained by solving the QP problem

$$\underset{d}{\text{minimize}} \quad \left\| Q^{1/2}(d + h^{(k)}) \right\|^2$$

$$\text{subject to:} \;\; Ad \le b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

4. Minimax Design of CQ Filters

Problem Formulation

- The formulation in this case is changed to

$$\underset{h}{\text{minimize}} \; \underset{\omega_a \le \omega \le \pi}{\text{maximize}} \quad \left| H_0(e^{j\omega}) \right|$$

$$\text{subject to:} \quad \sum_{n=0}^{N-1-2m} h_n \cdot h_{n+2m} = \delta(m) \quad \text{for } \; 0 \le m \le (N-2)/2$$

$$\sum_{n=0}^{N-1} (-1)^n n^l h_n = 0 \quad \text{for } \; l = 0, 1, \ldots, L-1$$

Constrained Linear Updates

- Like in the least squares design, the constrained linear update gives

$$\underset{d}{\text{minimize}}\ \underset{\omega_a \le \omega \le \pi}{\text{maximize}} \quad \left| H_0(e^{j\omega}) \right|$$

$$\text{subject to:} \qquad Ad \le b$$

$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

- Dealing with the objective function, we write

$$H_0(e^{j\omega}) = \sum_{n=0}^{N-1} h_n e^{-jn\omega} = h^T c(\omega) - jh^T s(\omega)$$

$$c(\omega) = \begin{bmatrix} 1 & \cos\omega & \cdots & \cos(N-1)\omega \end{bmatrix}^T, \; s(\omega) = \begin{bmatrix} 0 & \sin\omega & \cdots & \sin(N-1)\omega \end{bmatrix}^T$$

hence

$$\left|H_0(e^{j\omega})\right| = \sqrt{\left(h^T c(\omega)\right)^2 + \left(h^T s(\omega)\right)^2} = \left\| \begin{bmatrix} c(\omega)^T \\ s(\omega)^T \end{bmatrix} \cdot h \right\| \equiv \left\| T(\omega) \cdot h \right\|$$

$$\Rightarrow \quad \left|H_0(e^{j\omega}, h^{(k)} + d^{(k)})\right| = \left\| T(\omega) \cdot (h^{(k)} + d^{(k)}) \right\| = \left\| T(\omega) d^{(k)} + g^{(k)} \right\|$$

- This converts the minimax problem into

$$\text{minimize} \quad \eta$$

$$\text{subject to:} \quad \left\| T(\omega_i)d^{(k)} + g^{(k)} \right\| \le \eta \ \text{ for } \{\omega_i\} \subseteq [\omega_a, \pi]$$

$$Ad \le b$$

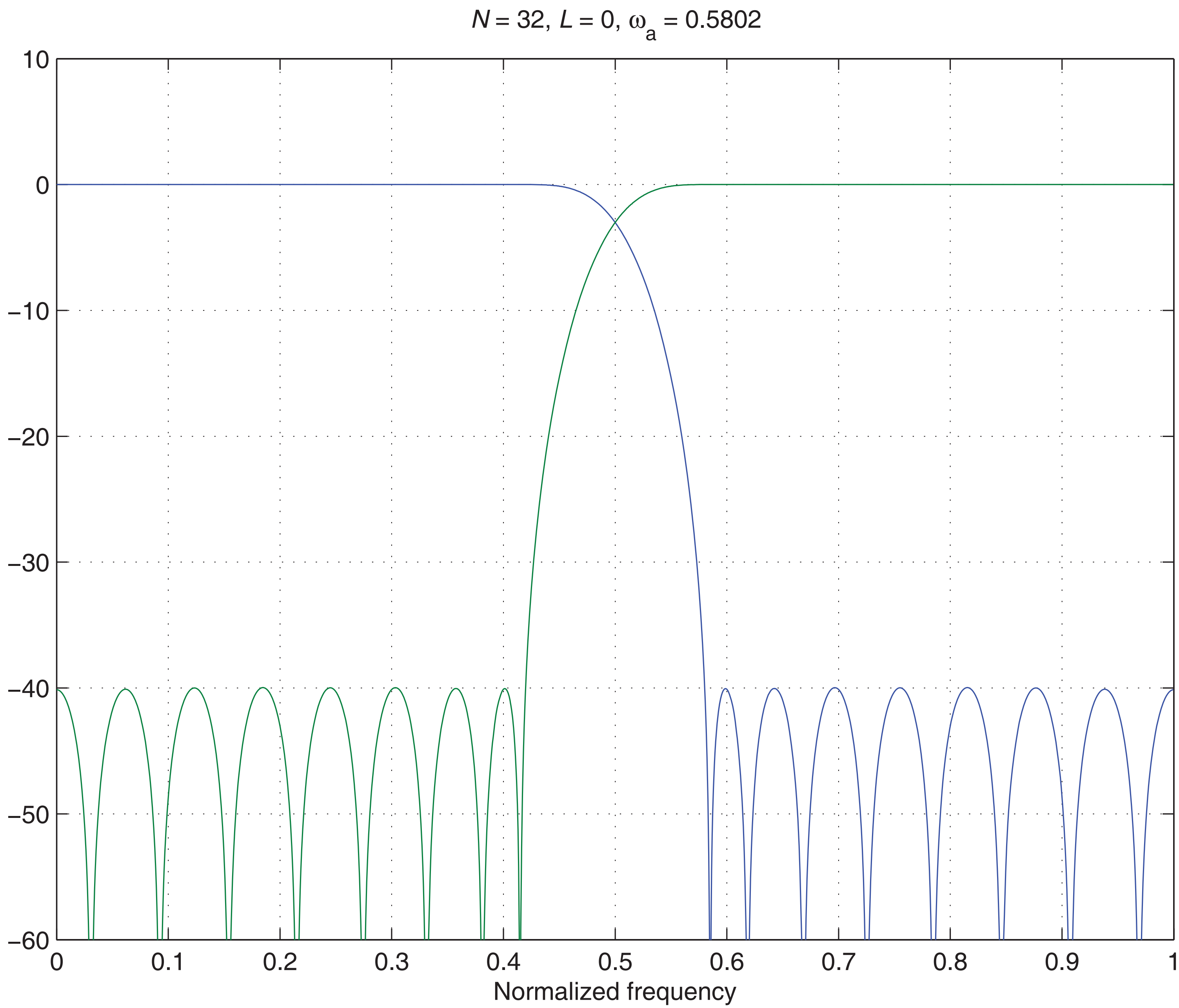
$$\begin{bmatrix} C^{(k)} \\ D \end{bmatrix} d = \begin{bmatrix} u^{(k)} \\ v^{(k)} \end{bmatrix}$$

which is an SOCP problem.

5. Experimental Results

5.1 Comparison with designs by Smith-Barnwell's method

Filter $H_0(z)$		Largest Eq. Error
$N = 8$	$H_0(z)$ of [2]	8.3168×10^{-8}
	Refined $H_0(z)$	$< 10^{-15}$
$N = 16$	$H_0(z)$ of [2]	2.6356×10^{-6}
	Refined $H_0(z)$	$< 10^{-15}$
$N = 32$	$H_0(z)$ of [2]	2.1623×10^{-6}
	Refined $H_0(z)$	$< 10^{-15}$



N = 32, L = 0, ωa = 0.5802
10
0
−10
−20
−30
−40
−50
−60
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Normalized frequency

5.2 LS and minimax designs with N = 96 and L = 0, 1, …, 5

Least squares with $N = 96$, $\omega_a = 0.56\pi$

L	Energy in Stopband	Largest Equation Error
0	5.6213×10^{-10}	$< 10^{-15}$
1	5.6660×10^{-10}	$< 10^{-15}$
2	5.6660×10^{-10}	$< 10^{-15}$
3	5.8954×10^{-10}	$< 10^{-15}$
4	5.8954×10^{-10}	$< 10^{-15}$
5	6.2901×10^{-10}	7.6190×10^{-10}

Minimax with $N = 96$, $\omega_a = 0.56\pi$

L	Instantaneous Energy in Stopband	Largest Equation Error
0	2.8649×10^{-9}	$< 10^{-15}$
1	3.0323×10^{-9}	8.8247×10^{-7}
2	3.0654×10^{-9}	2.5128×10^{-5}
3	3.4075×10^{-9}	1.0654×10^{-6}
4	3.1281×10^{-9}	4.0553×10^{-7}
5	3.7121×10^{-9}	1.0982×10^{-5}

Minimax design with $N = 96$, $L = 3$, $\omega_a = 0.56\pi$: