Reconstruction of Sparse Signals by Minimizing a Re-Weighted Approximate ℓ_0-Norm in the Null Space of the Measurement Matrix

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August 3, 2010

- Comperssive Sensing

Outline

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- Signal Recovery by ℓ_1 Minimization

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- Performance Evaluation

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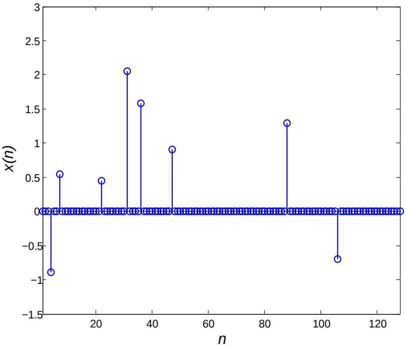
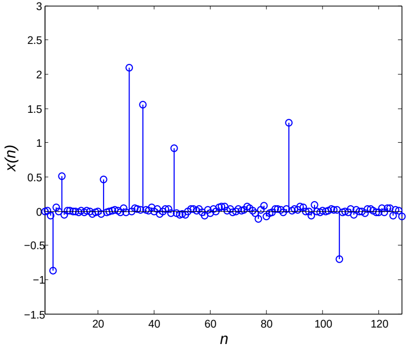
- A signal is near K-sparse if it contains K significant components.



A sparse signal



A near sparse signal

Compressive Sensing, cont'd

- Sparsity is a generic property of signals: A real-world signal always has a sparse or near-sparse representation with respect to an appropriate basis.

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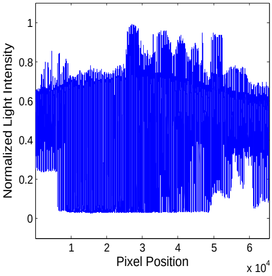
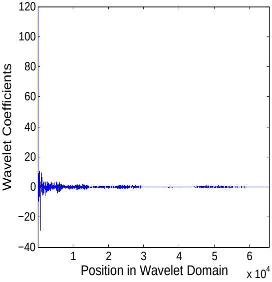
An Image



An equivalent 1-D signal



A wavelet representation of the image

Compressive Sensing, cont'd

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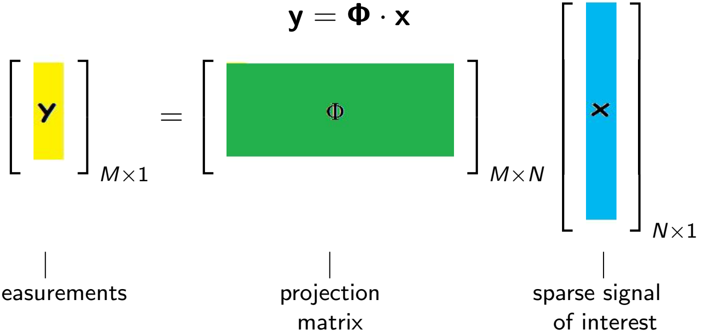
- Compressive sensing (CS) is a data acquisition process whereby a sparse signal $\mathbf{x}(n)$ represented by a vector $\mathbf{x}$ of length N is determined using a small number of projections represented by a matrix $\mathbf{\Phi}$ of dimension $M \times N$.
- In such a process, measurement vector $\mathbf{y}$ and signal vector $\mathbf{x}$ are interrelated by the equation

$$\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x}$$



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$$M \geq c \cdot K \cdot \log(N/K)$$

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- Typically,

$$K < M < N$$

Signal Recovery by ℓ_1 Minimization

- Recovering signal vector **x** from measurement vector **y** such that

$$\underset{M \times N}{\boldsymbol{\Phi}} \cdot \underset{N \times 1}{\mathbf{x}} = \underset{M \times 1}{\mathbf{y}}$$

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- Given that **x** is sparse, **x** can be reconstructed by solving the ℓ_1-minimization problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{x}||_1 \\ \text{subject to} \quad & \boldsymbol{\Phi}\mathbf{x} = \mathbf{y} \end{aligned}$$

  where $||\mathbf{x}||_1 = \sum_{i=1}^{N} |x_i|$.

Signal Recovery by ℓ_1 Minimization, cont'd

- Why ℓ_1-norm minimization?

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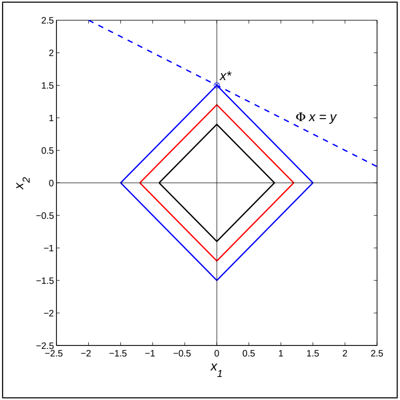

Contours for $||\mathbf{x}||_1 = c$

As c increases, the contour of $||\mathbf{x}||_1 = c$ grows and touches the hyperplane $\mathbf{\Phi x} = \mathbf{y}$, yielding a sparse solution

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

Signal Recovery by ℓ_1 Minimization, cont'd

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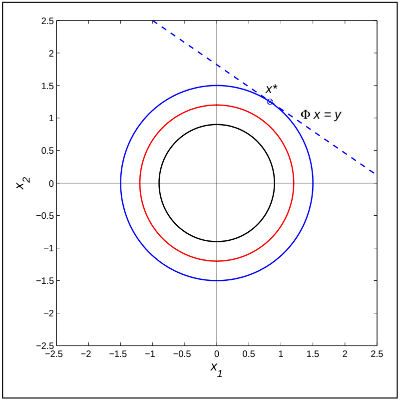

Contours of $||\mathbf{x}||_2 = r$

As r increases, the contour of $||\mathbf{x}||_2 = r$ grows and touches the hyperplane $\mathbf{\Phi x} = \mathbf{y}$.

The solution $\mathbf{x}^*$ obtained is not sparse.

Signal Recovery by ℓ_1 Minimization, cont'd

Theorem

If $\mathbf{\Phi} = \{\phi_{ij}\}$ where ϕ_{ij} are independent and identically distributed random variables with zero-mean and variance $1/N$ and $M \geq cK \log(N/K)$, the solution of the ℓ_1-minimization problem would recover exactly a K-sparse signal with high probability.

Signal Recovery by ℓ_1 Minimization, cont'd

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- For real-valued data $\{\mathbf{\Phi}, \mathbf{y}\}$, the ℓ_1-minimization problem is a linear programming problem.

- Example: $N = 512$, $M = 120$, $K = 26$

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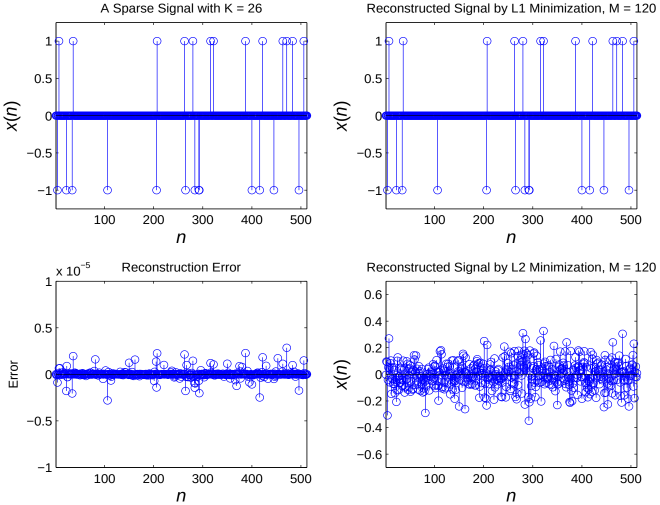

Signal Recovery by ℓ_p Minimization

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- Hence the sparsest solution of $\mathbf{\Phi x} = \mathbf{y}$ can be obtained by solving the ℓ_0-norm minimization problem

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & ||\mathbf{x}||_0 \\ \text{subject to} & \mathbf{\Phi x} = \mathbf{y} \end{array}$$

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- Unfortunately, the ℓ_0-norm minimization problem is nonconvex with combinatorial complexity.

- An effective signal recovery strategy is to solve the ℓ_p-minimization problem

$$\begin{array}{llll} \underset{\mathbf{x}}{\text{minimize}} & ||\mathbf{x}||_p^p & \text{with} & 0 < p < 1 \\ \text{subject to} & & \boldsymbol{\Phi}\mathbf{x} = \mathbf{y} & \end{array}$$

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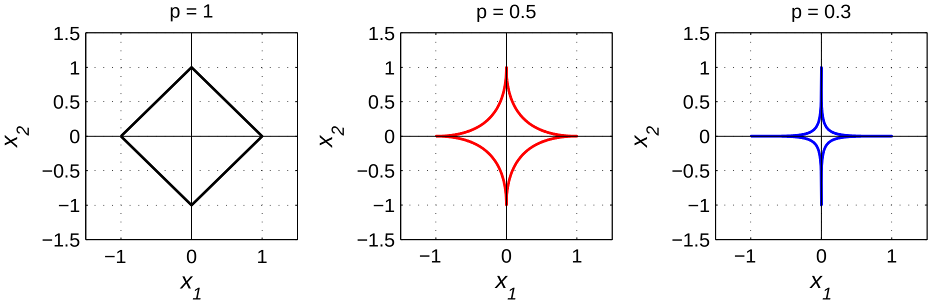
where $||\mathbf{x}||_p^p = \sum_{i=1}^{N} |x_i|^p$.

- The ℓ_p-norm minimization problem is nonconvex.

- Contours of $||\mathbf{x}||_p = 1$ with $p < 1$



$||\mathbf{x}||_1 = 1$ $\quad$ $||\mathbf{x}||_{0.5} = 1$ $\quad$ $||\mathbf{x}||_{0.3} = 1$

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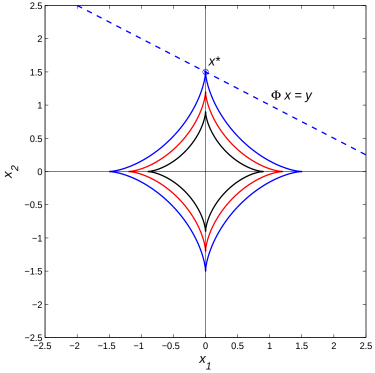

Contours of $||\mathbf{x}||_p^p = c$ with $p < 1$

As c increases, the contour $||\mathbf{x}||_p^p = c$ grows and touches the hyperplane $\mathbf{\Phi x} = \mathbf{y}$, yielding a sparse solution

$$\mathbf{x}^* = \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

The possibility that the contour will touch the hyperplane at another point is eliminated.

- We propose to minimize an approximate ℓ_0-norm

$$||\mathbf{x}||_{0,\sigma} = \sum_{i=1}^{N} \left(1 - e^{-x_i^2/2\sigma^2}\right)$$

where $\mathbf{x}$ lies in the solution space of $\mathbf{\Phi x} = \mathbf{y}$, namely,

$$\mathbf{x} = \mathbf{x}_s + \mathbf{V}_r \xi$$

where $\mathbf{x}_s$ is a solution of $\mathbf{\Phi x} = \mathbf{y}$ and $\mathbf{V}_r$ is an orthonormal basis of the null space of $\mathbf{\Phi}$.

Signal Recovery by ℓ_p Minimization, cont'd

- Why norm $||\mathbf{x}||_{0,\sigma}$ works?

Signal Recovery by ℓ_p Minimization, cont'd

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  With σ small,

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Therefore, for a K-sparse signal,

$$||\mathbf{x}||_{0,\sigma} = \sum_{i=1}^{N}\left(1 - e^{-x_i^2/2\sigma^2}\right) \approx K = ||\mathbf{x}||_0$$

Signal Recovery by ℓ_p Minimization, cont'd

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Signal Recovery by ℓ_p Minimization, cont'd

- Improved recovery rate can be achieved by using a re-weighting technique.
- This involves solving the optimization problem

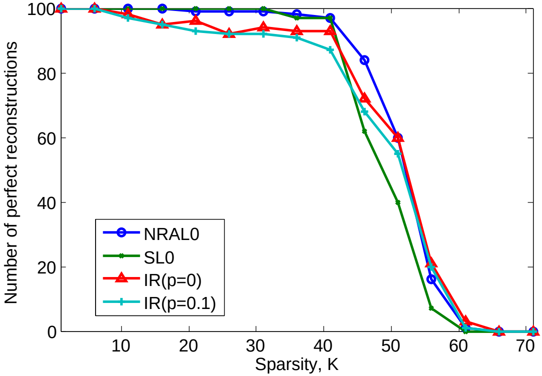
$$\underset{\boldsymbol{\xi}}{\text{minimize}} \sum_{i=1}^{n} w_i \left\{ 1 - e^{-[x_s(i) + \mathbf{v}_i^T \boldsymbol{\xi}]^2 / 2\sigma^2} \right\}$$

where

$$w_i^{(k+1)} = \frac{1}{|x_i^{(k)}| + \epsilon}$$

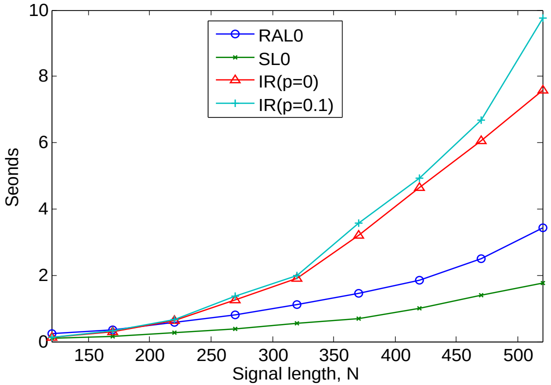
Performance Evaluation

Number of perfectly recovered instances versus sparsity K by various algorithms with $N = 256$ and $M = 100$ over 100 runs.



IR: Iterative re-weighting (Chartrand and Yin, 2008)
SL0: Smoothed ℓ_0-norm minimization (Mohimani et. al., 2009)
NRAL0: Proposed

Average CPU time versus signal length for various algorithms with $M = N/2$ and $K = M/2.5$.



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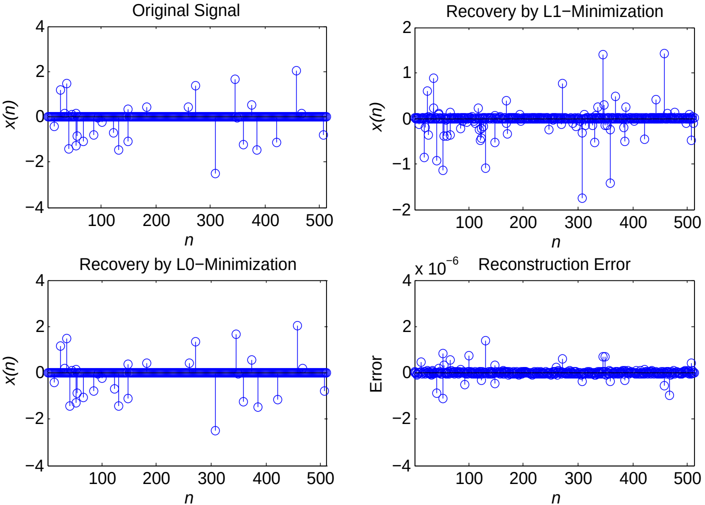
Performance comparison of ℓ_1 minimization with approximate ℓ_0 minimization for $N = 512$, $M = 80$, $K = 30$.



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- ℓ_1 minimization works in general for the reconstruction of sparse signals.
- ℓ_p minimization with $p < 1$ can improve the recovery performance for signals that are less sparse.
- Approximate ℓ_0-norm minimization offers good performance with improved complexity.

Thank you for your attention.

This presentation can be downloaded from:

http://www.ece.uvic.ca/∼andreas/RLectures/MWSCAS2010-Jeevan-Web.pdf