Unconstrained Regularized $\ell_p$-Norm Based Algorithm for the Reconstruction of Sparse Signals

J. K. Pant, W.-S. Lu, and A. Antoniou

University of Victoria

May 17, 2011
Compressive Sensing

Signal Recovery by Using $\ell_1$ and $\ell_p$ Minimizations

Signal Recovery by Using Regularized $\ell_p$ Minimization

Line Search Based on Banach’s Fixed-Point Theorem

Performance Evaluation
A signal $x(n)$ of length $N$ is $K$-sparse if it contains $K$ nonzero components with $K \ll N$.

A signal is near $K$-sparse if it contains $K$ significant components.
Compressive Sensing, cont’d

- Sparsity is a generic property of signals: A real-world signal always has a sparse or near-sparse representation with respect to an appropriate basis.

An Image

An equivalent 1-D signal

A wavelet representation of the image
Compressive sensing (CS) is a data acquisition process whereby a sparse signal $x(n)$ represented by a vector $x$ of length $N$ is determined using a small number of projections represented by a matrix $\Phi$ of dimension $M \times N$.

In such a process, measurement vector $y$ and signal vector $x$ are interrelated by the equation

$$y = \Phi \cdot x$$
Compressive Sensing, cont’d

- CS theory shows that these random projections contain much, sometimes all, the information content of signal $x$.

- If a sufficient number of such measurements is collected, recovering signal $x$ from measurements $y$ is possible.

- A condition for this to be possible is

  $$M \geq c \cdot K \cdot \log(N/K)$$

  where $c$ is a small constant.

- Typically,

  $$K < M < N$$
The inverse problem of recovering signal vector $x$ from measurement vector $y$ such that

$$\Phi \cdot x = y$$

is an ill-posed problem.

A classical approach for solving this problem is to find a vector $x^*$ with minimum $\ell_2$ norm in the translated null space of $\Phi$ such that

$$x^* = \arg \min_x \|x\|_2 \quad \text{subject to} \quad \Phi x = y$$

Unfortunately, the $\ell_2$ minimization fails to recover a sparse signal.
Why $\ell_2$-norm minimization fails to work?

As $r$ increases, the contour of $\|x\|_2 = r$ grows and touches the hyperplane $\Phi x = y$.

The solution $x^*$ obtained is not sparse.

Contours of $\|x\|_2 = r$
A sparse signal, say $x^*$, can be obtained by finding a vector with minimum $\ell_1$ norm in the translated null space of $\Phi$, i.e., using

$$x^* = \arg \min_x \|x\|_1 \quad \text{subject to} \quad \Phi x = y$$

As $c$ increases, the contour of $\|x\|_1 = c$ grows and touches the hyperplane $\Phi x = y$, yielding a sparse solution

$$x^* = \begin{bmatrix} 0 \\ c \end{bmatrix}$$
Theorem

If $\Phi = \{\phi_{ij}\}$ where $\phi_{ij}$ are independent and identically distributed random variables with zero-mean and variance $1/N$ and $M \geq cK \log(N/K)$, the solution of the $\ell_1$-minimization problem would recover exactly a $K$-sparse signal with high probability.

- For real-valued data $\{\Phi, y\}$, the $\ell_1$-minimization problem is a linear programming problem.
Example: $N = 512, M = 120, K = 26$

A Sparse Signal with $K = 26$

Reconstructed Signal by $L_1$ Minimization, $M = 120$

Reconstruction Error

Reconstructed Signal by $L_2$ Minimization, $M = 120$
The sparsity of a signal can be measured by using its $\ell_0$ pseudonorm

$$||x||_0 = \sum_{i=1}^{N} |x_i|^0$$

Hence the sparsest solution of $\Phi x = y$ can be obtained by finding the vector $x^*$ with the smallest value of the $\ell_0$ pseudonorm in the translated null space of $\Phi$, i.e.,

$$x^* = \arg \min_x ||x||_0 \quad \text{subject to } \Phi x = y$$

Unfortunately, the above $\ell_0$-pseudonorm minimization problem is nonconvex with combinatorial complexity.
An effective signal recovery strategy is to solve the \( \ell_p \)-minimization problem

\[
\begin{align*}
\text{minimize} & \quad \|x\|_p^p \\
\text{subject to} & \quad \Phi x = y
\end{align*}
\]

where \( \|x\|_p^p = \sum_{i=1}^{N} |x_i|^p \).

This \( \ell_p \)-norm minimization problem is nonconvex.
Contours of $|x|_p = 1$ with $p < 1$

- $p = 1$
  - $|x|_1 = 1$
- $p = 0.5$
  - $|x|_{0.5} = 1$
- $p = 0.3$
  - $|x|_{0.3} = 1$
Why $\ell_p$ minimization with $p < 1$?

As $c$ increases, the contour $\|x\|_p^p = c$ grows and touches the hyperplane $\Phi x = y$, yielding a sparse solution $x^* = \begin{bmatrix} 0 \\ c \end{bmatrix}$.

The possibility that the contour will touch the hyperplane at another point is eliminated.
We propose to minimize a regularized $\ell_p$ norm

$$||x||_{p,\epsilon}^p = \sum_{i=1}^{N} \left(x_i^2 + \epsilon^2\right)^{p/2}$$

where $x$ lies in the null space of $\Phi$ translated by the $\ell_2$-norm solution vector, say $x_s$, of $\Phi x = y$, namely,

$$x = x_s + V_r \xi$$

where $V_r$ is an orthonormal basis of the null space of $\Phi$. 
Signal Recovery by Using Regularized $\ell_p$ Minimization, cont’d

- Note that as $\epsilon \rightarrow 0$, we have

$$\left( x_i^2 + \epsilon^2 \right)^{p/2} \approx |x_i|^p$$

Therefore,

$$\|x\|_{p,\epsilon}^p \bigg|_{\epsilon \rightarrow 0} \approx \|x\|_p^p$$

i.e., the regularized $\ell_p$ norm closely approximates the $\ell_p$ norm.

- The reconstruction involves solving the optimization problem

$$(P1) \quad \text{minimize } \sum_{i=1}^{n} \left\{ [x_s(i) + v_i^T \xi]^2 + \epsilon^2 \right\}^{p/2}$$

for a small value of $\epsilon$. 
Signal Recovery by Using Regularized $\ell_p$ Minimization, cont’d

- **Optimization overview:**
  - Obtain an $\ell_2$-norm solution $\mathbf{x}$, set $\xi = 0$, and select an initial value of $\epsilon$ to satisfy the inequality
    \[ \epsilon \geq \sqrt{1 - p \cdot \max_{1 \leq i \leq N} |x_{si}|} \]
  - Using $\xi$ as an initializer, solve the optimization problem $\mathbf{P1}$ using a quasi-Newton algorithm such as Broyden-Fletcher-Goldfarb-Shanno algorithm. Set the resulting solution to $\xi$.
  - Reduce the value of $\epsilon$, use $\xi$ as an initializer, and solve problem $\mathbf{P1}$ again using the same quasi-Newton algorithm.
  - Repeat this procedure until problem $\mathbf{P1}$ is solved for a sufficiently small value of $\epsilon$. 
Line Search Based on Banach’s Fixed-Point Theorem:

The \((k + 1)\)th iterate is computed as

\[
\xi_{k+1} = \xi_k + \alpha d_k
\]

According to Banach’s fixed-point theorem, the step size \(\alpha\) can be computed using a finite number of iterations of

\[
\alpha_{l+1} = -\frac{\sum_{i=1}^{N} x_i \cdot v_i \cdot \gamma_i(\alpha_l, \epsilon)^{p/2-1}}{\sum_{i=1}^{N} v_i^2 \cdot \gamma_i(\alpha_l, \epsilon)^{p/2-1}}
\]

where

\[
\gamma_i(\alpha_l, \epsilon) = (x_i + \alpha v_i)^2 + \epsilon^2, \quad x_i = x_{si} + v_{i}^T \xi_k, \quad v_i = v_{i}^T d_k
\]
Performance Evaluation

Number of perfectly recovered instances versus sparsity $K$ by various algorithms with $N = 256$ and $M = 100$ over 100 runs.

URLP: Proposed
NRAL0: Null space re-weighted approximate $\ell_0$ (Pant, Lu, and Antoniou, 2010)
SL0: Smoothed $\ell_0$-norm minimization (Mohimani et. al., 2009)
IR: Iterative re-weighting (Chartrand and Yin, 2008)
Average CPU time versus signal length for various algorithms with $M = N/2$ and $K = M/2.5$.

URLP: Proposed
NRAL0: Null space re-weighted approximate $\ell_0$ (Pant, Lu, and Antoniou, 2010)
SL0: Smoothed $\ell_0$-norm minimization (Mohimani et. al., 2009)
IR: Iterative re-weighting (Chartrand and Yin, 2008)
Compressive sensing is an effective technique for sampling sparse signals.

\( \ell_1 \) minimization works in general for the reconstruction of sparse signals.

\( \ell_p \) minimization with \( p < 1 \) can improve the recovery performance for signals that are less sparse.

Regularized \( \ell_p \) minimization offers improved signal reconstruction performance.

A line search method based on Banach’s fixed-point theorem offers improved complexity.
Thank you for your attention.

This presentation can be downloaded from: