A New Algorithm for Compressive Sensing Based on Total-Variation Norm

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Outline

- Compressive Sensing and Signal Recovery
- Image Recovery Using Total-Variation Minimization
- Image Recovery Using Nonconvex Total-Variation Minimization
- Performance Evaluation
A signal $x(n)$ of length $N$ is $K$-sparse if it contains $K$ nonzero components with $K \ll N$.

A signal is near $K$-sparse if it contains $K$ significant components.
Compressive sensing (CS) is a data acquisition process whereby a sparse signal \( x \) or an image \( X \) represented by a vector \( x \) of length \( N \) can be determined using a small number of projections represented by a matrix \( \Phi \) of dimension \( M \times N \).

In CS, measurement vector \( y \) and signal vector \( x \) are interrelated by the equation:

\[
y = \Phi \cdot x
\]
A sparse signal \( x \) can be recovered by using an \( \ell_1 \)-norm minimization that solves the problem

\[
\begin{align*}
\text{minimize} & \quad \|x\|_1 = \sum_{i=1}^{N} |x_i| \\
\text{subject to} & \quad y = \Phi x
\end{align*}
\]

An \( \ell_p \)-pseudonorm minimization that solves the problem

\[
\begin{align*}
\text{minimize} & \quad \|x\|_p = \sum_{i=1}^{N} |x_i|^p \\
\text{subject to} & \quad y = \Phi x
\end{align*}
\]

where a \( p \) in the range \( 0 < p < 1 \) can be used to yield a sparser signal.
Many synthetic and natural images have a *spatially sparse* gradient.

The spatial gradient of an image $\mathbf{X}$ of size $n_1 \times n_2$ can be obtained as a matrix $\mathbf{G}$ of size $n_1 \times n_2$ whose $\{i, j\}$th component is given by

$$g_{i,j} = \begin{cases} \sqrt{(x_{i,j} - x_{i+1,j})^2 + (x_{i,j} - x_{i,j+1})^2} & \text{for } \begin{cases} 1 \leq i < n_1, \\ 1 \leq j < n_2 \end{cases} \\ |x_{i,j} - x_{i+1,j}| & \text{for } j = n_2, \\ |x_{i,j} - x_{i,j+1}| & \text{for } i = n_1, \\ 0 & \text{for } i = n_1, j = n_2 \end{cases}$$

where $x_{i,j}$ is the $\{i, j\}$th component of $\mathbf{X}$. 

The Shepp-Logan Phantom image has a sparse spatial gradient:

Phantom image

Sparse spatial gradient of Phantom image
The Cameraman image has near-sparse spatial gradient:

Cameraman image

Near-sparse spatial gradient of Cameraman image
The sparsity of the spatial gradient of an image $X$ can be measured in terms of the total-variation norm given by

$$TV(X) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} g_{i,j}$$

where $g_{i,j}$ is the $\{i,j\}$th element of matrix $G$.

The smaller the $TV(X)$, the sparser the gradient of $X$.

An image $X$ with sparse spatial gradient represented by a vector $x$ can be recovered from measurements $y$ by solving the optimization problem

$$\text{minimize} \quad \frac{1}{2} \| \Phi x - y \|^2_2 + \lambda TV(X)$$

where $\lambda$ a regularization parameter.
Inspired by the success of $\ell_p$ over $\ell_1$ minimization in CS, we consider the nonconvex version of the TV norm, called the $TV_p$ pseudonorm, given by

$$TV_p(X) = \left[ \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} \left( x_{i,j}^\\prime \right)^2 + x_{i,j}^\\prime^2 \right]^{p/2}$$

$$+ \sum_{i=1}^{n_1-1} \left( x_{i,n_2}^\\prime \right)^2^{p/2} + \sum_{j=1}^{n_2-1} \left( x_{n_1,j}^\\prime \right)^2^{p/2} \right]^{1/p}$$

where $x_{i,j}^\\prime = x_{i,j} - x_{i+1,j}$, $x_{i,j}^\\prime^\prime = x_{i,j} - x_{i,j+1}$, and $0 < p < 1$.

From the nonconvexity and nondifferentiability of the $\ell_p$ pseudonorm, it follows that function $TV_p(X)$ remains nonconvex and nondifferentiable for $p < 1$. 
To render the $TV_p$ pseudonorm differentiable and to facilitate its optimization, we consider the approximate $TV_p$ pseudonorm given by

$$TV_{p, \epsilon}^p(X) = \sum_{i=1}^{n_1-1} \sum_{j=1}^{n_2-1} \left( x_{i,j}^p + x_{i,j}^p + \epsilon^2 \right)^{p/2}$$

$$+ \sum_{i=1}^{n_1-1} \left( x_{i,n_2}^p + \epsilon^2 \right)^{p/2} + \sum_{j=1}^{n_2-1} \left( x_{n_1,j}^p + \epsilon^2 \right)^{p/2}$$

where $\epsilon$ is a nonzero parameter used to render it differentiable.

Note that $TV_{p, \epsilon}^p(X) \to TV(X)$ as $\epsilon \to 0, p \to 1.$
The reconstruction involves solving the optimization problem

$$\text{(P-TV}_p) \quad \min_x F_{\lambda, p, \epsilon}(X) = \frac{1}{2} \|\Phi x - y\|_2^2 + \lambda TV_{p, \epsilon}(X)$$

for a small values $\epsilon_T$ and $\lambda_T$ of $\epsilon$ and $\lambda$, respectively, and $p < 1$.

The gradient of the objective function $F_{\lambda, p, \epsilon}(X)$ can be evaluated as

$$g = \Phi^T (\Phi x - y) + \lambda pu$$

where $u$ is a vector representing the gradient of $TV_{p, \epsilon}(X)/p$. 
The problem $P_{-TV_p}$ can be solved by using the following sequential procedure:

- Select $\{\epsilon = \epsilon_1, \lambda = \lambda_1\}$ so that $\{\epsilon_1 > \epsilon_T, \lambda_1 > \lambda_T\}$, set the zero vector as initializer, and solve problem $P_{-TV_p}$. Denote the resulting solution as $x^*$.  

- Using $x^*$ as the initializer, solve problem $P_{-TV_p}$ again for smaller values of $\epsilon$ and $\lambda$.  

- Repeat this procedure until problem $P_{-TV_p}$ is solved for the pair $\{\epsilon = \epsilon_T, \lambda = \lambda_T\}$. Denote the final solution as $x_T^*$.  

- Construct image $X^*$ from the final solution $x_T^*$.  

- Output $X^*$ and stop.

The Fletcher-Reeves’ conjugate-gradient (FR-CG) technique can be applied to solve problem $P_{-TV_p}$ for a given pair of values of $\{\epsilon, \lambda\}$.  

In the FR-CG technique, iterate $x_k$ is updated to $x_{k+1}$ as

$$x_{k+1} = x_k + \alpha_k d_k$$

where

$$d_k = -g_k + \beta_{k-1} d_{k-1},$$

$$\beta_{k-1} = \frac{\|g_k\|_2^2}{\|g_{k-1}\|_2^2},$$

and $g_k$ is the gradient at $x = x_k$.

Step size $\alpha_k$ is obtained by using the recursion

$$\alpha_{l+1} = G(\alpha_l) \quad \text{for} \quad l = 2, 3, \ldots$$

with $\alpha_0 \geq 0$ where function $G(\alpha)$ depends on $x_k$, $\Phi$, $d_k$, $y$, $\epsilon$, and $p$. 
The performance of the proposed $TV_p$-RLS and conventional $TV$-RLS algorithms was tested using six images, namely,

- “Circles”, “Resolution Chart”, and “Shepp-Logan Phantom” having sparse spatial gradient and
- “Cameraman”, “Aeroplane”, and “Clock” having near-sparse spatial gradient.

The image reconstruction performance was measured in terms of the peak signal-to-noise ratio (PSNR) which is defined as

$$PSNR = 20 \log \left( \frac{I_{MAX}}{\sqrt{MSE}} \right) \text{dB}$$

where $I_{MAX} = 2^b - 1$ and $b = 8$ is the number of bits used to encode the components of image $X$.

The mean-square error is defined as

$$MSE = \frac{1}{n_1 n_2} \left\| X - \hat{X} \right\|_F^2$$
Experimental results:

<table>
<thead>
<tr>
<th>Images</th>
<th>$TV_p$-RLS ($p = 0.5$)</th>
<th>$TV$-RLS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PSNR (dB)</td>
<td>CPU time (seconds)</td>
</tr>
<tr>
<td>Cameraman</td>
<td>32.8</td>
<td>47.1</td>
</tr>
<tr>
<td>Aeroplane</td>
<td>41.7</td>
<td>49.1</td>
</tr>
<tr>
<td>Circles</td>
<td>90.1</td>
<td>43.6</td>
</tr>
<tr>
<td>Clock</td>
<td>38.4</td>
<td>48.1</td>
</tr>
<tr>
<td>Resolution Chart</td>
<td>74.6</td>
<td>45.0</td>
</tr>
<tr>
<td>Shepp-Logan</td>
<td>86.5</td>
<td>44.1</td>
</tr>
</tbody>
</table>
Reconstruction of an angiogram of size $256 \times 256$:

(a) Original angiogram
(b) Angiogram reconstructed using $TV_p$-RLS algorithm with $p = 0.5$
(c) Angiogram reconstructed using $TV$-RLS algorithm
Segments of the angiograms shown in Slide 17 for the range $120 \leq n_y \leq 220, 120 \leq n_x \leq 220$ where $n_y$ and $n_x$ are pixel indices for vertical and horizontal directions, respectively:

(a) Original angiogram  
(b) Angiogram reconstructed using $TV_p$-RLS algorithm with $p = 0.5$  
(c) Angiogram reconstructed using $TV$-RLS algorithm
Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.

- $\ell_1$ and $\ell_p$ minimizations work in general for the reconstruction of sparse signals.

- Total variation minimization is effective for the reconstruction of images.

- Nonconvex total-variation minimization offers improved reconstruction performance relative to the total-variation minimization for images with sparse spatial gradient.