# A New Algorithm for Compressive Sensing Based on Total-Variation Norm

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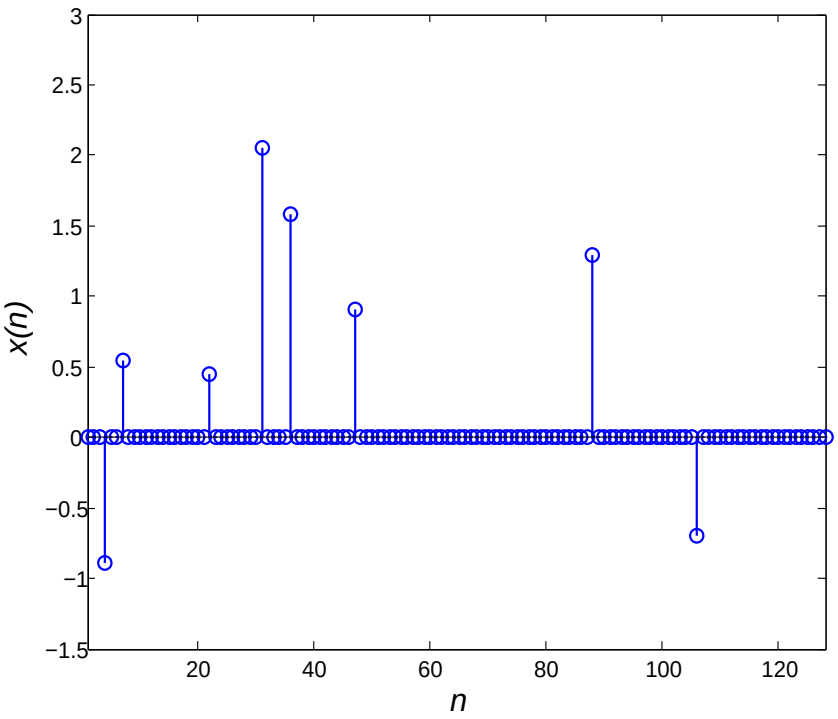
# Outline

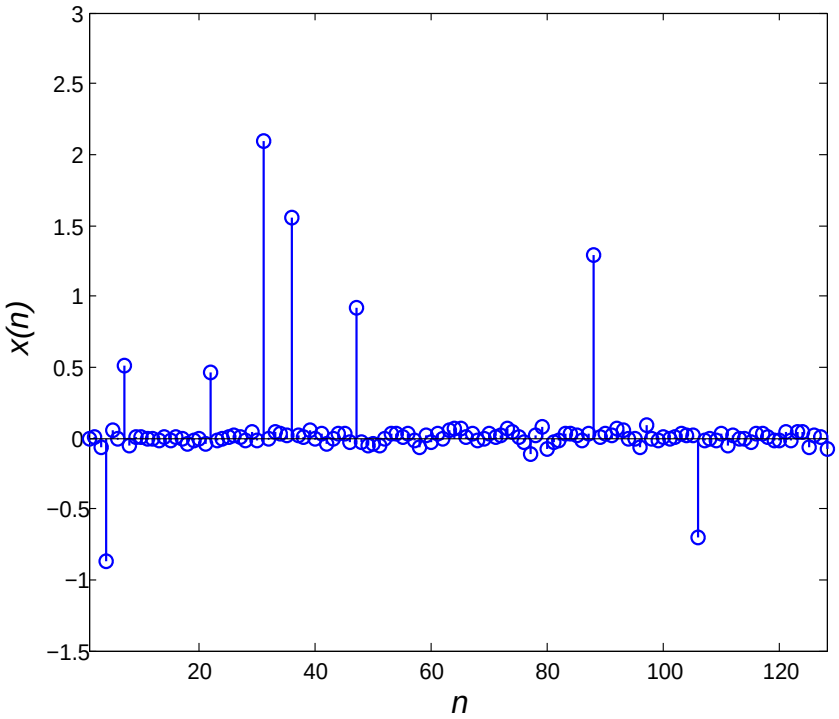
- Compressive Sensing and Signal Recovery
- Image Recovery Using Total-Variation Minimization
- Image Recovery Using Nonconvex Total-Variation Minimization
- Performance Evaluation

# Compressive Sensing and Signal Recovery

- A signal $\mathbf{x}(n)$ of length $N$ is $K$-sparse if it contains $K$ nonzero components with $K \ll N$.
- A signal is near $K$-sparse if it contains $K$ significant components.




A 8-sparse signal

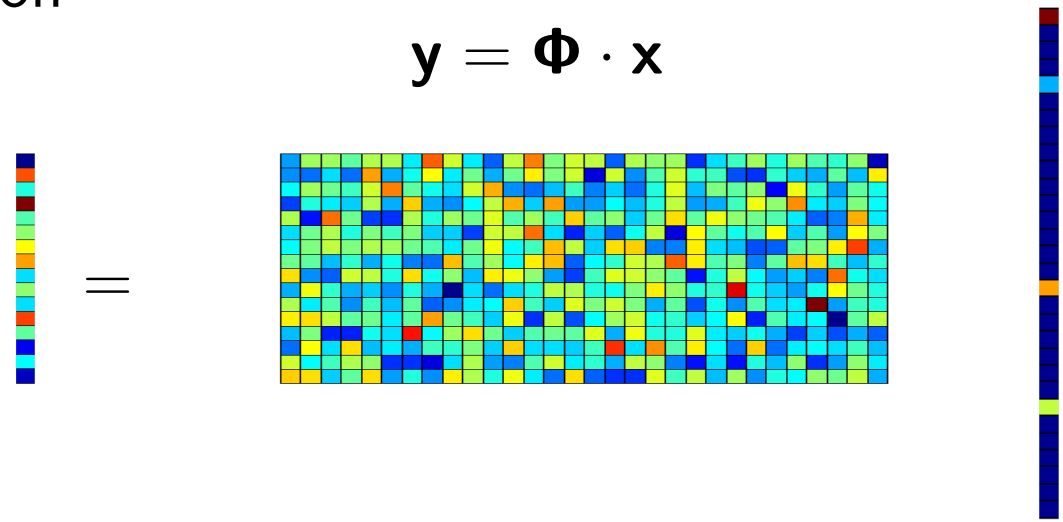



A near 8-sparse signal

# Compressive Sensing and Signal Recovery, cont'd

- Compressive sensing (CS) is a data acquisition process whereby a sparse signal $\mathbf{x}$ or an image $\mathbf{X}$ represented by a vector $\mathbf{x}$ of length $N$ can be determined using a small number of projections represented by a matrix $\mathbf{\Phi}$ of dimension $M \times N$.
- In CS, measurement vector $\mathbf{y}$ and signal vector $\mathbf{x}$ are interrelated by the equation

$$\mathbf{y} = \mathbf{\Phi} \cdot \mathbf{x}$$



16 measurements | projection matrix of size $16 \times 30$ | 4-sparse signal of length 30

# Compressive Sensing and Signal Recovery, cont'd

- A sparse signal $\mathbf{x}$ can be recovered by using an $\ell_1$-norm minimization that solves the problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{x}||_1 = \sum_{i=1}^{N} |x_i| \\ \text{subject to} \quad & \mathbf{y} = \mathbf{\Phi x} \end{aligned}$$

- An $\ell_p$-pseudonorm minimization that solves the problem

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & ||\mathbf{x}||_p^p = \sum_{i=1}^{N} |x_i|^p \\ \text{subject to} \quad & \mathbf{y} = \mathbf{\Phi x} \end{aligned}$$

where a $p$ in the range $0 < p < 1$ can be used to yield a sparser signal.

# Image Recovery Using Total-Variation Minimization

- Many synthetic and natural images have a *spatially sparse* gradient.
- The spatial gradient of an image $\mathbf{X}$ of size $n_1 \times n_2$ can be obtained as a matrix $\mathbf{G}$ of size $n_1 \times n_2$ whose $\{i, j\}$th component is given by

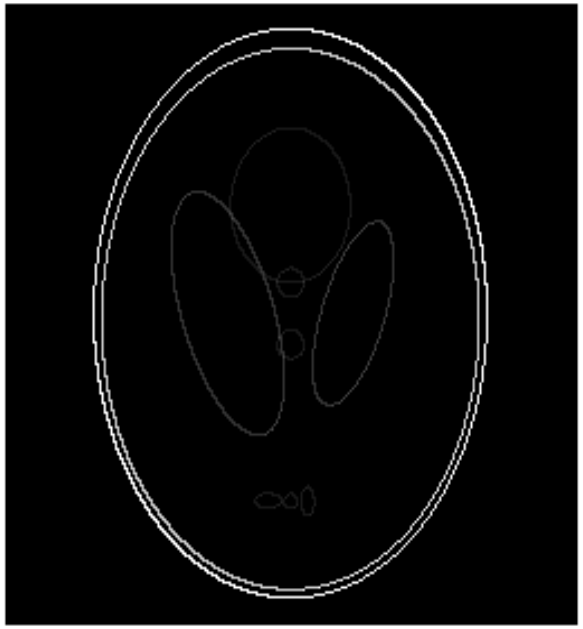
$$g_{i,j} = \begin{cases} \sqrt{\left(x_{i,j} - x_{i+1,j}\right)^2 + \left(x_{i,j} - x_{i,j+1}\right)^2} & \text{for} \quad \begin{cases} 1 \leq i < n_1, \\ 1 \leq j < n_2 \end{cases} \\ \left|x_{i,j} - x_{i+1,j}\right| & \text{for} \quad \begin{cases} j = n_2, \\ 1 \leq i < n_1 \end{cases} \\ \left|x_{i,j} - x_{i,j+1}\right| & \text{for} \quad \begin{cases} i = n_1, \\ 1 \leq j < n_2 \end{cases} \\ 0 & \text{for} \quad i = n_1, j = n_2 \end{cases}$$

where $x_{i,j}$ is the $\{i, j\}$th component of $\mathbf{X}$.

# Image Recovery Using Total-Variation Minimization, cont'd

- The Shepp-Logan Phantom image has a sparse spatial gradient:

Phantom image



Sparse spatial gradient of Phantom image

# Image Recovery Using Total-Variation Minimization, cont'd

- The Cameraman image has near-sparse spatial gradient:

Cameraman image

Near-sparse spatial gradient of Cameraman image

# Image Recovery Using Total-Variation Minimization, cont'd

- The sparsity of the spatial gradient of an image $\mathbf{X}$ can be measured in terms of the total-variation norm given by

$$TV(\mathbf{X}) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} g_{i,j}$$

where $g_{i,j}$ is the $\{i,j\}$th element of matrix $\mathbf{G}$.

- The smaller the $TV(\mathbf{X})$, the sparser the gradient of $\mathbf{X}$.
- An image $\mathbf{X}$ with sparse spatial gradient represented by a vector $\mathbf{x}$ can be recovered from measurements $\mathbf{y}$ by solving the optimization problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad \frac{1}{2}\,||\mathbf{\Phi x} - \mathbf{y}||_2^2 + \lambda TV(\mathbf{X})$$

where $\lambda$ a regularization parameter.

# Image Recovery Using Nonconvex Total-Variation Minimization

- Inspired by the success of $\ell_p$ over $\ell_1$ minimization in CS, we consider the nonconvex version of the TV norm, called the $TV_p$ pseudonorm, given by

$$TV_p(\mathbf{X}) = \left[\sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\left({x'^{i}_{i,j}}^{2} + {x'^{j}_{i,j}}^{2}\right)^{p/2} + \sum_{i=1}^{n_1-1}\left({x'^{i}_{i,n_2}}^{2}\right)^{p/2} + \sum_{j=1}^{n_2-1}\left({x'^{j}_{n_1,j}}^{2}\right)^{p/2}\right]^{1/p}$$

where $x'^{i}_{i,j} = x_{i,j} - x_{i+1,j}$, $x'^{j}_{i,j} = x_{i,j} - x_{i,j+1}$, and $0 < p < 1$.

- From the nonconvexity and nondifferentiability of the $\ell_p$ pseudonorm, it follows that function $TV_p(\mathbf{X})$ remains nonconvex and nondifferentiable for $p < 1$.

# Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- To render the $TV_p$ pseudonorm differentiable and to facilitate its optimization, we consider the approximate $TV_p$ pseudonorm given by

$$
\begin{aligned}
TV^p_{p,\epsilon}(\mathbf{X}) &= \sum_{i=1}^{n_1-1}\sum_{j=1}^{n_2-1}\left({x'^{i}_{i,j}}^{2} + {x'^{j}_{i,j}}^{2} + \epsilon^2\right)^{p/2} \\
&\quad + \sum_{i=1}^{n_1-1}\left({x'^{i}_{i,n_2}}^{2} + \epsilon^2\right)^{p/2} + \sum_{j=1}^{n_2-1}\left({x'^{j}_{n_1,j}}^{2} + \epsilon^2\right)^{p/2}
\end{aligned}
$$

  where $\epsilon$ is a nonzero parameter used to render it differentiable.

- Note that $\quad TV^p_{p,\epsilon}(\mathbf{X}) \to TV(\mathbf{X}) \quad$ as $\quad \epsilon \to 0, p \to 1.$

# Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The reconstruction involves solving the optimization problem

$$(\mathbf{P\text{-}TV}_p) \qquad \underset{\mathbf{x}}{\text{minimize}} \quad F_{\lambda,p,\epsilon}(\mathbf{X}) = \frac{1}{2}\,||\mathbf{\Phi x} - \mathbf{y}||_2^2 + \lambda\, TV_{p,\epsilon}^p(\mathbf{X})$$

  for a small values $\epsilon_T$ and $\lambda_T$ of $\epsilon$ and $\lambda$, respectively, and $p < 1$.

- The gradient of the objective function $F_{\lambda,p,\epsilon}(\mathbf{X})$ can be evaluated as

$$\mathbf{g} = \mathbf{\Phi}^T\left(\mathbf{\Phi x} - \mathbf{y}\right) + \lambda p \mathbf{u}$$

  where $\mathbf{u}$ is a vector representing the gradient of $TV_{p,\epsilon}^p(\mathbf{X})/p$.

# Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- The problem **P-TV**$_p$ can be solved by using the following sequential procedure:
  - Select $\{\epsilon = \epsilon_1, \lambda = \lambda_1\}$ so that $\{\epsilon_1 > \epsilon_T, \lambda_1 > \lambda_T\}$, set the zero vector as initializer, and solve problem **P-TV**$_p$. Denote the resulting solution as $\mathbf{x}^*$.
  - Using $\mathbf{x}^*$ as the initializer, solve problem **P-TV**$_p$ again for smaller values of $\epsilon$ and $\lambda$.
  - Repeat this procedure until problem **P-TV**$_p$ is solved for the pair $\{\epsilon = \epsilon_T, \lambda = \lambda_T\}$. Denote the final solution as $\mathbf{x}_T^*$.
  - Construct image $\mathbf{X}^*$ from the final solution $\mathbf{x}_T^*$.
  - Output $\mathbf{X}^*$ and stop.
- The Fletcher-Reeves' conjugate-gradient (FR-CG) technique can be applied to solve problem **P-TV**$_p$ for a given pair of values of $\{\epsilon, \lambda\}$.

# Image Recovery Using Nonconvex Total-Variation Minimization, cont'd

- In the FR-CG technique, iterate $\mathbf{x}_k$ is updated to $\mathbf{x}_{k+1}$ as

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$$

where

$$\mathbf{d}_k = -\mathbf{g}_k + \beta_{k-1}\mathbf{d}_{k-1},$$

$$\beta_{k-1} = \frac{||\mathbf{g}_k||_2^2}{||\mathbf{g}_{k-1}||_2^2},$$

and $\mathbf{g}_k$ is the gradient at $\mathbf{x} = \mathbf{x}_k$.

- Step size $\alpha_k$ is obtained by using the recursion

$$\alpha_{l+1} = G(\alpha_l) \quad \text{for} \quad l = 2, 3, \ldots$$

with $\alpha_0 \geq 0$ where function $G(\alpha)$ depends on $\mathbf{x}_k$, $\mathbf{\Phi}$, $\mathbf{d}_k$, $\mathbf{y}$, $\epsilon$, and $p$.

# Performance Evaluation

- The performance of the proposed $TV_p$-RLS and conventional $TV$-RLS algorithms was tested using six images, namely,
  - "Circles", "Resolution Chart", and "Shepp-Logan Phantom" having sparse spatial gradient and
  - "Cameraman", "Aeroplane", and "Clock" having near-sparse spatial gradient.
- The image reconstruction performance was measured in terms of the peak signal-to-noise ratio (PSNR) which is defined as

$$PSNR = 20 \log \left( \frac{I_{MAX}}{\sqrt{MSE}} \right) dB$$

where $I_{MAX} = 2^b - 1$ and $b = 8$ is the number of bits used to encode the components of image $\mathbf{X}$.

- The mean-square error is defined as

$$MSE = \frac{1}{n_1 n_2} \left\| \mathbf{X} - \hat{\mathbf{X}} \right\|_F^2$$

# Performance Evaluation, cont'd

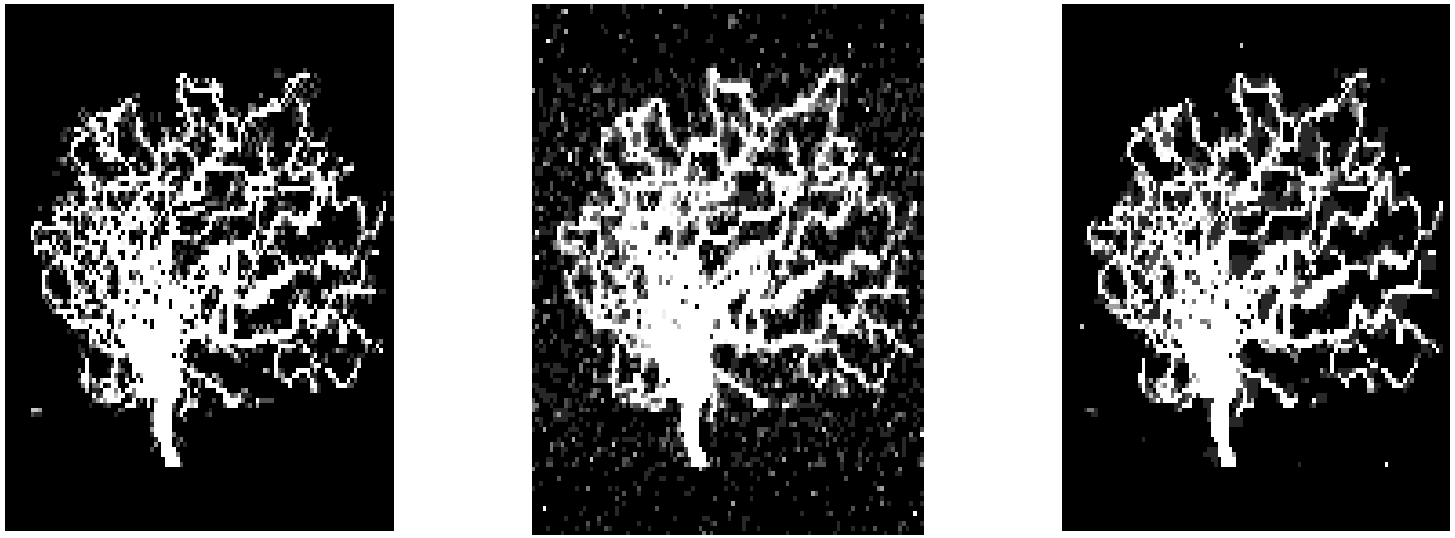
- Experimental results:

PSNR and CPU Time for $TV_p$-RLS and $TV$-RLS Algorithms

| Images | $TV_p$-RLS ($p = 0.5$) | | $TV$-RLS | |
|---|---|---|---|---|
| | PSNR (dB) | CPU time (seconds) | PSNR (dB) | CPU time (seconds) |
| Cameraman | 32.8 | 47.1 | 32.2 | 952.8 |
| Aeroplane | 41.7 | 49.1 | 41.5 | 767.0 |
| Circles | 90.1 | 43.6 | 58.4 | 483.0 |
| Clock | 38.4 | 48.1 | 37.3 | 911.4 |
| Resolution Chart | 74.6 | 45.0 | 49.7 | 1201.7 |
| Shepp-Logan | 86.5 | 44.1 | 76.2 | 121.2 |

# Performance Evaluation, cont'd

- Reconstruction of an angiogram of size $256 \times 256$:

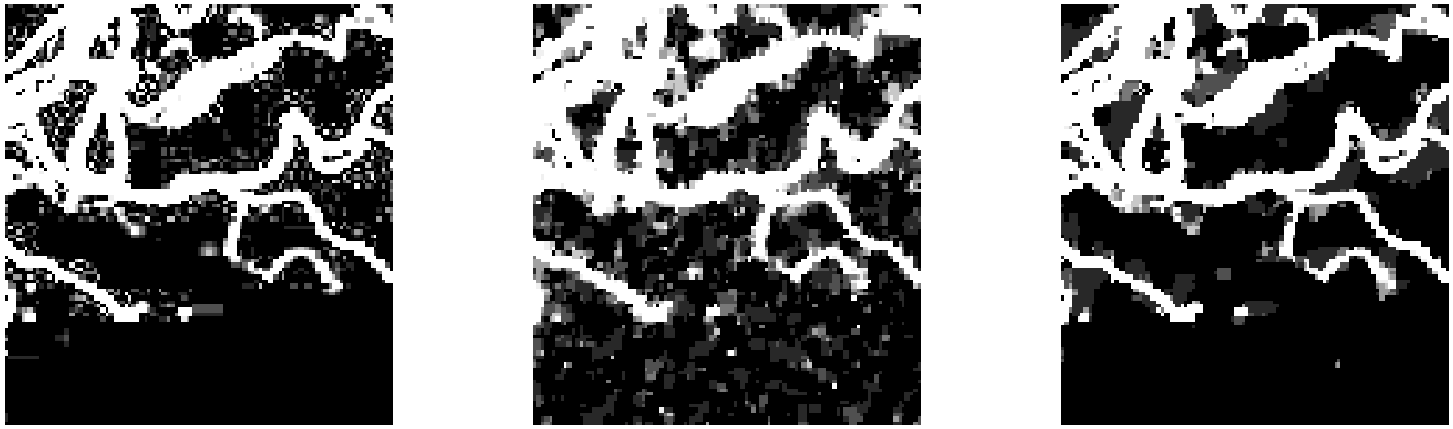


(a) (b) (c)

(a) Original angiogram
(b) Angiogram reconstructed using $TV_p$-RLS algorithm with $p = 0.5$
(c) Angiogram reconstructed using $TV$-RLS algorithm

# Performance Evaluation, cont'd

- Segments of the angiograms shown in Slide 17 for the range $120 \leq n_y \leq 220, 120 \leq n_x \leq 220$ where $n_y$ and $n_x$ are pixel indices for vertical and horizontal directions, respectively:



(a) (b) (c)

(a) Original angiogram
(b) Angiogram reconstructed using $TV_p$-RLS algorithm with $p = 0.5$
(c) Angiogram reconstructed using $TV$-RLS algorithm

# Conclusions

- Compressive sensing is an effective technique for sampling sparse signals.
- $\ell_1$ and $\ell_p$ minimizations work in general for the reconstruction of sparse signals.
- Total variation minimization is effective for the reconstruction of images.
- Nonconvex total-variation minimization offers improved reconstruction performance relative to the total-variation minimization for images with sparse spatial gradient.