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Asymptotic Solutions of the Equations for a Viscous Heat-Conducting Compressible Medium

E. P. Stolyarov

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Abstract — Small nonstationary perturbations in a viscous heat-conducting compressible medium are analyzed on the basis of the linearization of the complete system of hydrodynamic equations for small Knudsen numbers ($\mathrm{Kn} \ll 1$). It is shown that the density and temperature perturbations (elastic perturbations) satisfy the same wave equation which is an asymptotic limit of the hydrodynamic equations far from the inhomogeneity regions of the medium (rigid, elastic or fluid boundaries) as $\mathrm{M}_a = v/a \to 0$, where v is the perturbed velocity and a is the adiabatic speed of sound. The solutions of the new equation satisfy the first and second laws of thermodynamics and are valid up to the frequencies determined by the applicability limits of continuum models. Fundamental solutions of the equation are obtained and analyzed. The boundary conditions are formulated and the problem of the interaction of a spherical elastic harmonic wave with an infinite flat surface is solved. Important physical effects which cannot be described within the framework of the ideal fluid model are discussed.

Keywords: viscosity, heat conduction, compressible fluid, elastic waves, dispersion relation, diffusion, dissipation, nonequilibrium, induced emission, asymptotic expansions.

Every motion of a compressible continuum, excluding the trivial motion as a rigid body at a velocity constant in magnitude and direction or in rotation at a constant angular velocity, is accompanied by a violation of its equilibrium state. The hydrodynamic description of processes in systems not very remote from equilibrium is based on the mass, momentum and energy conservation laws with flux vectors linearly dependent on the spatial derivatives of the hydrodynamic flow functions: density, velocity, and temperature.

The transport laws were obtained from the solution of the kinetic Boltzmann equation by the Chapman-Enskog method [1, 2] in the first approximation in the Knudsen number $\mathrm{Kn} = l_m/L \ll 1$, where l_m is the mean molecular free path and L is a characteristic scale on which the hydrodynamic flow functions change by their characteristic value. This restriction is basic when describing weakly inhomogeneous hydrodynamic processes. In particular, for a perfect gas with the equation of state $p = \rho RT$ under normal conditions the quantity l_m equals about $2 \cdot 10^{-7}$ m and the mean free path time, close to the equilibrium establishment time, is $\tau_m \approx 5 \cdot 10^{-10}$ s. This bounds the range of elastic oscillation frequencies from above at about 20 MHz and the characteristic scales from below at $L \approx 10^{-4}$ m. At higher frequencies and lower scales we must use the equations of hydrodynamics of fast processes, which take into account the time dependence of the transfer coefficients [3]. Acoustic (sound) oscillations are only a small part of the spectrum of elastic continuum oscillations (from 20 Hz to 20 kHz). For example, in ordinary industrial aerodynamic plant the mean-square amplitudes of the fluctuating pressure component amount to 2–4% of the mean static pressure. The corresponding velocity fluctuation amplitude referred to the speed of sound (acoustic Mach number M_a) may, under the normal conditions, be equal to 10^{-4}. Since in a viscous fluid on a rigid wall the velocity vanishes, in the wall layer there always exists a region where the fluctuating velocity component is of the same order as the mean hydrodynamic velocity. Hence, this region can crucially affect the fluctuating motion near a wall and especially in the neighborhood of critical points.

In this study we will consider certain features of the propagation of small perturbations in a compressible medium at rest due to the presence of viscosity and heat conduction. The effects that arise in the presence of non-moving boundaries in the medium will be illustrated with reference to the solution of the problem of the interaction between an elastic spherical wave emitted by a point periodic source and an infinite rigid plane.

1. One of the simplest nonstationary motions is the small oscillations of a body of finite dimensions moving in a homogeneous medium at rest at a mean velocity $\mathbf{v}_0$ constant in magnitude and direction. At a large distance from the body there is always a region where the hydrodynamic oscillations, decreasing as r^{-2}, become fairly small as compared with the elastic nonstationary perturbations, whose amplitude decreases inversely to the distance from the body (as r^{-1}). The equations of propagation of small nonstationary perturbations in such a flow have a very simple form and in fact coincide with those considered by D. I. Blokhintsev [4]. The characteristic linear scale of the wave motion is the wavelength λ and the time scale is λ/a_0. Elementary estimates show that in the hydrodynamic equations the terms nonlinear in the perturbations are of the order of the acoustic Mach number $\mathrm{M}_a = v/a_0$ as compared with the leading linear terms and the leading viscous terms are of the order of Re_a^{-1}, where the Reynolds number is calculated from the wavelength and the speed of sound. Therefore, we can neglect the nonlinear terms in the equations but must retain the leading viscous terms if

$$\mathrm{Re}_a \cdot \mathrm{M}_a \sim \frac{\lambda v}{\nu} \ll 1, \text{ or } \lambda \ll \frac{\nu}{v}$$

Under normal conditions, at a sound level of about 160 dB, this estimate gives $\lambda \ll 0.3$ m for the wavelength and $f \gg 1$ kHz for the frequency. The linearized equations of nonstationary motion in a viscous heat-conducting compressible fluid can thus be considered asymptotic as $\mathrm{M}_a \to 0$. For a one-component perfect gas (in the absence of chemical reactions and phase transfers), neglecting the temperature dependence of the viscosity and thermal conductivity, we can write down the equations of continuum motion as follows:

$$\begin{aligned}
&\frac{d\mathbf{v}}{dt} + \frac{a^2}{\gamma}\frac{\boldsymbol{\nabla} p}{p} = \nu_1 \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \nu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{v}\\
&\frac{1}{\rho}\frac{d\rho}{dt} + \boldsymbol{\nabla}\cdot\mathbf{v} = 0\\
&\frac{1}{T}\frac{dT}{dt} = -(\gamma-1)\boldsymbol{\nabla}\cdot\mathbf{v} + \gamma\nu_2\frac{\Delta T}{T} + \frac{\gamma(\gamma-1)}{a^2}\Phi\\
&\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}, \quad \nu = \frac{\eta}{\rho}, \quad \nu_1 = \frac{1}{\rho}\left(\frac{4}{3}\eta + \zeta\right), \quad \nu_2 = \frac{\chi}{\rho c_p} = \frac{\nu}{\mathrm{Pr}}
\end{aligned} \tag{1.1}$$

Here, Φ is the dissipative function (see, for example, [6]), χ is the thermal conductivity, η is the shear viscosity, and ζ is the bulk (second) viscosity associated with nonequilibrium processes in the elastic wave. The need to introduce this quantity for processes with large relaxation times was justified by L. I. Mandelstam and M. A. Leontovich [5] and discussed in detail in [6]. From Lighthill's estimates it follows that the second viscosity is equal to 0.8η for nitrogen N_2 and is due to the increased equilibrium establishment time in a diatomic as compared with a monatomic gas resulting from the presence of rotational degrees of freedom in diatomic molecules [2].

Let us introduce the following new variables by dividing the pressure, density and temperature by their characteristic values:

$$\pi = \ln\frac{p}{p_0}, \quad \sigma = \ln\frac{\rho}{\rho_0}, \quad \tau = \ln\frac{T}{T_0}, \quad \boldsymbol{\nabla}\pi = \boldsymbol{\nabla}\sigma + \boldsymbol{\nabla}\tau$$

$$\mathbf{v} = \mathbf{V}_0 + \mathbf{v}_p + \mathbf{v}_s, \quad \mathbf{v}_p = -\boldsymbol{\nabla}\varphi, \quad q = \boldsymbol{\nabla}\cdot\mathbf{v} = -\Delta\varphi, \quad \boldsymbol{\psi} = \boldsymbol{\nabla}\times\mathbf{v}_s$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V}_0\cdot\boldsymbol{\nabla}, \quad a_0^2 = \gamma R T_0$$

In the new variables the linearized equations of disturbed motion can be written in the form:

$$\frac{d\boldsymbol{\psi}}{dt} + \nu\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{\psi} = 0 \tag{1.2}$$

$$\frac{d\sigma}{dt} = -q = \Delta\varphi$$
$$\left(\frac{d}{dt} - \nu_1\Delta\right)\frac{d\sigma}{dt} = \frac{a_0^2}{\gamma}(\Delta\sigma + \Delta\tau) \tag{1.3}$$

$$\left(\frac{d}{dt} - \gamma\nu_2\Delta\right)\tau = (\gamma - 1)\frac{d\sigma}{dt} \tag{1.4}$$

The equation for the rotational velocity component can be separated and is determined by the perturbations of the other hydrodynamic flow functions via the boundary conditions that follow from Eq. (1.1). Eliminating the variable τ from (1.3) and (1.4), we obtain the following equation describing the propagation of small density perturbations σ in a flow with hydrodynamic parameters constant in the mean:

$$\left(\frac{d}{dt} - \gamma\nu_2\Delta\right)\left[\frac{\gamma}{a_0^2}\left(\frac{d}{dt} - \nu_1\Delta\right)\frac{d}{dt} - \Delta\right]\sigma = (\gamma - 1)\Delta\frac{d\sigma}{dt}$$

The temperature perturbations τ satisfy the same equation fourth-order in the spatial variables and second-order in time. The small coefficient of the highest derivative indicates the possible existence of regions with singular perturbations where the spatial derivatives are maximal (boundary layers, neighborhoods of sharp edges, etc.). The first operator on the right side of the equation is the heat-conduction operator and the second (in square brackets) is the wave operator with a dissipative term which depends on the shear and bulk viscosities.

If we neglect the viscosity and the thermal conductivity, we obtain the equation of perturbation propagation in a constant ideal fluid flow

$$\frac{d}{dt}\left(\frac{\gamma}{a_0^2}\frac{d^2}{dt^2} - \gamma\Delta\right)\sigma = 0, \quad \text{or} \quad \left(\frac{1}{a_0^2}\frac{d^2}{dt^2} - \Delta\right)\sigma = F(x - V_0t, y, z)$$

In spite of the presence of a wave operator, this is not a wave equation. It describes density perturbations that “float” at a constant velocity together with the flow and in a medium at rest, in the absence of gravity forces, are stationary and satisfy the Poisson equation. This apparent paradox can be attributed to the fact that the density and temperature perturbations split into two modes, one of which propagates at the speed of sound while the other is entrained by the flow at its convective velocity.

Let us consider certain features of the simplest solutions in free space in a homogeneous medium at rest. The equation

$$\left(\frac{\partial}{\partial t} - \gamma\nu_2\Delta\right)\left[\frac{\gamma}{a_0^2}\left(\frac{\partial}{\partial t} - \nu_1\Delta\right)\frac{\partial}{\partial t} - \Delta\right]\sigma - (\gamma - 1)\Delta\frac{\partial\sigma}{\partial t} = 0 \tag{1.5}$$

admits particular stationary solutions, namely, spherical waves of the form

$$\sigma = A\frac{e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}}{r} \tag{1.6}$$

where A is the complex amplitude, ω is the frequency, and $\mathbf{k}$ is a wave vector whose modulus satisfies the dispersion relation

$$\nu_2k^4(1 - i\gamma\varepsilon_1) - \omega k^2(i + \varepsilon_1 + \gamma\varepsilon_2) + i\frac{\omega^3}{a_0^2} = 0$$
$$\varepsilon_1 = \frac{\nu_1\omega}{a_0^2}, \qquad \varepsilon_2 = \frac{\nu_2\omega}{a_0^2} \tag{1.7}$$

In somewhat different notation, Eq. (1.7) was obtained by Kirchhoff in 1868. It has two pairs of complex conjugate roots

$$k_{1,2} = \pm \frac{\omega}{a_0\sqrt{1 - i\gamma\varepsilon_1}} + O(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2) \tag{1.8}$$

$$k_{3,4} = \pm \sqrt{\frac{\omega}{\nu_2}[i - (\gamma - 1)(\varepsilon_1 - \varepsilon_2)]} + O(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2) \tag{1.9}$$

For a Prandtl number $\mathrm{Pr} = {}^3\!/_4$, $\zeta = 0.8\eta$, and $\gamma = 1.4$

$$k_{1,2} \approx \pm\frac{\omega}{a_0}\sqrt{1 + i\frac{\gamma\omega}{\rho_0 a_0^2}\left(\frac{4}{3}\eta + \zeta\right)} = \pm\frac{\omega}{a_0}\sqrt{1 + i\frac{18.8}{\mathrm{Re}_a}}$$

$$k_{3,4} \approx \pm\sqrt{\frac{\omega\rho_0\mathrm{Pr}}{\eta}\left[i - \omega\frac{\gamma - 1}{\rho_0 a_0^2}\left(\frac{4}{3}\eta + \zeta - \frac{\eta}{\mathrm{Pr}}\right)\right]} = \pm\sqrt{\frac{3}{4}\frac{\omega}{\nu}\left(i - \frac{2}{\mathrm{Re}_a}\right)}$$

Here the Reynolds number Re_a is calculated from the shear viscosity η, the wavelength, the density and the adiabatic speed of sound in the undisturbed medium.

The first pair of complex roots $k_{1,2}$ (1.8) characterizes the density and temperature modes propagating at almost the adiabatic speed of sound (damped elastic waves). The damping factor related with the diffusion processes in the wave is determined by the imaginary part of the wave number and is equal to

$$\alpha \approx \frac{\gamma\omega^2}{2\rho_0 a_0^3}\left(\frac{4}{3}\eta + \zeta\right)$$

while the mechanical energy loss per unit time is $\langle dE_m/dt\rangle \approx 2\alpha a_0\langle E\rangle$. The mechanical energy loss in the wave due to viscous dissipation, absent from the linearized equations, can be approximately estimated by calculating the mechanical energy dissipated per unit volume in unit time, as for a plane acoustic wave (see [6]). Correct to higher-order quantities, we obtain the same estimate

$$\frac{\langle dE_m/dt\rangle}{a_0\langle E\rangle} \approx \frac{\omega^2}{\rho_0 a_0^3}\left[\frac{4}{3}\eta + \zeta + \frac{(\gamma - 1)\chi}{c_p}\right]$$

The total mechanical energy loss in the elastic wave is thus twice as great as in a wave without account for damping. Part of this energy is transformed into elastic oscillation energy: if in Eq. (1.4) we retain the nonlinear term with viscous dissipation, then on the right side of Eq. (1.5) we must add the nonlinear term

$$Q = \Delta\Phi'$$

$$\Phi' = \gamma\left[\nu_2(\nabla\tau)^2 + (\gamma - 1)\frac{\nu_1}{a_0^2}\left(\frac{\partial\sigma}{\partial t}\right)^2\right]$$

The second pair of roots $k_{3,4}$ (1.9) is related with the temperature and density perturbations that remain in the medium in the form of a nonequilibrium entropy wake after the elastic wave has passed. The cause of this phenomenon is known [2, 6] and can easily be explained if we consider the wave in the form of a wave packet [7]. As long as the wave front has not arrived, there are no perturbations in the medium. The acoustic wave disturbs the thermodynamic equilibrium in the homogeneous medium and produces rapid changes of different signs in the temperature and density over a wavelength. The transition to an equilibrium state with an increase in entropy due to viscosity and heat conduction starts inside the wave but does not finish during its passage. After the wave has gone, in the fluid an entropy wake remains in the form of density and temperature inhomogeneities distributed over the medium, disturbing its equilibrium state which continues to be restored with increase in entropy. In a monochromatic wave that continues for an infinitely long time the process of the equilibrium disturbance and restoration is continuous. The second term under the root sign in expression (1.9) characterizes the secondary emission from the wake.

An important consequence of the solution obtained is that within the framework of the continuum equations there is no paradox of an infinite perturbation propagation velocity, as discussed in [3] in connection with fast processes with frequencies of the order of a_0/l_m.

Using the expressions for the complex roots (1.8) and (1.9), we can rewrite the small-perturbation propagation equations in another form equivalent to the initial equation, correct to terms quadratic in ε. Examples of this sort can be found in monograph [7]. The equation describing the velocity and pressure in an elastic wave differs from the classical acoustic equation in containing a dissipative term with a higher derivative:

$$\left(1 + \frac{\gamma \nu_1}{a_0^2}\frac{\partial}{\partial t}\right)\Delta\sigma - \frac{1}{a_0^2}\frac{\partial^2\sigma}{\partial t^2} = 0 \tag{1.10}$$

Equation (1.10) formally coincides with the equation obtained by Stokes (see [8]), differing only with respect to the larger coefficient associated with damping. The temperature and density perturbations break down into two components, one of which (wave component) can be described by the above equation and the other by a diffusion equation which can be written in the form:

$$\nu_2\Delta\tau_d - \frac{\partial}{\partial t}\left[1 - \frac{\gamma - 1}{a_0^2}(\nu_1 - \nu_2)\frac{\partial}{\partial t}\right]\tau_d = 0 \tag{1.11}$$

Equation (1.11) differs from the classical heat conduction equation with respect to the term containing the second time derivative with a small coefficient proportional to the bulk viscosity. This equation is of the elliptic type. It is worth noting that without account for the bulk viscosity at $\mathrm{Pr} = {}^3\!/_4$ this equation goes over into the classical heat conduction equation and with no account for viscosity at all ($\nu_1 = 0$) into the wave (telegraph) equation of hyperbolic type to which the equations of hydrodynamics of fast processes reduce [3].

The general solution of Eq. (1.5) for periodic perturbations can be written following Rayleigh [8]. We will assume the hydrodynamic functions to be proportional to e^{int} and write down the general solution for the temperature in the form:

$$\tau = A_1 q_1 + A_2 q_2 \tag{1.12}$$

where the functions q_1 and q_2 are solutions of the equations

$$\begin{aligned} \Delta q_1 &= \lambda_1^2 q_1, \qquad \lambda_1^2 = \frac{n^2}{a_0^2}\left[1 - \frac{n}{a_0^2}(\nu_1 + \gamma\nu_2)\right] \\ \Delta q_2 &= \lambda_2^2 q_2, \qquad \lambda_2^2 = \frac{n}{\nu_2}\left[1 - \frac{n}{a_0^2}(\gamma - 1)(\nu_1 - \nu_2)\right] \end{aligned} \tag{1.13}$$

The perturbed velocity vector is determined by the relations

$$\mathbf{v} = B_1\nabla q_1 + B_2\nabla q_2 + A_3\mathbf{q} \tag{1.14}$$

where the vector $\mathbf{q}$ is a solution of the equations (with constant B_1 and B_2)

$$\Delta\mathbf{q} = \lambda_3^2\mathbf{q}, \qquad \nabla\cdot\mathbf{q} = 0, \qquad \lambda_3^2 = \frac{n}{\nu} \tag{1.15}$$

$$B_1 = A_1\left(\nu - \frac{n}{\lambda_1^2}\right), \qquad B_2 = A_2\left(\nu - \frac{n}{\lambda_2^2}\right)$$

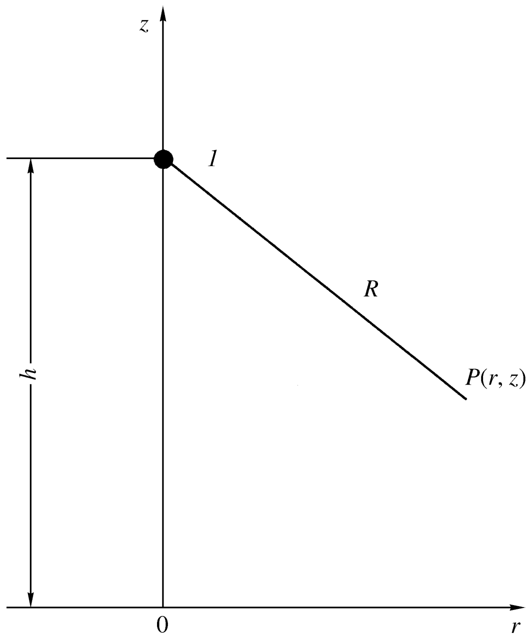

Fig. 1. Point source *1* above an absolutely rigid plane.

The density perturbations can be expressed in terms of the temperature perturbations by the relation (C_1 and C_2 are constants)

$$\sigma = C_1 q_1 + C_2 q_2 \tag{1.16}$$

$$C_1 = A_1 \frac{n - \gamma \nu_2 \lambda_1^2}{n(\gamma - 1)}, \qquad C_2 = A_2 \frac{n - \gamma \nu_2 \lambda_2^2}{n(\gamma - 1)}$$

These results were applied by Kirchhoff and Rayleigh to the problems of plane wave propagation in channels and can be used in solving other similar problems.

In conclusion, we note that the elastic waves describe the asymptotic nonstationary hydrodynamic field far from the source. Therefore, an asymptotic solution can be obtained only correct to certain constants which can be found from the requirement of matching with the solution near the source. In order to construct the global solution, we must construct a local solution near the source and then use the method of matched asymptotic expansions [9] or other asymptotic methods [10].

2. As an example, we will consider the solution of the problem of a spherical wave impinging on an unbounded rigid wall. The interaction of a plane wave with an absolutely rigid and heat-conducting wall was considered in detail in [11]. From an analysis of the solution it follows that at angles of incidence close to 90° anomalously high (up to 60%) acoustic absorption should be observed. In [12], where the problem of the acoustic field of a point source near a rigid boundary was considered without account for heat conduction or second viscosity, using the reflection coefficient for plane waves taken from [11], the above-mentioned anomalously high absorption effect was attributed to the existence of a cylindrical (inhomogeneous) wave which propagates along the plane as in a waveguide and entrains part of the energy emitted from the source. In the present study we will consider the complete problem of the temperature, density and velocity fields near a rigid surface without the simplifications used in [12] and will give a somewhat different interpretation of the resulting effects.

Let us consider the steady-state, that is, lasting for an infinitely long time, field of a spherically symmetric source located in a homogeneous medium at rest at a distance h from a plane surface and oscillating with the frequency ω. We will use a cylindrical coordinate system with the symmetry axis perpendicular to the plane and passing through the source center (see Fig. 1). We will assume that the distance h is much greater than the characteristic dimension of the source L ($h \gg L$) and that the reflection from the plane does not affect the source. We will relate the linear dimensions to the scale h, velocities to the speed of sound a_0, and amplitudes to the amplitude in the incident wave. The temperature in the spherical wave can be described by the function

$$q_0 = \frac{\exp\{\lambda_1 R_0 - ikt\}}{R_0} \tag{2.1}$$

and in the wave reflected from the surface by the function

$$A_1 q_1 = A_1 \frac{\exp\{\lambda_1 R_1 - ikt\}}{R_1} \tag{2.2}$$

$$R_0 = \sqrt{r^2 + (z-1)^2}, \quad R_1 = \sqrt{r^2 + (z+1)^2}, \quad \lambda_1 = ik\sqrt{1 + ik(\varepsilon_1 + \gamma\varepsilon_2)}$$

$$\varepsilon_1 = \frac{\nu_1}{a_0 h}, \qquad \varepsilon_2 = \frac{\nu_2}{a_0 h}, \qquad k = \frac{\omega h}{a_0}$$

Here, A_1 is the complex reflection coefficient. Due to the symmetry of the problem the velocity perturbation field has two components: tangential u parallel to the plane and v normal to it. On the rigid wall we assign the no-slip conditions and require the heat flux to be equal to zero, and at infinity require the solution to be bounded. Thus, in the case of a thermally insulated wall, the boundary conditions can be written in the form:

$$z = 0, \qquad u = v = 0, \quad \frac{\partial \tau}{\partial z} = 0$$

$$R \to \infty, \qquad u \to 0, \quad v \to 0, \quad \tau \to 0$$

In the case of a heat-conducting wall, only the boundary condition for the temperature on the surface will change:

$$z = 0, \qquad u = v = 0, \quad \tau = 0$$

If the source emission is steady, the problem formulated can be reduced to solving a system of ordinary differential equations by the Hankel transform method. To do this we introduce the following transforms of the unknown functions:

$$Q_i = \int_0^\infty rJ_0(\xi r) q_i \, dr, \qquad T = \int_0^\infty rJ_0(\xi r)\tau \, dr, \qquad R = \int_0^\infty rJ_0(\xi r)\sigma \, dr$$

$$U = \int_0^\infty rJ_1(\xi r) u \, dr, \qquad V = \int_0^\infty rJ_0(\xi r) v \, dr$$

where $J_0(x)$ and $J_1(x)$ are Bessel functions of zeroth and first order, respectively. Multiplying relations (1.12)–(1.16) by $J_0(\xi r)$, integrating from zero to infinity, and using the well-known relations for Bessel functions [13] and the expansions of functions (2.1) and (2.2) in cylindrical waves [7], we obtain:

$$\frac{d^2 Q_i}{dz^2} = \alpha_i Q_i, \quad \alpha_i = \sqrt{\xi^2 + \lambda_i^2}, \quad \lambda_1^2 = -k^2[1 + ik(\varepsilon_1 + \gamma\varepsilon_2)] \tag{2.3}$$

$$\lambda_2^2 = -\frac{ik}{\varepsilon_2}[1 + ik(\gamma - 1)(\varepsilon_1 - \varepsilon_2)], \quad \lambda_3^2 = -\frac{ik}{\varepsilon}, \quad \varepsilon = \frac{\nu}{a_0 h}$$

$$Q_0 = \frac{\exp\{\alpha_1 |z-1|\}}{\alpha_1}, \quad Q_1 = \frac{\exp\{-\alpha_1(z-1)\}}{\alpha_1}, \quad Q_2 = \exp(\alpha_2 z), \quad Q_3 = \exp(\alpha_3 z)$$

$$T = Q_0 + A_1 Q_1 + A_2 Q_2$$
$$U = -b_1 \xi Q_0 - A_1 b_1 \xi Q_1 - A_2 b_2 \xi Q_2 - A_3 \frac{\alpha_3}{\xi} Q_3$$
$$V = \alpha_1 b_1 Q_0 - A_1 \alpha_1 b_1 Q_1 + A_2 \alpha_2 b_2 Q_2 + A_3 Q_3$$
$$R = c_1 Q_0 + A_1 c_1 Q_1 + A_2 c_2 Q_2$$
$$b_1 = \varepsilon + \frac{ik}{\lambda_1^2}, \quad b_2 = \varepsilon + \frac{ik}{\lambda_2^2}, \quad c_1 = \frac{k - i\gamma\varepsilon_2\lambda_1^2}{k(\gamma-1)}, \quad c_2 = \frac{k - i\gamma\varepsilon_2\lambda_2^2}{k(\gamma-1)}$$

Using the boundary conditions, we obtain expressions for the coefficients A_i. In the case of a thermally insulated wall ($dT_w/dz = 0$)

$$A_1 = \frac{1}{D}\left[\alpha_1\alpha_3\left(1 - \frac{b_2}{b_1}\right) - \xi^2\left(1 - \frac{\alpha_1}{\alpha_2}\frac{b_2}{b_1}\right)\right], \quad A_2 = -\frac{2}{\alpha_2 D}\exp\{-\alpha_1\}$$
$$A_3 = -\frac{2b_1\xi^2\exp(-\alpha_1)}{D}\left(1 - \frac{b_2}{b_1}\right), \quad D = \alpha_1\alpha_3\left(1 - \frac{b_2}{b_1}\right) + \xi^2\left(1 + \frac{\alpha_1}{\alpha_2}\frac{b_2}{b_1}\right)$$

For an absolutely heat-conducting wall ($T_w = 0$), the expressions for A_1, A_2, and D change:

$$A_1 = \frac{1}{D}\left[\alpha_3\left(\alpha_1 - \alpha_2\frac{b_2}{b_1}\right) - \xi^2\left(1 - \frac{b_2}{b_1}\right)\right], \quad A_2 = -\frac{2\alpha_3}{D}\exp\{-\alpha_1\}$$
$$D = \alpha_3\left(\alpha_1 + \alpha_2\frac{b_2}{b_1}\right) + \xi^2\left(1 - \frac{b_2}{b_1}\right)$$

In the leading approximation in ε_i the expressions for the Hankel transforms of the unknown functions are as follows:

$$T = \frac{\exp\{-\alpha_1|z-1|\}}{\alpha_1} + A_1\frac{\exp\{-\alpha_1(z+1)\}}{\alpha_1} + A_2\exp\{\alpha_2 z\}$$
$$U = \frac{i\xi\exp\{-\alpha_1|z-1|\}}{k\alpha_1} + A_1\frac{i\xi\exp\{-\alpha_1(z+1)\}}{k\alpha_1} - A_3\frac{\alpha_3\exp\{\alpha_3 z\}}{\xi}$$
$$V = -\frac{i\exp\{-\alpha_1|z-1|\}}{k} + A_1\frac{i\exp\{-\alpha_1(z+1)\}}{k} + A_3\exp\{\alpha_3 z\}$$
$$R = \frac{\exp\{-\alpha_1|z-1|\}}{\alpha_1(\gamma-1)} + A_1\frac{\exp\{-\alpha_1(z+1)\}}{\alpha_1(\gamma-1)} - A_2\exp\{\alpha_2 z\}$$

where in the case of a thermally insulated wall ($dT/dz = 0$) the constants are determined by the formulas

$$A_1 = \frac{\alpha_1\alpha_3 - \xi^2}{D}, \quad A_2 = -\frac{2\xi^2}{\alpha_2 D}\exp\{-\alpha_1\}, \quad A_3 = \frac{2i\xi^2}{kD}\exp\{-\alpha_1\}, \quad D = \alpha_1\alpha_3 + \xi^2$$

In the case of an absolutely heat-conducting wall ($T_w = 0$), only the expression for A_2 changes:

$$A_2 = -\frac{2\alpha_3}{D}\exp\{-\alpha_1\}$$

The temperature, density and velocity fields are determined by the inverse Hankel transform in accordance with the relations

$$\tau(r,\ z;\ k) = \int_0^\infty \xi J_0(\xi r) T(z;\ k,\ \xi)\, d\xi, \quad \sigma(r,\ z;\ k) = \int_0^\infty \xi J_0(\xi r) R(z;\ k,\ \xi)\, d\xi$$

$$u(r,\ z;\ k) = \int_0^\infty \xi J_1(\xi r) U(z;\ k,\ \xi)\, d\xi, \quad v(r,\ z;\ k) = \int_0^\infty \xi J_0(\xi r) V(z;\ k,\ \xi)\, d\xi$$

Far from the wall the reflected temperature field can be expressed by the integral

$$\tau_1 \int_0^\infty \frac{\sqrt{\xi^2 + \lambda_1^2}\sqrt{\varepsilon\xi^2 - ik} - \sqrt{\varepsilon}\xi^2}{\sqrt{\xi^2 + \lambda_1^2}\sqrt{\varepsilon\xi^2 - ik} + \sqrt{\varepsilon}\xi^2} \exp\left\{-(z+1)\sqrt{\xi^2 + \lambda_1^2}\right\} \xi J_0(\xi r)\, d\xi \tag{2.4}$$

and the density field differs from (2.4) in the same approximation only with respect to the multiplier $(\gamma - 1)^{-1}$.

From the solution there follow the expressions for the induced vorticity field

$$\psi = (\nabla \times \mathbf{v})_\varphi = -\frac{2}{\sqrt{\varepsilon}} \int_0^\infty \frac{\xi^2 J_1(\xi r) \exp\left\{-\sqrt{\xi^2 + \lambda_1^2}\right\}}{\sqrt{\xi^2 + \lambda_1^2}\sqrt{\varepsilon\xi^2 - ik} + \sqrt{\varepsilon}\xi^2} \exp\left\{\frac{z}{\sqrt{\varepsilon}}\sqrt{\varepsilon\xi^2 - ik}\right\} d\xi \tag{2.5}$$

and for the density gradient normal to the wall

$$\left(\frac{\partial \sigma}{\partial z}\right)\bigg|_{z=0} = \frac{2\gamma\sqrt{\varepsilon}}{\gamma - 1} \int_0^\infty \frac{\xi^3 J_0(\xi r) \exp\left\{-\sqrt{\xi^2 + \lambda_1^2}\right\}}{\sqrt{\xi^2 + \lambda_1^2}\sqrt{\varepsilon\xi^2 - ik} + \sqrt{\varepsilon}\xi^2} d\xi \tag{2.6}$$

where $\varepsilon = \eta/\rho_0 a_0 h$ is a small parameter and the quantity λ_1^2 is defined in (2.3) .

The integral expressions (2.4)–(2.6) are very complicated for finding a simple analytical solution as the integrands have poles and branch points. A similar problem arises, for example, in connection with the propagation of electromagnetic waves over the earth's surface [7]. In principle, for arbitrary fixed r, z, and k a solution can be found by numerical integration, since all the singularities lie outside the real axis and the integrals exist and converge. We will restrict ourselves to a few qualitative conclusions which can be made without calculations, on the basis of the form of the solutions obtained.

As follows from expression (2.5), the solution for the vorticity is singular: it tends to infinity as $\varepsilon \to 0$ but remains finite for any $\varepsilon > 0$ and exponentially decreases with distance from the wall. These are nothing other than the transverse shear waves, well-known from hydrodynamics courses, which appear due to the no-slip boundary condition. The vorticity excited leads to the appearance of a non-zero derivative of the density normal to the wall (2.6) and, as a result, to secondary induced dipole emission. The temperature and density variation is determined by a complex reflection coefficient (multiplier of the exponential in integral (2.4)) which differs considerably from that assumed in [12]. The presence of heat transfer on the wall ($q_w \neq 0$) or the wall elasticity ($v_w \neq 0$) may be expected to have a considerable effect on the reflectivity and the induced emission in the wall region.

Due the linearity of the problem the method developed can be generalized to include an arbitrary distribution of sources for stationary boundary conditions. If the boundary conditions are nonstationary or the source emission is nonsteady, then instead of the Fourier transform the Laplace transform must be used.

The propagation of perturbations in a moving inhomogeneous medium differs considerably from that considered. In a viscous heat-conducting fluid, in addition to the effects described by classical acoustics (interference, diffraction, resonance effects, etc.), novel effects are manifested: density and temperature perturbations in the entropy wave wake, which are transported at the local flow velocity; secondary emission of elastic waves from the flow inhomogeneity region, accompanied by the interaction of the base flow with the density and temperature perturbations; formation of a nonstationary vortex layer near the homogeneity boundaries, which inevitably affects the flow development in the boundary layer; heat-transfer effect on the transformation of an ordered (wave) motion into chaotic (molecular) form. All these problems go well beyond the framework of a journal paper and need to be discussed separately.

Summary. A linearized equation, the asymptotic limit of the initial hydrodynamic equations for a viscous heat-conducting compressible medium, is derived. Its particular solutions describing the propagation of elastic waves and the formation of an entropy wake behind them, far from the body and other homogeneity boundaries, are analyzed. An analytical expression for the damping coefficient of elastic waves in a homogeneous medium at rest is found. The solution of the problem of the interaction between a spherical harmonic wave and an infinite flat surface is constructed. Integral expressions for the temperature, density and velocity fields are obtained. It is shown that in a thin layer near a rigid boundary an induced nonstationary vortex field and on the boundary a non-zero normal pressure gradient arise.

The results obtained can be applied in the asymptotic analysis of nonstationary boundary layer flows, for developing methods of delaying or preventing flow transition from the laminar to the turbulent regime, and for constructing global solutions of nonstationary problems on the basis of the method of matched asymptotic expansions.

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