Continuous-Time Signals and Systems
Annotated Lecture Examples
Edition 2020-05-01

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Formulas and Tables

Useful Formulae and Other Information

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$\mathscr{F}^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0)$$

$$a_k = \tfrac{1}{T} X_T(k\omega_0)$$

$$\mathscr{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\mathscr{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

$$\mathscr{U}\mathscr{L}\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \tfrac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

$$\sin\theta = \tfrac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

x	$\cos x$	$\sin x$
0	1	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{3\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
π	-1	0
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
$\frac{3\pi}{2}$	0	-1
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$

$$A_k = (v - p_k)F(v)\big|_{v=p_k}$$

$$A_{kl} = \frac{1}{(q_k - l)!}\left[\frac{d^{q_k - l}}{dv^{q_k - l}}\left[(v-p_k)^{q_k}F(v)\right]\right]\bigg|_{v=p_k}$$

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Fourier Series Properties

Property	Time Domain	Fourier Domain
Linearity	$Ax(t)+By(t)$	Aa_k+Bb_k
Time-Domain Shifting	$x(t-t_0)$	$e^{-jk\omega_0 t_0}a_k$
Time Reversal	$x(-t)$	a_{-k}

Fourier Transform Properties

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1(t)*x_2(t)$	$X_1(\omega)X_2(\omega)$
Frequency-Domain Convolution	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Parseval's Relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2\,d\omega$	

Fourier Transform Pairs

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert\operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}Bt$	$\operatorname{rect}\frac{\omega}{2B}$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$

Bilateral Laplace Transform Properties

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1\cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}\{s_0\}$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1\cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R\cap\{\mathrm{Re}\{s\}>0\}$

Property	
Initial Value Theorem	$x(0^+)=\lim_{s\to\infty}sX(s)$
Final Value Theorem	$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$

Unilateral Laplace Transform Properties

Property	Time Domain	Laplace Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$
Time/Laplace-Domain Scaling	$x(at)$, $a>0$	$\frac{1}{a}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$X^*(s^*)$
Time-Domain Convolution	$x_1(t)*x_2(t)$, $x_1(t)$ and $x_2(t)$ are causal	$X_1(s)X_2(s)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)-x(0^-)$
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$
Time-Domain Integration	$\int_{0^-}^{t}x(\tau)d\tau$	$\frac{1}{s}X(s)$

Property	
Initial Value Theorem	$x(0^+)=\lim_{s\to\infty}sX(s)$
Final Value Theorem	$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$

Bilateral Laplace Transform Pairs

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}>0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\}<0$
4	$t^nu(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}>0$
5	$-t^nu(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\}<0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}>-a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\}<-a$
8	$t^ne^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}>-a$
9	$-t^ne^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\}<-a$
10	$[\cos\omega_0t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
11	$[\sin\omega_0t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\}>0$
12	$[e^{-at}\cos\omega_0t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$
13	$[e^{-at}\sin\omega_0t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\}>-a$

Unilateral Laplace Transform Pairs

Pair	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
6	$\cos\omega_0t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Unit:
Complex Analysis

Example A.10. Determine for what values of z the function $f(z) = z^2$ is analytic.

Solution. First, we observe that f is a polynomial function. Then, we recall that polynomial functions are analytic everywhere. Therefore, f is analytic everywhere.

Example A.11. Determine for what values of z the function $f(z) = 1/z$ is analytic.

Solution. We can deduce the analyticity properties of f as follows. First, we observe that f is a rational function. Then, we recall that a rational function is analytic everywhere except at points where its denominator polynomial becomes zero. Since the denominator polynomial of f only becomes zero at 0, f is analytic everywhere except at 0.

Example A.12 (Poles and zeros of a rational function)**.** Find and plot the poles and (finite) zeros of the function

$$f(z) = \frac{z^2(z^2+1)(z-1)}{(z+1)(z^2+3z+2)(z^2+2z+2)}.$$

Solution. We observe that f is a rational function, so we can easily determine the poles and zeros of f from its factored form. We now proceed to factor f. First, we factor z^2+3z+2. To do this, we solve for the roots of $z^2+3z+2=0$ to obtain

or factor by hand

$$z = \frac{-3 \pm \sqrt{3^2-4(1)(2)}}{2(1)} = -\tfrac{3}{2} \pm \tfrac{1}{2} = \{-1,-2\}.$$

$z^2+3z+2=(z+2)(z+1)$

(For additional information on how to find the roots of a quadratic equation, see Section A.16.) So, we have

$$z^2+3z+2 = (z+1)(z+2). \quad (1)$$

Second, we factor z^2+2z+2. To do this, we solve for the roots of $z^2+2z+2=0$ to obtain

$$z = \frac{-2 \pm \sqrt{2^2-4(1)(2)}}{2(1)} = -1 \pm j = \{-1+j,-1-j\}.$$

So, we have

$$z^2+2z+2 = (z+1-j)(z+1+j). \quad (2)$$

Lastly, we factor z^2+1. Using the well-known factorization for a sum of squares, we obtain

$$z^2+1 = (z+j)(z-j). \quad (3)$$

$a^2+b^2=(a+jb)(a-jb)$

Combining the above results, we can rewrite f as

(1), (2), (3)

$$f(z) = \frac{z^2(z+j)(z-j)(z-1)}{(z+1)^2(z+2)(z+1-j)(z+1+j)}. = \frac{z^2(z+j)^1(z-j)^1(z-1)^1}{(z+1)^2(z+2)^1(z+1-j)^1(z+1+j)^1}$$

From this expression, we can trivially deduce that f has:

- first order zeros at 1, j, and $-j$,
- a second order zero at 0,

} from numerator

- first order poles at $-1+j$, $-1-j$, -2, and
- a second order pole at -1.

} from denominator

The zeros and poles of this function are plotted in Figure A.9. In such plots, the poles and zeros are typically denoted by the symbols "x" and "o", respectively.

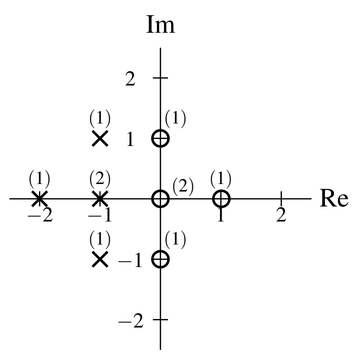


Figure A.9: Plot of the poles and zeros of f (with their orders indicated in parentheses).

■

Unit: Preliminaries

Example 2.2. For two functions x_1 and x_2, the expression $x_1 + x_2$ denotes the function that is the sum of the functions x_1 and x_2. The expression $(x_1 + x_2)(t)$ denotes the function $x_1 + x_2$ evaluated at t. Since the addition of functions can be defined pointwise (i.e., we can add two functions by adding their values at corresponding pairs of points), the following relationship always holds:

$$\overset{\text{adding functions}}{(x_1 + x_2)(t)} = \overset{\text{adding numbers}}{x_1(t) + x_2(t)} \quad \text{for all } t.$$

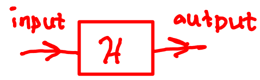
Similarly, since subtraction, multiplication, and division can also defined pointwise, the following relationships also hold:

$$\begin{aligned} \overset{\text{subtracting functions}}{(x_1 - x_2)(t)} &= \overset{\text{subtracting numbers}}{x_1(t) - x_2(t)} \quad \text{for all } t, \\ \text{multiplying functions} \rightarrow (x_1x_2)(t) &= x_1(t)x_2(t) \leftarrow \text{multiplying numbers} \quad \text{for all } t, \quad \text{and} \\ \text{dividing functions} \rightarrow (x_1/x_2)(t) &= x_1(t)/x_2(t) \leftarrow \text{dividing numbers} \quad \text{for all } t. \end{aligned}$$

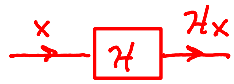
It is important to note, however, that not all mathematical operations involving functions can be defined in a pointwise manner. That is, some operations fundamentally require that their operands be functions. The convolution operation (for functions), which will be considered later, is one such example. If some operator, which we denote for illustrative purposes as "$\diamond$", is defined in such a way that it can only be applied to functions, then the expression $(x_1 \diamond x_2)(t)$ is mathematically valid, but the expression $x_1(t) \diamond x_2(t)$ is not. The latter expression is not valid since the $\diamond$ operator requires two functions as operands, but the provided operands $x_1(t)$ and $x_2(t)$ are numbers (namely, the values of the functions x_1 and x_2 each evaluated at t). Due to issues like this, one must be careful in the use of mathematical notation related to functions. Otherwise, it is easy to fall into the trap of writing expressions that are ambiguous, contradictory, or nonsensical. ■

Example 2.6. For a system operator $\mathcal{H}$, a function x, a real variable t, and a real constant t_0, the expression $\mathcal{H}x(t-t_0)$ denotes the result obtained by taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x and then evaluating y at $t-t_0$. ■

$\mathcal{H}$ is a system.



$\underbrace{\mathcal{H}x}_{\text{function}}$ is the output of the system $\mathcal{H}$ when the input is $\underbrace{x}_{\text{function}}$.

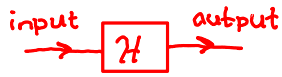


Since $\mathcal{H}x$ is a function, we can evaluate it at some point such as $t-t_0$.

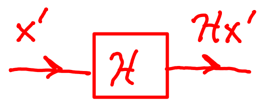
$$\overbrace{\underbrace{\mathcal{H}x}_{\text{function}}\underbrace{(t-t_0)}_{\text{point at which function is evaluated}}}^{\text{number}}$$

Example 2.7. For a system operator $\mathcal{H}$, function x', and real number t, the expression $\mathcal{H}x'(t)$ denotes result of taking the function y produced as the output of the system $\mathcal{H}$ when the input is the function x' and then evaluating y at t. ■

$\mathcal{H}$ is a system.



$\underbrace{\mathcal{H}x'}_{\text{function}}$ is the output of the system $\mathcal{H}$ when the input is $\underbrace{x'}_{\text{function}}$.



Since $\mathcal{H}x'$ is a function, we can evaluate it at a point such as t.

$$\overbrace{\underbrace{\mathcal{H}x'}_{\text{function}}\underbrace{(t)}_{\substack{\text{point at}\\ \text{which}\\ \text{function is}\\ \text{evaluated}}}}^{\text{number}}$$

Unit:
CT Signals and Systems

3.3 Suppose that we have two functions x and y related as

$$y(t) = x(at - b),$$

where a and b are real constants and $a \neq 0$.
(a) Show that y can be formed by first time shifting x by b and then time scaling the result by a.
(b) Show that y can also be formed by first time scaling x by a and then time shifting the result by $\frac{b}{a}$.

Answer (a). (shift then scale)

When working with time transformed functions, always give each transformed function a name

Let f denote the result of time shifting x by b. So, by definition, we have

$$f(t) = x(t - b). \quad (1)$$

Let g denote the result of time scaling f by a. So, by definition, we have

$$g(t) = f(at).$$

Substituting the above formula for f into the equation for g, we obtain

$$\begin{aligned} g(t) &= f(at) \\ &= x(at - b) \\ &= y(t). \end{aligned}$$

substituting (1)

Therefore, y can be formed in the manner specified in the problem statement.

Answer (b). (scale then shift)

Let f denote the result of time scaling x by a. So, by definition, we have

$$f(t) = x(at).$$

Let g denote the result of time shifting f by $\frac{b}{a}$. So, by definition, we have

$$g(t) = f\left(t - \tfrac{b}{a}\right).$$

Substituting the above formula for f into the equation for g, we obtain

$$\begin{aligned} g(t) &= f\left(t - \tfrac{b}{a}\right) \\ &= x\left(a\left[t - \tfrac{b}{a}\right]\right) \\ &= x(at - b) \\ &= y(t). \end{aligned}$$

substituting (1)

Therefore, y can be formed in the manner specified in the problem statement.

Theorem 3.1 (Decomposition of function into even and odd parts)**.** *Any arbitrary function x can be uniquely represented as the sum of the form*

$$x(t) = x_e(t) + x_o(t), \tag{3.7}$$

where x_e and x_o are even and odd, respectively, and given by

$$x_e(t) = \tfrac{1}{2}\left[x(t) + x(-t)\right] \quad \text{and} \tag{3.8}$$

$$x_o(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right]. \tag{3.9}$$

*As a matter of terminology, x_e is called the **even part** of x and is denoted* Even$\{x\}$*, and x_o is called the **odd part** of x and is denoted* Odd$\{x\}$.

Partial *Proof.* From (3.8) and (3.9), we can easily confirm that $x_e + x_o = x$ as follows:

$$\begin{aligned} x_e(t) + x_o(t) &= \overbrace{\boxed{\tfrac{1}{2}[x(t)+x(-t)]}}^{x_e} + \overbrace{\boxed{\tfrac{1}{2}[x(t)-x(-t)]}}^{x_o} \\ &= \tfrac{1}{2}x(t) + \tfrac{1}{2}x(-t) + \tfrac{1}{2}x(t) - \tfrac{1}{2}x(-t) \\ &= x(t). \end{aligned}$$

from the definition of xe and xo

x(-t) terms cancel

Furthermore, we can easily verify that x_e is even and x_o is odd. From the definition of x_e in (3.8), we have

$$\begin{aligned} x_e(-t) &= \tfrac{1}{2}[x(-t) + x(-[-t])] \\ &= \tfrac{1}{2}[x(t) + x(-t)] \\ &= x_e(t). \end{aligned}$$

substitute -t for t in definition of xe

even

Thus, x_e is even. From the definition of x_o in (3.9), we have

$$\begin{aligned} x_o(-t) &= \tfrac{1}{2}[x(-t) - x(-[-t])] \\ &= \tfrac{1}{2}[-x(t) + x(-t)] \\ &= -x_o(t). \end{aligned}$$

substitute -t for t in definition of xo

odd

Thus, x_o is odd.

Example 3.2. Let $x_1(t) = \sin(\pi t)$ and $x_2(t) = \sin t$. Determine whether the function $y = x_1 + x_2$ is periodic.

Solution. Denote the fundamental periods of x_1 and x_2 as T_1 and T_2, respectively. We then have

$$T_1 = \frac{2\pi}{\pi} = 2 \quad \text{and} \quad T_2 = \frac{2\pi}{1} = 2\pi.$$

Here, we used the fact that the fundamental period of $\sin(\alpha t)$ is $\frac{2\pi}{|\alpha|}$. Thus, we have

$$\frac{T_1}{T_2} = \frac{2}{2\pi} = \frac{1}{\pi}.$$

Since π is an irrational number, $\frac{T_1}{T_2}$ is not rational. Therefore, y is not periodic. ■

Example 3.4. Let $x_1(t) = \cos(6\pi t)$ and $x_2(t) = \sin(30\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. We have

$$T_1 = \frac{2\pi}{6\pi} = \frac{1}{3} \quad \text{and} \quad T_2 = \frac{2\pi}{30\pi} = \frac{1}{15}.$$

Thus, we have

$$\frac{T_1}{T_2} = \left(\frac{1}{3}\right)/\left(\frac{1}{15}\right) = \frac{15}{3} = \frac{5}{1}.$$

5 and 1 are coprime

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 5 and 1 are coprime, we have

$$T = 1T_1 = 5T_2 = \frac{1}{3}.$$

■

$$\frac{T_1}{T_2} = \frac{p}{q}$$

cross multiplication pattern (p, q coprime)

Example 3.8 (Sifting property example)**.** Evaluate the integral

$$\int_{-\infty}^{\infty} [\sin t]\delta(t - \pi/4)dt.$$

Solution. Using the sifting property of the unit impulse function, we have

$$= \sin t \big|_{t=\pi/4}$$

$$\begin{aligned} \int_{-\infty}^{\infty} [\sin t]\delta(t - \pi/4)dt &= \sin\left(\tfrac{\pi}{4}\right) \\ &= \tfrac{1}{\sqrt{2}}. \end{aligned}$$

■

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$$

in this example, $x(t) = \sin t$ and $t_0 = \frac{\pi}{4}$

Example 3.9 (Sifting property example). Evaluate the integral

does not have form of sifting property due to "4"

$$\int_{-\infty}^{\infty}[\sin(2\pi t)]\delta(\underbrace{4t}_{\tau}-1)dt.$$

sifting property

Solution. First, we observe that the integral to be evaluated does not quite have the same form as (3.24). So, we need to perform a change of variable. Let $\tau = 4t$ so that $t = \tau/4$ and $dt = d\tau/4$. Performing the change of variable, we obtain

change of variable

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin(2\pi t)]\delta(4t-1)dt &= \int_{-\infty}^{\infty}\tfrac{1}{4}[\sin(2\pi\tau/4)]\delta(\tau-1)d\tau\\ &= \int_{-\infty}^{\infty}\underbrace{\left[\tfrac{1}{4}\sin(\pi\tau/2)\right]}_{x(\tau)}\delta(\tau-1)d\tau.\end{aligned}$$

Now the integral has the desired form, and we can use the sifting property of the unit-impulse function to write

$$\begin{aligned}\int_{-\infty}^{\infty}[\sin(2\pi t)]\delta(4t-1)dt &= \left[\tfrac{1}{4}\sin(\pi\tau/2)\right]\big|_{\tau=1}\\ &= \tfrac{1}{4}\sin(\pi/2)\\ &= \tfrac{1}{4}.\end{aligned}$$

■

Sifting property

$$\int_{-\infty}^{\infty}x(\tau)\delta(\tau-t_0)d\tau = x(t_0)$$

in this example, $x(\tau) = \frac{1}{4}\sin\left(\frac{\pi}{2}\tau\right)$ and $t_0 = 1$

Example 3.10. Evaluate the integral $\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau$.

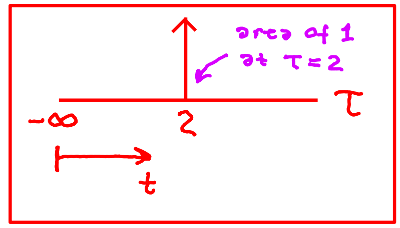
Solution. Using the equivalence property of the delta function given by (3.23), we can write

equivalence property

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \int_{-\infty}^{t}(2^2+1)\delta(\tau-2)d\tau \\ &= 5\underbrace{\int_{-\infty}^{t}\delta(\tau-2)d\tau}.\end{aligned}$$

consider simplification of the underlined integral

Using the defining properties of the delta function given by (3.22), we have that



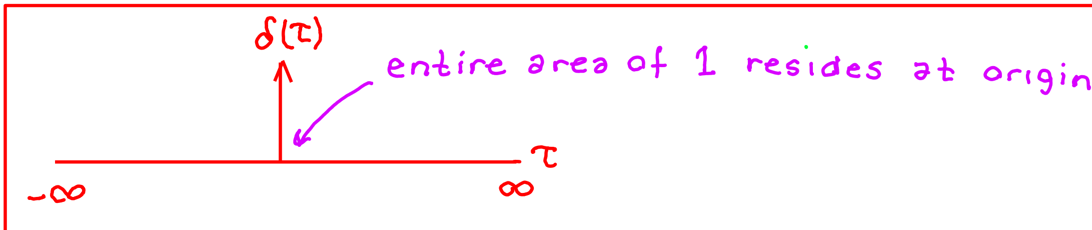
$$\begin{aligned}\int_{-\infty}^{t}\delta(\tau-2)d\tau &= \begin{cases}1 & t\geq 2\\ 0 & t<2\end{cases} \\ &= u(t-2).\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\int_{-\infty}^{t}(\tau^2+1)\delta(\tau-2)d\tau &= \begin{cases}5 & t\geq 2\\ 0 & t<2\end{cases} \\ &= 5u(t-2).\end{aligned}$$

$$= 5\int_{-\infty}^{t}\delta(\tau-2)\,d\tau$$

■



$$\int_a^b \delta(\tau)\,d\tau = \begin{cases}1 & 0\in[a,b]\\ 0 & \text{otherwise}\end{cases}$$

(where $a \leq b$)

Example 3.11 (Rectangular function)**.** Show that the rect function can be expressed in terms of u as

$$\operatorname{rect} t = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right).$$

Solution. Using the definition of u and time-shift transformations, we have

$$u\left(t+\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u\left(t-\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have

$$\begin{aligned} u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right) &= \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & t \geq \frac{1}{2} \end{cases} \\ &= \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\ &= \operatorname{rect} t. \end{aligned}$$

$$= \begin{cases} 0-0 & t < -\frac{1}{2} \\ 1-0 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 1-1 & t \geq \frac{1}{2} \end{cases}$$

Graphically, we have the scenario depicted in Figure 3.24.

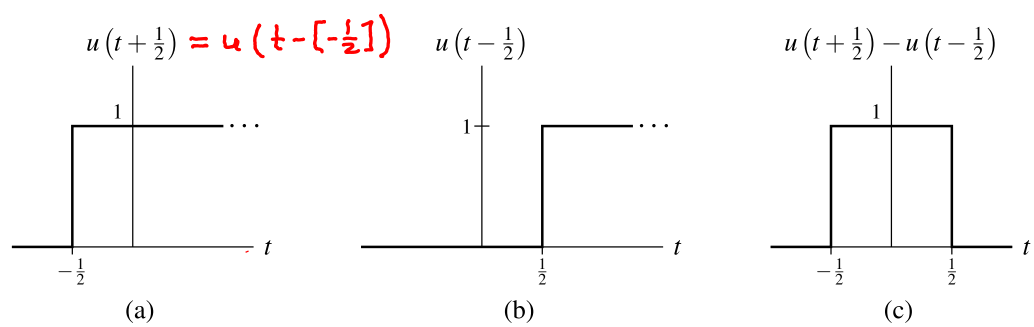
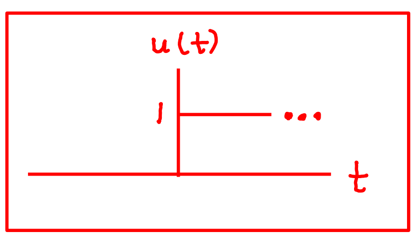


Figure 3.24: Representing the rectangular function using unit-step functions. (a) A shifted unit-step function, (b) another shifted unit-step function, and (c) their difference (which is the rectangular function).

■

recall:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Example 3.12 (Piecewise-linear function)**.** Consider the piecewise-linear function x given by

$$x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution. A plot of x is shown in Figure 3.25(a). We consider each segment of the piecewise-linear function separately. The first segment (i.e., for $0 \le t < 1$) can be expressed as

$$v_1(t) = t[u(t) - u(t-1)].$$

This function is plotted in Figure 3.25(b). The second segment (i.e., for $1 \le t < 2$) can be expressed as

$$v_2(t) = [u(t-1) - u(t-2)]\,(1)$$

This function is plotted in Figure 3.25(c). The third segment (i.e., for $2 \le t < 3$) can be expressed as

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$

This function is plotted in Figure 3.25(d). Now, we observe that $x = v_1 + v_2 + v_3$. That is, we have

$$\begin{aligned} x(t) &= v_1(t) + v_2(t) + v_3(t) \\ &= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\ &= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\ &= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3). \end{aligned}$$

Thus, we have found a single expression for $x(t)$ that is valid for all t.

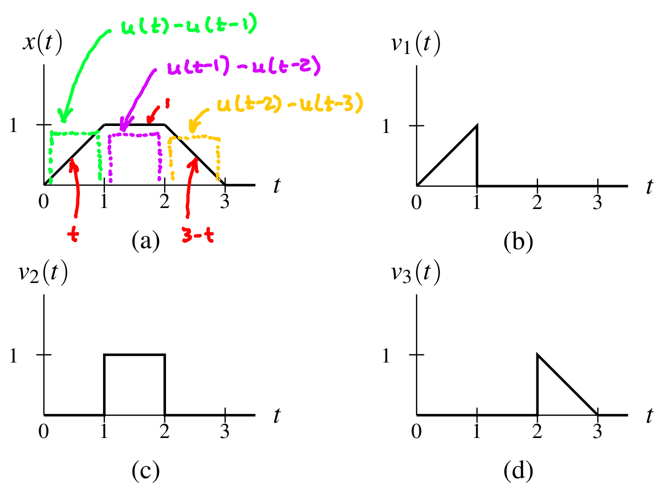


Figure 3.25: Representing a piecewise-linear function using unit-step functions. (a) The function x. (b), (c), and (d) Three functions whose sum is x.

■

Example 3.15 (Ideal amplifier). Determine whether the system $\mathcal{H}$ is memoryless, where

$$\mathcal{H}x(t) = Ax(t)$$

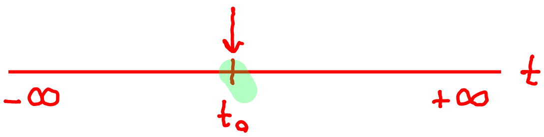
and A is a nonzero real constant.

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = Ax(t_0).$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ only for $t = t_0$. Therefore, the system is memoryless. ■

compute output
at this point



at what points must
input be known?

Example 3.16 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is memoryless, where

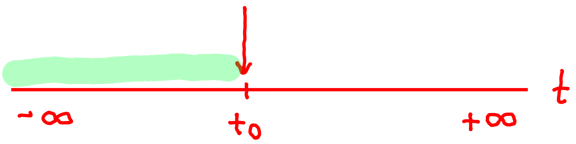
$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t)$ at any arbitrary point $t = t_0$. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, $\mathcal{H}x(t_0)$ depends on $x(t)$ for $-\infty < t \leq t_0$. So, $\mathcal{H}x(t_0)$ is dependent on $x(t)$ for some $t \neq t_0$ (e.g., $t_0 - 1$). Therefore, the system has memory (i.e., is not memoryless). ■

consider computation
of output at this point



at what points must
this input be known?

Example 3.19 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is causal, where

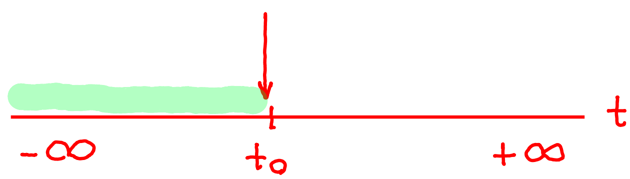
$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{-\infty}^{t_0} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ depends only on $x(t)$ for $-\infty < t \leq t_0$. Since all of the values in this interval are less than or equal to t_0, the system is causal. ■

consider computation
of output at this point



at what points must
input be known?

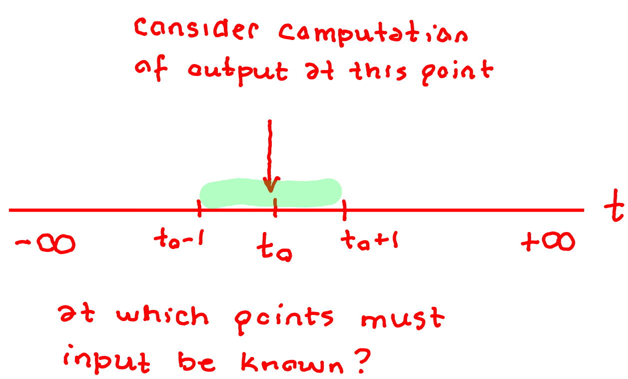
Example 3.20. Determine whether the system $\mathcal{H}$ is causal, where

$$\mathcal{H}x(t) = \int_{t-1}^{t+1} x(\tau)d\tau.$$

Solution. Consider the calculation of $\mathcal{H}x(t_0)$ for arbitrary t_0. We have

$$\mathcal{H}x(t_0) = \int_{t_0-1}^{t_0+1} x(\tau)d\tau.$$

Thus, we can see that $\mathcal{H}x(t_0)$ only depends on $x(t)$ for $t_0 - 1 \leq t \leq t_0 + 1$. Since some of the values in this interval are greater than t_0 (e.g., $t_0 + 1$), the system is not causal. ■



Example 3.23. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = x(t - t_0)$$

y = Hx

and t_0 is a real constant.

substitute t+t0 for t

Solution. Let $y = \mathcal{H}x$. By substituting $t + t_0$ for t in $y(t) = x(t - t_0)$, we obtain

$$\begin{aligned} y(t + t_0) &= x(t + t_0 - t_0) \\ &= x(t). \end{aligned}$$

Thus, we have shown that

$$x(t) = y(t + t_0).$$

This, however, is simply the equation of the inverse system $\mathcal{H}^{-1}$. In particular, we have that

$$x(t) = \mathcal{H}^{-1}y(t)$$

where

$$\mathcal{H}^{-1}y(t) = y(t + t_0).$$

Thus, we have found $\mathcal{H}^{-1}$. Therefore, the system $\mathcal{H}$ is invertible. ■

Example 3.24. Determine whether the system $\mathcal{H}$ is invertible, where

$$\mathcal{H}x(t) = \sin[x(t)].$$

Solution. Consider an input of the form $x(t) = 2\pi k$ where k is an arbitrary integer. The response $\mathcal{H}x$ to such an input is given by

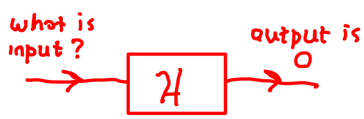
①

$$\begin{aligned}\mathcal{H}x(t) &= \sin[x(t)] \\ &= \sin 2\pi k \\ &= 0.\end{aligned}$$

substitute ①

sin function is zero at all integer multiples of π

Thus, we have found an infinite number of distinct inputs (i.e., $x(t) = 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$) that all result in the same output. Therefore, the system is not invertible. ■



we don't know what the input is. input could be $x(t) = 0$ or $x(t) = 2\pi$ or $x(t) = -2\pi$ or ...

Example 3.27 (Ideal integrator)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau.$$

Solution. Suppose that we choose the input $x = u$ (where u denotes the unit-step function). Clearly, u is bounded (i.e., $|u(t)| \leq 1$ for all t). Calculating the response $\mathcal{H}x$ to this input, we have

$$\begin{aligned}\mathcal{H}x(t) &= \int_{-\infty}^{t} u(\tau)d\tau \\ &= \int_{0}^{t} d\tau \\ &= [\tau]|_0^t \\ &= t.\end{aligned}$$

$u(\tau) = 0$ for $\tau < 0$

From this result, however, we can see that as $t \to \infty$, $\mathcal{H}x(t) \to \infty$. Thus, the output $\mathcal{H}x$ is unbounded for the bounded input x. Therefore, the system is not BIBO stable. ■

A system $\mathcal{H}$ is said to be BIBO stable if, for every bounded function x, $\mathcal{H}x$ is bounded. That is,

$$|x(t)| \leq A < \infty \text{ for all } t \implies |\mathcal{H}x(t)| \leq B < \infty \text{ for all } t.$$

To show that a system is not BIBO stable, we simply need to find a counterexample (i.e., an example of a bounded input that yields an unbounded output).

Example 3.28 (Squarer)**.** Determine whether the system $\mathcal{H}$ is BIBO stable, where

$$\mathcal{H}x(t) = x^2(t).$$

Solution. Suppose that the input x is bounded such that (for all t)

$$|x(t)| \le A,$$

where A is a finite real constant. Squaring both sides of the inequality, we obtain

$$|x(t)|^2 \le A^2.$$

Interchanging the order of the squaring and magnitude operations on the left-hand side of the inequality, we have

$$|x^2(t)| \le A^2.$$

Using the fact that $\mathcal{H}x(t) = x^2(t)$, we can write

$$|\mathcal{H}x(t)| \le A^2.$$

Since A is finite, A^2 is also finite. Thus, we have that $\mathcal{H}x$ is bounded (i.e., $|\mathcal{H}x(t)| \le A^2 < \infty$ for all t). Therefore, the system is BIBO stable. ■

squaring a finite number always yields a finite result

A system $\mathcal{H}$ is said to be BIBO stable if, for every bounded function x, $\mathcal{H}x$ is bounded. That is,

$$|x(t)| \le A < \infty \text{ for all } t \implies |y(t)| \le B < \infty \text{ for all } t.$$

To show a system is BIBO stable, we must show that every bounded input produces a bounded output.

Example 3.32. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \sin[x(t)]. \quad (1)$$

(2)

Solution. Let $x'(t) = x(t - t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we can easily deduce that

equal for all x and all t_0

by substituting $t - t_0$ for t in (1)

$$\mathcal{H}x(t - t_0) = \sin[x(t - t_0)] \quad \text{and}$$

$$\mathcal{H}x'(t) = \sin x'(t) \quad \text{from definition of } \mathcal{H} \text{ in (1)}$$

$$= \sin[x(t - t_0)]. \quad \text{from definition of } x' \text{ in (2)}$$

Since $\mathcal{H}x(t - t_0) = \mathcal{H}x'(t)$ for all x and t_0, the system is time invariant. ■

A system $\mathcal{H}$ is said to be time invariant if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t - t_0) = \mathcal{H}x'(t) \text{ for all } t, \text{ where } x'(t) = x(t - t_0)$$

Example 3.33. Determine whether the system $\mathcal{H}$ is time invariant, where

$$\mathcal{H}x(t) = \mathrm{Odd}(x)(t) = \tfrac{1}{2}\left[x(t) - x(-t)\right]. \quad ①$$

②

Solution. Let $x'(t) = x(t - t_0)$, where t_0 is an arbitrary real constant. From the definition of $\mathcal{H}$, we have

$$\begin{aligned}
\mathcal{H}x(t-t_0) &= \tfrac{1}{2}[x(t-t_0) - x(-(t-t_0))] \quad \leftarrow \text{by substituting } t-t_0 \text{ for } t \text{ in ①} \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t+t_0)] \quad \text{and} \\
\mathcal{H}x'(t) &= \tfrac{1}{2}[x'(t) - x'(-t)] \quad \leftarrow \text{from definition of } \mathcal{H} \text{ in ①} \\
&= \tfrac{1}{2}[x(t-t_0) - x(-t-t_0)]. \quad \text{from definition of } x' \text{ in ②}
\end{aligned}$$

not equal if $t_0 \neq 0$

Since $\mathcal{H}x(t-t_0) = \mathcal{H}x'(t)$ does not hold for all x and t_0, the system is not time invariant. ■

only equal if $t_0 = 0$

A system $\mathcal{H}$ is said to be time invariant if, for every function x and every real constant t_0, the following condition holds:

$$\mathcal{H}x(t-t_0) = \mathcal{H}x'(t) \text{ for all } t, \text{ where } x'(t) = x(t-t_0).$$

Example 3.35. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = tx(t). \quad ①$$

②

Solution. Let $x'(t) = a_1x_1(t) + a_2x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we can write

equal for all x_1, x_2, a_1, a_2

$$a_1\mathcal{H}x_1(t) + a_2\mathcal{H}x_2(t) = a_1tx_1(t) + a_2tx_2(t) \quad \text{and}$$

from definition of $\mathcal{H}$ in ①

$$\begin{aligned}\mathcal{H}x'(t) &= tx'(t) \\ &= t[a_1x_1(t) + a_2x_2(t)] \\ &= a_1tx_1(t) + a_2tx_2(t).\end{aligned}$$

from definition of $\mathcal{H}$ in ①

from definition of x' in ②

Since $\mathcal{H}(a_1x_1 + a_2x_2) = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$ for all x_1, x_2, a_1, and a_2, the superposition property holds and the system is linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1x_1 + a_2x_2\} = a_1\mathcal{H}x_1 + a_2\mathcal{H}x_2$$

Example 3.36. Determine whether the system $\mathcal{H}$ is linear, where

$$\mathcal{H}x(t) = |x(t)|. \quad ①$$

②

Solution. Let $x'(t) = a_1 x_1(t) + a_2 x_2(t)$, where x_1 and x_2 are arbitrary functions and a_1 and a_2 are arbitrary complex constants. From the definition of $\mathcal{H}$, we have

$$a_1 \mathcal{H}x_1(t) + a_2 \mathcal{H}x_2(t) = a_1 |x_1(t)| + a_2 |x_2(t)| \quad \text{and}$$

from definition of $\mathcal{H}$ in ①

$$\mathcal{H}x'(t) = |x'(t)|$$

from definition of $\mathcal{H}$ in ①

$$= |a_1 x_1(t) + a_2 x_2(t)|.$$

from definition of x' in ②

At this point, we recall the triangle inequality (i.e., for $a, b \in \mathbb{C}$, $|a+b| \leq |a| + |b|$). Thus, $\mathcal{H}(a_1 x_1 + a_2 x_2) = a_1 \mathcal{H}x_1 + a_2 \mathcal{H}x_2$ cannot hold for all x_1, x_2, a_1, and a_2 due, in part, to the triangle inequality. For example, this condition fails to hold for

$$a_1 = -1, \quad x_1(t) = 1, \quad a_2 = 0, \quad \text{and} \quad x_2(t) = 0,$$

in which case

$$a_1 \mathcal{H}x_1(t) + a_2 \mathcal{H}x_2(t) = -1 \quad \text{and} \quad \mathcal{H}x'(t) = 1.$$

Counterexample

Therefore, the superposition property does not hold and the system is not linear. ■

A system $\mathcal{H}$ is said to be linear if, for all functions x_1 and x_2 and all complex constants a_1 and a_2, the following condition holds:

$$\mathcal{H}\{a_1 x_1 + a_2 x_2\} = a_1 \mathcal{H}x_1 + a_2 \mathcal{H}x_2.$$

Example 3.41. Consider the system $\mathcal{H}$ characterized by the equation

$$\mathcal{H}x(t) = \mathcal{D}^2 x(t), \quad (1)$$

where $\mathcal{D}$ denotes the derivative operator. For each function x given below, determine if x is an eigenfunction of $\mathcal{H}$, and if it is, find the corresponding eigenvalue.
(a) $x(t) = \cos 2t$; and
(b) $x(t) = t^3$.

Solution. (a) We have

from definition of $\mathcal{H}$ in (1)

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{\cos 2t\}(t) \\ &= \mathcal{D}\{-2\sin 2t\}(t) \\ &= -4\cos 2t \\ &= -4x(t).\end{aligned}$$

$\frac{d}{dt} \cos t = -\sin t$

$\frac{d}{dt} \sin t = \cos t$

from definition of x

So, we have $\mathcal{H}x = -4x$.

Therefore, x is an eigenfunction of $\mathcal{H}$ with the eigenvalue -4.

(b) We have

from definition of $\mathcal{H}$ in (1)

$$\begin{aligned}\mathcal{H}x(t) &= \mathcal{D}^2\{t^3\}(t) \\ &= \mathcal{D}\{3t^2\}(t) \\ &= 6t \\ &= \tfrac{6}{t^2}x(t).\end{aligned}$$

$\frac{d}{dt} t^3 = 3t^2$

$\frac{d}{dt} 3t^2 = 6t$

from definition of x

$\left(\frac{6t\, x(t)}{x(t)} = \frac{6t\, x(t)}{t^3}\right)$

not a constant

Therefore, x is not an eigenfunction of $\mathcal{H}$. ■

A function x is said to be an eigenfunction of the system $\mathcal{H}$ with eigenvalue λ if

$\mathcal{H}x = \lambda x$.

Unit:
CT LTI Systems

Example X.4.1

Let x and h denote functions, and let t denote a real number.

$x * h$

This expression denotes the function resulting from convolving the function x with the function h.

$(x * h)(t)$

$x * h(t)$

Both of these expressions denote the number resulting from convolving the function x with the function h and then evaluating the resulting function at the point t.

$(x + h)(t)$

$x(t) + h(t)$

These expressions have slightly different meanings (i.e., the former is adding functions while the latter is adding numbers), but they are both valid mathematical expressions and, by definition, they are always equal since the addition of functions is defined pointwise (i.e., $(x+h)(t) = x(t) + h(t)$).

$x(t) * h(t)$

Strictly speaking, this expression is not mathematically valid, as it is attempting to convolve the number x(t) with the number h(t). Both operands of a convolution operation, however, must be functions. Convolution cannot be defined in a pointwise manner. In other words, $(x*h)(t)$ does not equal $x(t) * h(t)$ because the latter expression is not even mathematically valid. Sadly, many engineering textbooks abuse notation in this way, and this often leads to confusion for students. Sometimes this abused notation $x(t) * h(t)$ is intended to mean $x * h$; sometimes it might mean $x * h(t)$; and yet other times it may mean something else entirely (and the reader is simply forced to guess the intended meaning).

Example 4.1. Compute the convolution $x * h$ where

$$x(t) = \begin{cases} -1 & -1 \le t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = e^{-t}u(t).$$

Solution. We begin by plotting the functions x and h as shown in Figures 4.1(a) and (b), respectively. Next, we proceed to determine the time-reversed and time-shifted version of h. We can accomplish this in two steps. First, we time-reverse $h(\tau)$ to obtain $h(-\tau)$ as shown in Figure 4.1(c). Second, we time-shift the resulting function by t to obtain $h(t-\tau)$ as shown in Figure 4.1(d).

At this point, we are ready to begin considering the computation of the convolution integral. For each possible value of t, we must multiply $x(\tau)$ by $h(t-\tau)$ and integrate the resulting product with respect to τ. Due to the form of x and h, we can break this process into a small number of cases. These cases are represented by the scenarios illustrated in Figures 4.1(e) to (h).

First, we consider the case of $t < -1$. From Figure 4.1(e), we can see that

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0. \tag{4.2}$$

Second, we consider the case of $-1 \le t < 0$. From Figure 4.1(f), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{t} -e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{t} \\ &= -e^{-t}[e^{t} - e^{-1}] \\ &= e^{-t-1} - 1. \end{aligned} \tag{4.3}$$

Third, we consider the case of $0 \le t < 1$. From Figure 4.1(g), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{t} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{t} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{t} \\ &= -e^{-t}[1 - e^{-1}] + e^{-t}[e^{t} - 1] \\ &= e^{-t}[e^{-1} - 1 + e^{t} - 1] \\ &= 1 + (e^{-1} - 2)e^{-t}. \end{aligned} \tag{4.4}$$

Fourth, we consider the case of $t \ge 1$. From Figure 4.1(h), we can see that

$$\begin{aligned} x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau &= \int_{-1}^{0} -e^{\tau-t}d\tau + \int_{0}^{1} e^{\tau-t}d\tau \\ &= -e^{-t}\int_{-1}^{0} e^{\tau}d\tau + e^{-t}\int_{0}^{1} e^{\tau}d\tau \\ &= -e^{-t}[e^{\tau}]|_{-1}^{0} + e^{-t}[e^{\tau}]|_{0}^{1} \\ &= e^{-t}[e^{-1} - 1 + e - 1] \\ &= (e - 2 + e^{-1})e^{-t}. \end{aligned} \tag{4.5}$$

Combining the results of (4.2), (4.3), (4.4), and (4.5), we have that

$$x * h(t) = \begin{cases} 0 & t < -1 \\ e^{-t-1} - 1 & -1 \leq t < 0 \\ (e^{-1} - 2)e^{-t} + 1 & 0 \leq t < 1 \\ (e - 2 + e^{-1})e^{-t} & 1 \leq t. \end{cases}$$

The convolution result $x * h$ is plotted in Figure 4.1(i).

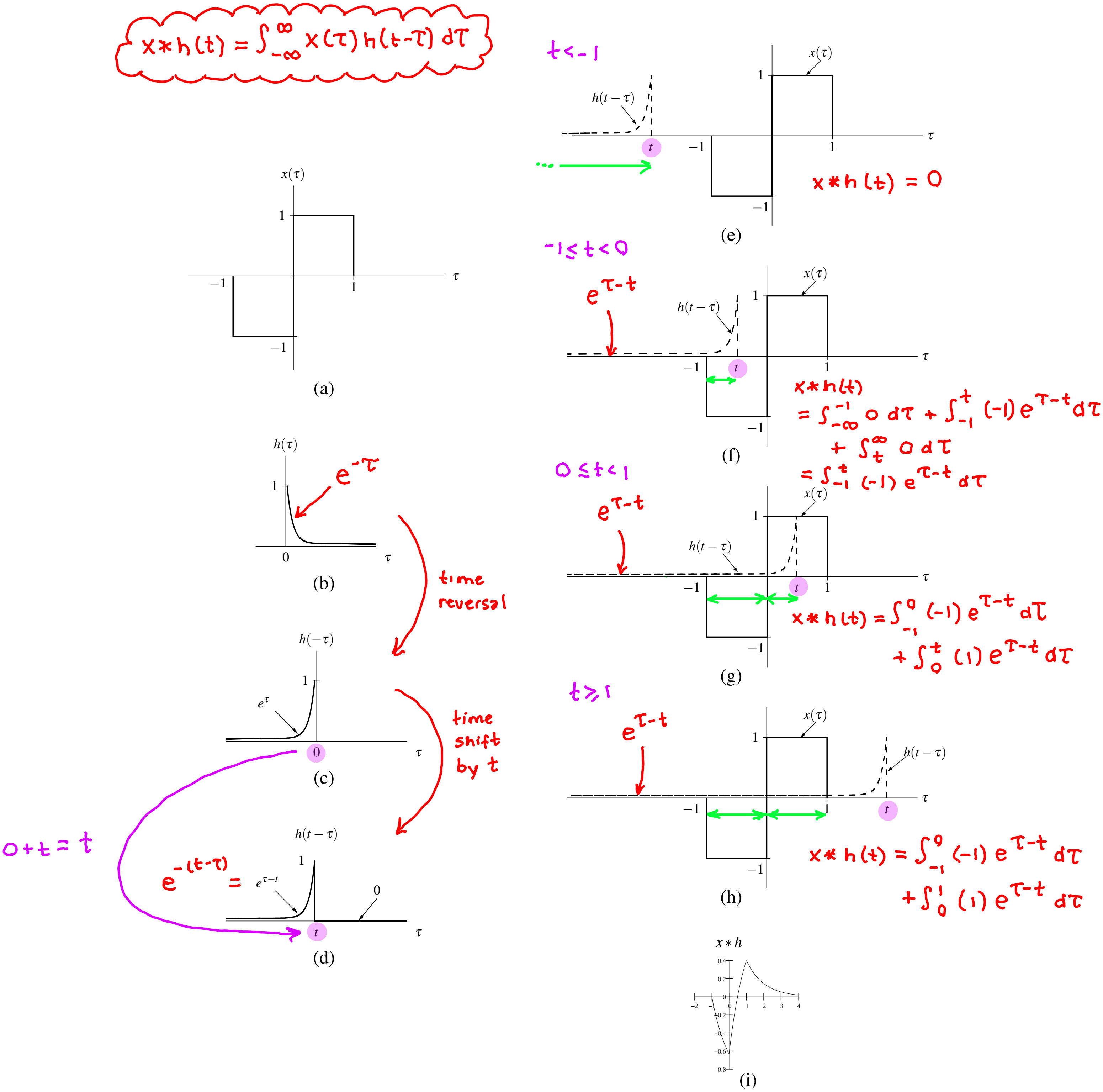


Figure 4.1: Evaluation of the convolution $x*h$. (a) The function x; (b) the function h; plots of (c) $h(-\tau)$ and (d) $h(t-\tau)$ versus τ; the functions associated with the product in the convolution integral for (e) $t<-1$, (f) $-1\le t<0$, (g) $0\le t<1$, and (h) $t\ge 1$; and (i) the convolution result $x*h$.

■

Answer (u).

We need to compute $x * h$, where $\boxed{x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau}$

$$x(t) = \begin{cases} 2-t & 1 \le t < 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} -t-2 & -3 \le t < -2 \\ 0 & \text{otherwise.} \end{cases}$$

First, we plot $x(\tau)$ and $h(t-\tau)$ versus τ in Figures (a) and (d), respectively.

Figure (e): $t < -2$

$x * h(t) = 0$

Figure (f): $-2 \le t < -1$

$x * h(t)$

$= \int_1^{t+3} (2-\tau)(\tau-t-2)\,d\tau$

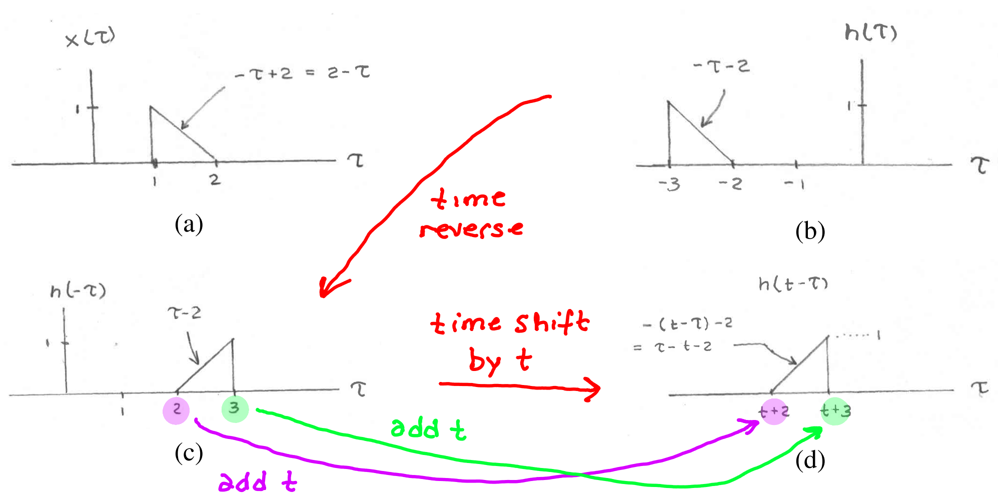
Figure (g): $-1 \le t < 0$

$x * h(t)$

$= \int_{t+2}^{2} (2-\tau)(\tau-t-2)\,d\tau$

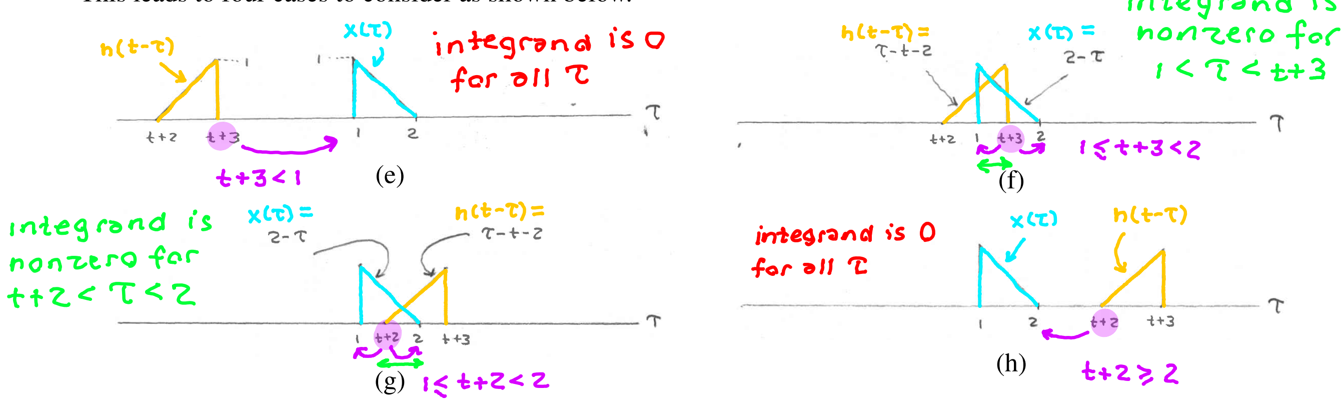
Figure (h): $t \ge 0$

$x * h(t) = 0$



(a) (b) (c) (d)

This leads to four cases to consider as shown below.



(e) (f) (g) (h)

From Figure (e), for $t < -2$ (i.e., $t+3 < 1$), we have

$$x * h(t) = 0.$$

From Figure (f), for $-2 \le t < -1$ (i.e., $1 \le t+3 < 2$), we have

$$x * h(t) = \int_1^{t+3} \underbrace{(2-\tau)}_{x(\tau)}\underbrace{(\tau-t-2)}_{h(t-\tau)}\,d\tau.$$

From Figure (g), for $-1 \le t < 0$ (i.e., $1 \le t+2 < 2$), we have

$$x * h(t) = \int_{t+2}^{2} \underbrace{(2-\tau)}_{x(\tau)}\underbrace{(\tau-t-2)}_{h(t-\tau)}\,d\tau.$$

From Figure (h), for $t \ge 0$ (i.e., $t+2 \ge 2$), we have

$$x * h(t) = 0.$$

Simplifying, we obtain

$$x * h(t) = \begin{cases} \frac{1}{6}t^3 - t - \frac{2}{3} & -2 \leq t < -1 \\ -\frac{1}{6}t^3 & -1 \leq t < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.1 (Commutativity of convolution)**.** *Convolution is commutative. That is, for any two functions x and h,*

$$x * h = h * x. \tag{4.16}$$

In other words, the result of a convolution is not affected by the order of its operands.

Proof. We now provide a proof of the commutative property stated above. To begin, we expand the left-hand side of (4.16) as follows:

from definition of convolution

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(\underbrace{t-\tau}_{v})d\tau.$$

$$\boxed{h * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau}$$

Next, we perform a change of variable. Let $v = t - \tau$ which implies that $\tau = t - v$ and $d\tau = -dv$. Using this change of variable, we can rewrite the previous equation as

$$\begin{aligned} x * h(t) &= \int_{t+\infty}^{t-\infty} x(t-v)h(v)(-dv) && \text{from change of variable} \\ &= \int_{\infty}^{-\infty} x(t-v)h(v)(-dv) && \text{infinity dominates sums} \\ &= \int_{-\infty}^{\infty} x(t-v)h(v)dv && \textstyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \\ &= \int_{-\infty}^{\infty} h(v)x(t-v)dv && \text{rearrange factors} \\ &= h * x(t). && \text{definition of convolution} \end{aligned}$$

Remember that changing integration variable changes limits!

(Note that, above, we used the fact that, for any function f, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.) Thus, we have proven that convolution is commutative. ∎

Theorem 4.5 (LTI systems and convolution)**.** *A LTI system $\mathcal{H}$ with impulse response h is such that*

$$\mathcal{H}x = x * h.$$

In other words, a LTI system computes a convolution. In particular, the output of the system is given by the convolution of the input and impulse response.

Proof. Using the fact that δ is the convolutional identity, we can write

convolutional identity

$$\mathcal{H}x(t) = \mathcal{H}\{x * \delta\}(t).$$

rewrite convolution as integral

Rewriting the convolution in terms of an integral, we have

$$\mathcal{H}x(t) = \mathcal{H}\left\{\int_{-\infty}^{\infty} x(\tau)\delta(\cdot - \tau)d\tau\right\}(t).$$

Since $\mathcal{H}$ is a linear operator, we can pull the integral and $x(\tau)$ (which is a constant with respect to the operation performed by $\mathcal{H}$) outside $\mathcal{H}$ to obtain

interchange $\mathcal{H}$ with both $x(\tau)$ and integral (linearity)

$$\mathcal{H}x(t) = \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\{\delta(\cdot - \tau)\}(t)d\tau.$$

Since $\mathcal{H}$ is time invariant, we can interchange the order of $\mathcal{H}$ and the time shift of δ by τ (i.e., $\underbrace{\mathcal{H}\{\delta(\cdot - \tau)\}}_{\text{shift by } \tau \text{ then } \mathcal{H}} = \underbrace{H\delta(\cdot - \tau)}_{\mathcal{H} \text{ then shift by } \tau}$) and then use the fact that $h = \mathcal{H}\delta$ to obtain

interchange $\mathcal{H}$ and time shift (time invariance)

$$\begin{aligned}
\mathcal{H}x(t) &= \int_{-\infty}^{\infty} x(\tau)\mathcal{H}\delta(t - \tau)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \\
&= x * h(t).
\end{aligned}$$

$h = \mathcal{H}\delta$ (by definition)

Thus, we have shown that $\mathcal{H}x = x * h$, where $h = \mathcal{H}\delta$. ∎

Example 4.5. Consider a LTI system $\mathcal{H}$ with impulse response

$$h(t) = u(t). \tag{4.23}$$

Show that $\mathcal{H}$ is characterized by the equation

$$\mathcal{H}x(t) = \int_{-\infty}^{t} x(\tau)d\tau \tag{4.24}$$

(i.e., $\mathcal{H}$ corresponds to an ideal integrator).

Solution. Since the system is LTI, we have that

$$\mathcal{H}x(t) = x * h(t). \qquad (1)$$

Substituting (4.23) into the preceding equation, and simplifying we obtain

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{from (1)} \\
&= x * u(t) && \text{substitute given function } h \\
&= \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau && \text{definition of convolution} \\
&= \int_{-\infty}^{t} x(\tau)\underbrace{u(t-\tau)}_{1}d\tau + \int_{t^+}^{\infty} x(\tau)\underbrace{u(t-\tau)}_{0}d\tau && \text{split into two integrals} \\
&= \int_{-\infty}^{t} x(\tau)d\tau. && \text{second integral is 0}
\end{aligned}$$

Therefore, the system with the impulse response h given by (4.23) is, in fact, the ideal integrator given by (4.24). ■

Example 4.7. Consider the system with input x, output y, and impulse response h as shown in Figure 4.9. Each subsystem in the block diagram is LTI and labelled with its impulse response. Find h.

Solution. From the left half of the block diagram, we can write

To begin, we label all signals in Figure 4.9.

①
$$\begin{aligned} v(t) &= x(t) + x * h_1(t) + x * h_2(t) \\ &= x * \underline{\delta(t)} + x * h_1(t) + x * h_2(t) \\ &= (x * [\delta + h_1 + h_2])(t). \end{aligned}$$

δ is convolutional identity

distributive property

Similarly, from the right half of the block diagram, we can write

$$y(t) = v * h_3(t).$$ ②

Substituting the expression for v into the preceding equation we obtain

$$\begin{aligned} y(t) &= v * h_3(t) \\ &= (x * [\delta + h_1 + h_2]) * h_3(t) \\ &= x * [\underbrace{h_3 + h_1 * h_3 + h_2 * h_3}_{h}](t). \end{aligned}$$

from ②

substituting ① for v

distributive and associative properties and convolutional identity

Thus, the impulse response h of the overall system is

$$h(t) = h_3(t) + h_1 * h_3(t) + h_2 * h_3(t).$$ ■

Recall that, for any LTI system with input x, output y, and impulse response h, y = x * h.

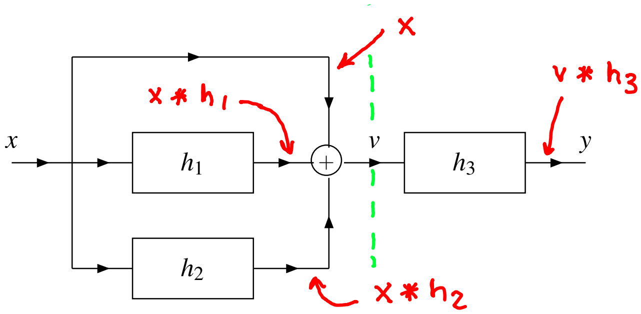


Figure 4.9: System interconnection example.

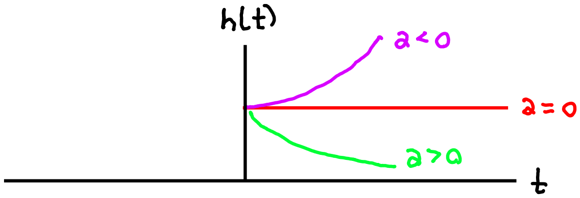
Example 4.8. Consider the LTI system with the impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system has memory.

Solution. The system has memory since $h(t) \neq 0$ for some $t \neq 0$ (e.g., $h(1) = e^{-a} \neq 0$). ■

condition for memorylessness violated



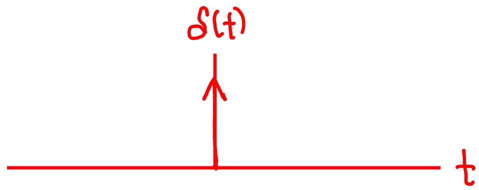
memoryless $\Leftrightarrow$ $h(t) = 0$ for all $t \neq 0$

Example 4.9. Consider the LTI system with the impulse response h given by

$$h(t) = \delta(t).$$

Determine whether this system has memory.

Solution. Clearly, h is only nonzero at the origin. This follows immediately from the definition of the unit-impulse function δ. Therefore, the system is memoryless (i.e., does not have memory). ■



memoryless $\Longleftrightarrow$ $h(t) = 0$ for all $t \neq 0$

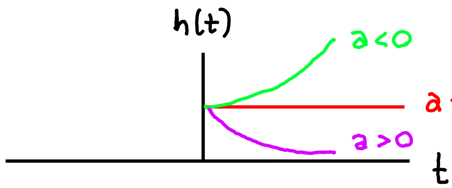
Example 4.10. Consider the LTI system with impulse response h given by

$$h(t) = e^{-at}u(t),$$

where a is a real constant. Determine whether this system is causal.

Solution. Clearly, $h(t) = 0$ for $t < 0$ (due to the $u(t)$ factor in the expression for $h(t)$). Therefore, the system is causal. ■

this is true regardless of a



causal $\iff$ $h(t)=0$ for all $t<0$

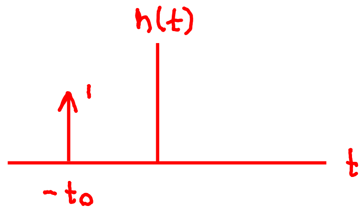
Example 4.11. Consider the LTI system with impulse response h given by

$$h(t) = \delta(t + t_0),$$

$t_0 > 0$

where t_0 is a strictly positive real constant. Determine whether this system is causal.

Solution. From the definition of δ, we can easily deduce that $h(t) = 0$ except at $t = -t_0$. Since $-t_0 < 0$, the system is not causal. ■



causal : $h(t) = 0$ for all $t < 0$

Example 4.12. Consider the LTI system $\mathcal{H}$ with impulse response h given by

$$h(t) = A\delta(t - t_0),$$

where A and t_0 are real constants and $A \neq 0$. Determine if $\mathcal{H}$ is invertible, and if it is, find the impulse response h_{inv} of the system $\mathcal{H}^{-1}$.

Solution. If the system $\mathcal{H}^{-1}$ exists, its impulse response h_{inv} is given by the solution to the equation

$$h * h_{\mathrm{inv}} = \delta. \tag{4.34}$$

$\mathcal{H}$ is invertible if and only if a solution for h_{inv} exists

So, let us attempt to solve this equation for h_{inv}. Substituting the given function h into (4.34) and using straightforward algebraic manipulation, we can write

$$\begin{aligned}
& h * h_{\mathrm{inv}}(t) = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} h(\tau) h_{\mathrm{inv}}(t-\tau)\, d\tau = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} A\delta(\tau - t_0) h_{\mathrm{inv}}(t-\tau)\, d\tau = \delta(t) \\
\Rightarrow\quad & \int_{-\infty}^{\infty} \delta(\tau - t_0) \underline{h_{\mathrm{inv}}(t-\tau)}\, d\tau = \tfrac{1}{A}\delta(t).
\end{aligned}$$

definition of convolution

substitute given function h

divide both sides by $A \neq 0$

Using the sifting property of the unit-impulse function, we can simplify the integral expression in the preceding equation to obtain

$h_{inv}(t-\tau)\big|_{\tau=t_0} = \frac{1}{A}\delta(t)$ sifting property

$$h_{\mathrm{inv}}(t - t_0) = \tfrac{1}{A}\delta(t). \tag{4.35}$$

Substituting $t + t_0$ for t in the preceding equation yields

$$\begin{aligned}
h_{\mathrm{inv}}([t + t_0] - t_0) = \tfrac{1}{A}\delta(t + t_0) \quad & \Leftrightarrow \\
h_{\mathrm{inv}}(t) = \tfrac{1}{A}\delta(t + t_0). &
\end{aligned}$$

impulse response of inverse system

Since $A \neq 0$, the function h_{inv} is always well defined. Thus, $\mathcal{H}^{-1}$ exists and consequently $\mathcal{H}$ is invertible. ■

Example 4.14. Consider the LTI system with impulse response h given by

$$h(t) = e^{at}u(t),$$

where a is a real constant. Determine for what values of a the system is BIBO stable.

Solution. We need to determine for what values of a the impulse response h is absolutely integrable. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} \left|e^{at}u(t)\right| dt \\
&= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} e^{at}\,dt \\
&= \int_{0}^{\infty} e^{at}\,dt \\
&= \begin{cases} \int_0^\infty e^{at}\,dt & a \neq 0 \\ \int_0^\infty 1\,dt & a = 0 \end{cases} \\
&= \begin{cases} \left[\frac{1}{a}e^{at}\right]\Big|_0^\infty & a \neq 0 \\ [t]\big|_0^\infty & a = 0. \end{cases}
\end{aligned}$$

split integration interval
and use fact that
$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

drop zero integral

identify two cases for
integration

integrate

Now, we simplify the preceding equation for each of the cases $a \neq 0$ and $a = 0$. Suppose that $a \neq 0$. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= \left[\tfrac{1}{a}e^{at}\right]\Big|_0^\infty \\
&= \tfrac{1}{a}\left(e^{a\infty} - 1\right).
\end{aligned}$$

what is $e^{a\infty}$?

We can see that the result of the above integration is finite if $a < 0$ and infinite if $a > 0$. In particular, if $a < 0$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= 0 - \tfrac{1}{a} \\
&= -\tfrac{1}{a}.
\end{aligned}$$

assuming $a<0$

Suppose now that $a = 0$. In this case, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} |h(t)|\,dt &= [t]\big|_0^\infty \\
&= \infty.
\end{aligned}$$

Thus, we have shown that

combining above
results

$$\int_{-\infty}^{\infty} |h(t)|\,dt = \begin{cases} -\frac{1}{a} & a < 0 \\ \infty & a \geq 0. \end{cases}$$

In other words, the impulse response h is absolutely integrable if and only if $a < 0$. Consequently, the system is BIBO stable if and only if $a < 0$. ■

Example 4.15. Consider the LTI system with input x and output y defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \quad 1$$

(i.e., an ideal integrator). Determine whether this system is BIBO stable.

Solution. First, we find the impulse response h of the system. We have

using 1 and $h = \mathcal{H}\delta$

$$\begin{aligned} h(t) &= \int_{-\infty}^{t} \delta(\tau)\,d\tau \\ &= \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \\ &= u(t). \end{aligned}$$

integral is 1 if integration interval includes origin

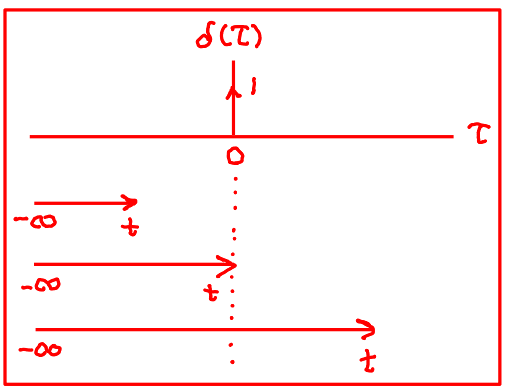
definition of unit-step function

Using this expression for h, we now check to see if h is absolutely integrable. We have

$$\begin{aligned} \int_{-\infty}^{\infty} |h(t)|\,dt &= \int_{-\infty}^{\infty} |u(t)|\,dt \\ &= \int_{0}^{\infty} 1\,dt \\ &= \infty. \end{aligned}$$

$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Thus, h is not absolutely integrable. Therefore, the system is not BIBO stable. ■



Theorem 4.12 (Eigenfunctions of LTI systems)**.** *For an arbitrary LTI system $\mathcal{H}$ with impulse response h and a function of the form $x(t) = e^{st}$, where s is an arbitrary complex constant (i.e., x is an arbitrary complex exponential), the following holds:*

$$\mathcal{H}x(t) = H(s)e^{st},$$

where

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau. \tag{4.49}$$

That is, x is an eigenfunction of $\mathcal{H}$ with the corresponding eigenvalue $H(s)$.

Proof. We have

$$\begin{aligned}
\mathcal{H}x(t) &= x * h(t) && \text{commutative property of convolution} \\
&= h * x(t) && \text{definition of convolution} \\
&= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau && \text{substitute given function } x \\
&= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau && \text{factor out } e^{st} \\
&= e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \\
&= H(s)e^{st}. && \blacksquare
\end{aligned}$$

call this H(s)

Suppose that we have a LTI system $\mathcal{H}$ with input x, output y, impulse response h, and system function H. Suppose now that we can express some arbitrary input signal x as a sum of complex exponentials as follows:

$$x(t) = \sum_k a_k e^{s_k t}. \quad 1$$

(As it turns out, many functions can be expressed in this way.) From the eigenfunction properties of LTI systems, the response of the system to the input $a_k e^{s_k t}$ is $a_k H(s_k) e^{s_k t}$. By using this knowledge and the superposition property, we can write

$$\begin{aligned} y(t) &= \mathcal{H}x(t) && \text{substitute 1 for x} \\ &= \mathcal{H}\left\{\sum_k a_k e^{s_k t}\right\}(t) && \text{linearity of } \mathcal{H} \\ &= \sum_k a_k \mathcal{H}\{e^{s_k t}\}(t) && \text{complex exponentials are eigenfunctions of LTI systems} \\ &= \sum_k a_k H(s_k) e^{s_k t}. \end{aligned}$$

Thus, we have that

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{4.48}$$

Thus, if an input to a LTI system can be represented as a linear combination of complex exponentials, the output can also be represented as linear combination of the same complex exponentials. Furthermore, observe that the relationship between the input $x(t) = \sum_k a_k e^{s_k t}$ and output y in (4.48) does not involve convolution (such as in the equation $y = x * h$). In fact, the formula for y is identical to that for x except for the insertion of a constant multiplicative factor $H(s_k)$. In effect, we have used eigenfunctions to replace convolution with the much simpler operation of multiplication by a constant.

Example 4.16. Consider the LTI system $\mathcal{H}$ with the impulse response h given by

$$h(t) = \delta(t-1).$$

(a) Find the system function H of the system $\mathcal{H}$. (b) Use the system function H to determine the response y of the system $\mathcal{H}$ to the particular input x given by

$$x(t) = e^t \cos(\pi t).$$

Solution. (a) We find the system function H using (4.49). Substituting the given function h into (4.49), we obtain

$$\begin{aligned} H(s) &= \int_{-\infty}^{\infty} h(t)e^{-st}dt && \text{(4.49)} \\ &= \int_{-\infty}^{\infty} \delta(t-1)e^{-st}dt && \text{substitute given } h \\ &= \left[e^{-st}\right]\big|_{t=1} && \text{sifting property} \\ &= e^{-s}. \end{aligned}$$

(b) We can rewrite x to obtain

$$\begin{aligned} x(t) &= e^t \cos(\pi t) && \text{Euler} \\ &= e^t \left[\tfrac{1}{2}\left(e^{j\pi t} + e^{-j\pi t}\right)\right] && \text{exponent rules} \\ &= \tfrac{1}{2}e^{(1+j\pi)t} + \tfrac{1}{2}e^{(1-j\pi)t}. \end{aligned}$$

So, the input x is now expressed in the form

$$x(t) = \sum_{k=0}^{1} a_k e^{s_k t},$$

where

$$a_k = \tfrac{1}{2} \quad \text{and} \quad s_k = \begin{cases} 1+j\pi & k=0 \\ 1-j\pi & k=1. \end{cases}$$

Now, we use H and the eigenfunction properties of LTI systems to find y. Calculating y, we have

$$\begin{aligned} y(t) &= \sum_{k=0}^{1} a_k H(s_k) e^{s_k t} && \mathcal{H}\{a_k e^{s_k t}\}(t) = a_k H(s_k) e^{s_k t} \\ &= a_0 H(s_0)e^{s_0 t} + a_1 H(s_1) e^{s_1 t} && \text{expand summation} \\ &= \tfrac{1}{2}H(1+j\pi)e^{(1+j\pi)t} + \tfrac{1}{2}H(1-j\pi)e^{(1-j\pi)t} && \text{substitute for } a_k, s_k \\ &= \tfrac{1}{2}e^{-(1+j\pi)}e^{(1+j\pi)t} + \tfrac{1}{2}e^{-(1-j\pi)}e^{(1-j\pi)t} && \text{evaluate } H(\ldots) \\ &= \tfrac{1}{2}e^{t-1+j\pi t-j\pi} + \tfrac{1}{2}e^{t-1-j\pi t+j\pi} \\ &= \tfrac{1}{2}e^{t-1}e^{j\pi(t-1)} + \tfrac{1}{2}e^{t-1}e^{-j\pi(t-1)} && \text{rearrange} \\ &= e^{t-1}\left[\tfrac{1}{2}\left(e^{j\pi(t-1)} + e^{-j\pi(t-1)}\right)\right] && \text{Euler} \\ &= e^{t-1}\cos\left[\pi(t-1)\right]. \end{aligned}$$

Observe that the output y is just the input x time shifted by 1. This is not a coincidence because, as it turns out, a LTI system with the system function $H(s) = e^{-s}$ is an ideal unit delay (i.e., a system that performs a time shift of 1). ■

NOTE: THIS SOLUTION DID NOT REQUIRE THE COMPUTATION OF A CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

Interlude

Interlude

1) LTI systems are relatively simple mathematically and are extremely useful in practice (e.g., for modelling real-world systems).

2) LTI systems, while relatively simpler, involve convolution.

3) Are we doomed to directly face convolution in every problem we solve that involves LTI systems?

4) Often, there is a better way. Employ transform-based solution techniques that utilize mathematical tools such as:

- CT Fourier series
- CT Fourier transform
- Laplace transform

Unit:
CT Fourier Series

Example 5.1 (Fourier series of a periodic square wave)**.** Find the Fourier series representation of the periodic square wave x shown in Figure 5.1.

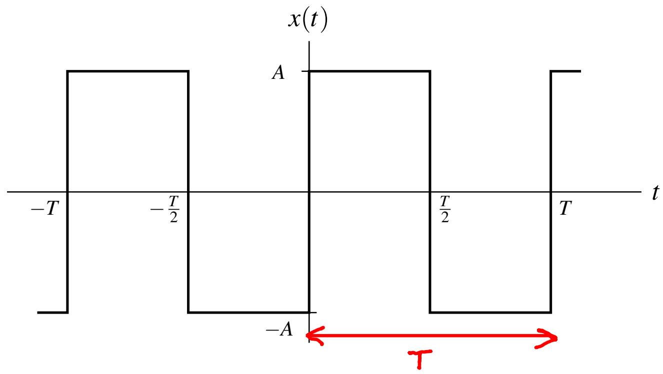


Figure 5.1: Periodic square wave.

Solution. Let us consider the single period of $x(t)$ for $0 \le t < T$. For this range of t, we have

$$x(t) = \begin{cases} A & 0 \le t < \frac{T}{2} \\ -A & \frac{T}{2} \le t < T. \end{cases}$$

Let $\omega_0 = \frac{2\pi}{T}$. From the Fourier series analysis equation, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ &= \tfrac{1}{T} \left(\int_0^{T/2} A e^{-jk\omega_0 t} dt + \int_{T/2}^{T} (-A) e^{-jk\omega_0 t} dt \right) \\ &= \begin{cases} \dfrac{1}{T} \left(\left[\dfrac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\dfrac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) & k \ne 0 \\ \dfrac{1}{T} \left([At]|_0^{T/2} + [-At]|_{T/2}^{T} \right) & k = 0. \end{cases} \end{aligned}$$

Fourier series analysis equation

split into 2 integrals and substitute given x

integrate

1

Now, we simplify the expression for c_k for each of the cases $k \ne 0$ and $k = 0$ in turn. First, suppose that $k \ne 0$. We have

$$\begin{aligned} c_k &= \frac{1}{T} \left(\left[\frac{-A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_0^{T/2} + \left[\frac{A}{jk\omega_0} e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{-A}{j2\pi k} \left(\left[e^{-jk\omega_0 t} \right] \Big|_0^{T/2} - \left[e^{-jk\omega_0 t} \right] \Big|_{T/2}^{T} \right) \\ &= \frac{jA}{2\pi k} \left(\left[e^{-j\pi k} - 1 \right] - \left[e^{-j2\pi k} - e^{-j\pi k} \right] \right) \\ &= \frac{jA}{2\pi k} \left[2e^{-j\pi k} - e^{-j2\pi k} - 1 \right] \\ &= \frac{jA}{2\pi k} \left[2(e^{-j\pi})^k - (e^{-j2\pi})^k - 1 \right]. \end{aligned}$$

from 1 above

factor out constant and $T\omega_0 = 2\pi$

simplify

(2)

Now, we observe that $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$. So, we have

from (2)

$$\begin{aligned}
c_k &= \frac{jA}{2\pi k}[2(-1)^k - 1^k - 1] \\
&= \frac{jA}{2\pi k}[2(-1)^k - 2] \\
&= \frac{jA}{\pi k}[(-1)^k - 1] \\
&= \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}, k \neq 0. \end{cases}
\end{aligned}$$

simplify

$(-1)^k - 1 = \begin{cases} -2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$

Now, suppose that $k = 0$. We have

from (1) above

$$\begin{aligned}
c_0 &= \tfrac{1}{T}\left([At]\big|_0^{T/2} + [-At]\big|_{T/2}^{T} \right) \\
&= \tfrac{1}{T}\left[\tfrac{AT}{2} - \tfrac{AT}{2} \right] \\
&= 0.
\end{aligned}$$

simplify

Thus, the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j(2\pi/T)kt},$$

where

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even}. \end{cases}$$

■

Example 5.3. Consider the periodic function x with fundamental period $T = 3$ as shown in Figure 5.3. Find the Fourier series representation of x.

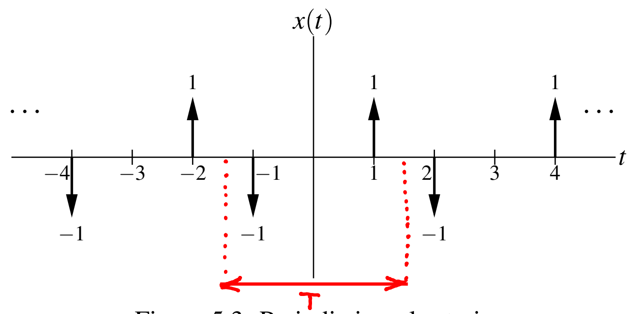


Figure 5.3: Periodic impulse train.

Solution. The function x has the fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{3}$. Let us consider the single period of $x(t)$ for $-\frac{T}{2} \le t < \frac{T}{2}$ (i.e., $-\frac{3}{2} \le t < \frac{3}{2}$). From the Fourier series analysis equation, we have

$$
\begin{aligned}
c_k &= \tfrac{1}{T}\int_T x(t)e^{-jk\omega_0 t}\,dt && \text{Fourier series analysis equation}\\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} x(t)e^{-j(2\pi/3)kt}\,dt && \text{consider interval } [-T/2, T/2)\\
&= \tfrac{1}{3}\int_{-3/2}^{3/2} \left[-\delta(t+1)+\delta(t-1)\right]e^{-j(2\pi/3)kt}\,dt && \text{substitute given } x\\
&= \tfrac{1}{3}\left[\int_{-3/2}^{3/2} -\delta(t+1)e^{-j(2\pi/3)kt}\,dt + \int_{-3/2}^{3/2}\delta(t-1)e^{-j(2\pi/3)kt}\,dt\right] && \text{split into 2 integrals}\\
&= \tfrac{1}{3}\left[-e^{-jk(2\pi/3)(-1)} + e^{-jk(2\pi/3)(1)}\right] && \text{extend limits and apply sifting property}\\
&= \tfrac{1}{3}\left[e^{-j(2\pi/3)k} - e^{j(2\pi/3)k}\right] && \text{simplify}\\
&= \tfrac{1}{3}\left[2j\sin\left(-\tfrac{2\pi}{3}k\right)\right] && \text{Euler } \left[\sin\theta = \tfrac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)\right]\\
&= \tfrac{2j}{3}\sin\left(-\tfrac{2\pi}{3}k\right) && \text{simplify}\\
&= -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right). && \text{sin is odd}
\end{aligned}
$$

Thus, x has the Fourier series representation

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\\
&= \sum_{k=-\infty}^{\infty} -\tfrac{2j}{3}\sin\left(\tfrac{2\pi}{3}k\right)e^{j(2\pi/3)kt}.
\end{aligned}
$$

■

Example 5.6. Consider the periodic function x with period $T = 2$ as shown in Figure 5.4. Let $\hat{x}$ denote the Fourier series representation of x (i.e., $\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$, where $\omega_0 = \pi$). Determine the values $\hat{x}(0)$ and $\hat{x}(1)$.

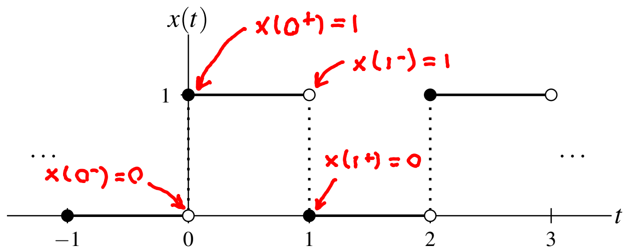


Figure 5.4: Periodic function x.

theorem for function satisfying Dirichlet conditions

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 5.4 applies. Thus, we have that

$$\hat{x}(0) = \tfrac{1}{2}\left[x(0^-) + x(0^+)\right]$$

average of left and right limits

$$\begin{aligned} &= \tfrac{1}{2}(0+1) \\ &= \tfrac{1}{2} \quad \text{and} \end{aligned}$$

$$\hat{x}(1) = \tfrac{1}{2}\left[x(1^-) + x(1^+)\right]$$

average of left and right limits

$$\begin{aligned} &= \tfrac{1}{2}(1+0) \\ &= \tfrac{1}{2}. \end{aligned}$$

■

Suppose that we have a complex periodic function x with period T and Fourier series coefficient sequence c. One can easily show that the coefficient c_0 is the average value of x over a single period T. The proof is trivial. Consider the Fourier series analysis equation given by (5.2). Substituting $k = 0$ into this equation, we obtain

$$\begin{aligned} c_0 &= \left[\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt\right]\bigg|_{k=0} \\ &= \frac{1}{T}\int_T x(t)e^0 dt \\ &= \frac{1}{T}\int_T x(t)dt. \end{aligned}$$

from Fourier series analysis equation

evaluate at $k=0$

$e^0 = 1$

Thus, c_0 is simply the average value of x over a single period.

Example 5.7. The periodic square wave x in Example 5.1 has fundamental period T, fundamental frequency ω_0, and the Fourier series coefficient sequence given by

$$c_k = \begin{cases} \frac{-j2A}{\pi k} & k \text{ odd} \\ 0 & k \text{ even,} \end{cases}$$

where A is a positive constant. Find and plot the magnitude and phase spectra of x. Determine at what frequency (or frequencies) x has the most information.

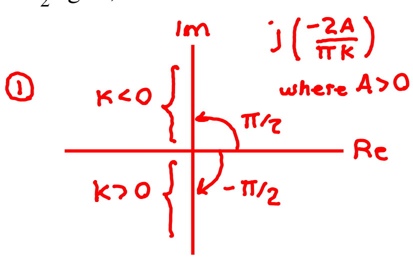
Solution. First, we compute the magnitude spectrum of x, which is given by $|c_k|$. We have

$$\begin{aligned} |c_k| &= \begin{cases} \left|\frac{-j2A}{\pi k}\right| & k \text{ odd} \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{2A}{\pi|k|} & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

$\left|\frac{-j2A}{\pi k}\right| = \frac{|-j2A|}{|\pi k|} = \frac{2A}{\pi|k|}$

(since $|a/b| = |a|/|b|$ and $|ab| = |a||b|$)

Next, we compute the phase spectrum of x, which is given by $\arg c_k$. Using the fact that $\arg 0 = 0$ and $\arg \frac{-j2A}{\pi k} = -\frac{\pi}{2} \operatorname{sgn} k$, we have

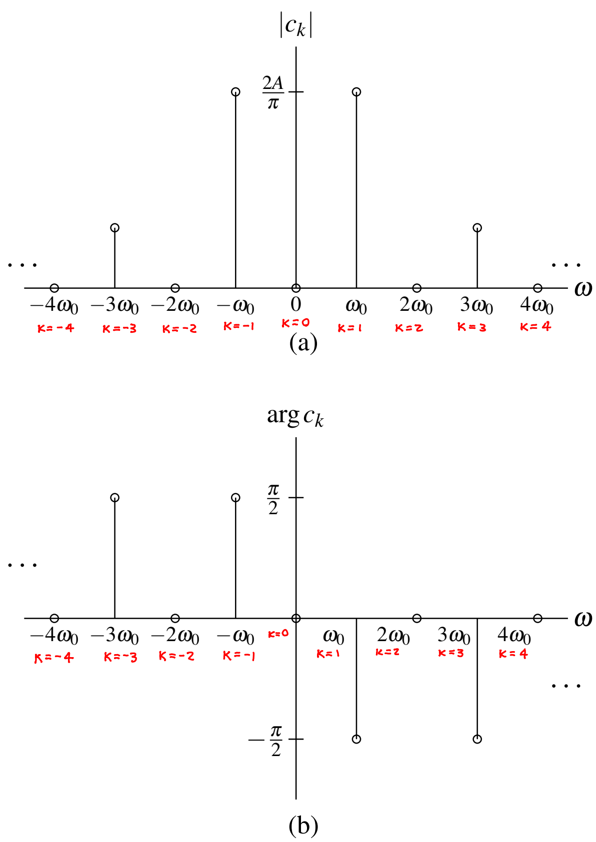


$$\begin{aligned} \arg c_k &= \begin{cases} \arg \frac{-j2A}{\pi k} & k \text{ odd} \\ \arg 0 & k \text{ even} \end{cases} \\ &= \begin{cases} \frac{\pi}{2} & k \text{ odd}, k < 0 \\ -\frac{\pi}{2} & k \text{ odd}, k > 0 \\ 0 & k \text{ even} \end{cases} \\ &= \begin{cases} -\frac{\pi}{2} \operatorname{sgn} k & k \text{ odd} \\ 0 & k \text{ even.} \end{cases} \end{aligned}$$

see ① for $\arg\left[j\left(\frac{-2A}{\pi k}\right)\right]$

$\operatorname{sgn} k = \begin{cases} 1 & k>0 \\ 0 & k=0 \\ -1 & k<0 \end{cases}$

The magnitude and phase spectra of x are plotted in Figures 5.7(a) and (b), respectively. Note that the magnitude spectrum is an even function, while the phase spectrum is an odd function. This is what we should expect, since x is real. Since $|c_k|$ is largest for $k = -1$ and $k = 1$, the function x has the most information at frequencies $-\omega_0$ and ω_0. ■



magnitude spectrum

- $|c_k|$ is largest for $k=-1$ ($-\omega_0$) and $k=1$ (ω_0)
- $|c_k|$ is even since x is real

phase spectrum

- $\arg c_k$ is odd since x is real

Figure 5.7: Frequency spectrum of the periodic square wave. (a) Magnitude spectrum and (b) phase spectrum.

Example 5.9. Consider a LTI system with the frequency response

$$H(\omega) = e^{-j\omega/4}.$$

Find the response y of the system to the input x, where

$$x(t) = \tfrac{1}{2}\cos(2\pi t).$$

Euler $\left[\cos\theta = \tfrac{1}{2}(e^{j\theta}+e^{-j\theta})\right]$

Solution. To begin, we rewrite x as

$$x(t) = \tfrac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}).$$

Thus, the Fourier series for x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$c_k = \begin{cases} \frac{1}{4} & k \in \{-1,1\} \\ 0 & \text{otherwise.} \end{cases}$$

Fourier series with only two nonzero terms

Thus, we can write

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t} \\
&= c_{-1}H(-\omega_0)e^{-j\omega_0 t} + c_1 H(\omega_0)e^{j\omega_0 t} \\
&= \tfrac{1}{4}H(-2\pi)e^{-j2\pi t} + \tfrac{1}{4}H(2\pi)e^{j2\pi t} \\
&= \tfrac{1}{4}e^{j\pi/2}e^{-j2\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j2\pi t} \\
&= \tfrac{1}{4}[e^{-j(2\pi t - \pi/2)} + e^{j(2\pi t - \pi/2)}] \\
&= \tfrac{1}{4}\left(2\cos(2\pi t - \tfrac{\pi}{2})\right) \\
&= \tfrac{1}{2}\cos(2\pi t - \tfrac{\pi}{2}) \\
&= \tfrac{1}{2}\cos\left(2\pi\left[t - \tfrac{1}{4}\right]\right).
\end{aligned}$$

from eigenfunction properties of LTI systems

expand summation

substitute for c_{-1}, c_1, ω_0

evaluate $H(\ldots)$

combine exponentials

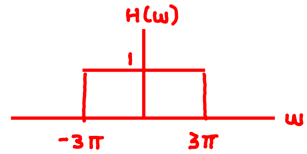
express in terms of cos (Euler)

Observe that $y(t) = x\left(t - \tfrac{1}{4}\right)$. This is not a coincidence because, as it turns out, a LTI system with the frequency response $H(\omega) = e^{-\omega/4}$ is an ideal delay of $\tfrac{1}{4}$ (i.e., a system that performs a time shift of $\tfrac{1}{4}$). ■

NOTE: THE APPROACH USED IN THE SOLUTION TO THIS PROBLEM DID NOT REQUIRE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

Example 5.10 (Lowpass filtering)**.** Suppose that we have a LTI system with input x, output y, and frequency response H , where



$$H(\omega) = \begin{cases} 1 & |\omega| \le 3\pi \\ 0 & \text{otherwise.} \end{cases}$$

Further, suppose that the input x is the periodic function

$$x(t) = 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t).$$

(a) Find the Fourier series representation of x. (b) Use this representation in order to find the response y of the system to the input x. (c) Plot the frequency spectra of x and y.

don't use Fourier series analysis equation!

Solution. (a) We begin by finding the Fourier series representation of x. Using Euler's formula, we can re-express x as

Euler

simplify and reorder terms

rewrite exponentials as $jk\omega_0$

ω_0 must be as large as possible

$$\begin{aligned}
x(t) &= 1 + 2\cos(2\pi t) + \cos(4\pi t) + \tfrac{1}{2}\cos(6\pi t) \\
&= 1 + 2\left[\tfrac{1}{2}(e^{j2\pi t} + e^{-j2\pi t})\right] + \left[\tfrac{1}{2}(e^{j4\pi t} + e^{-j4\pi t})\right] + \tfrac{1}{2}\left[\tfrac{1}{2}(e^{j6\pi t} + e^{-j6\pi t})\right] \\
&= 1 + e^{j2\pi t} + e^{-j2\pi t} + \tfrac{1}{2}[e^{j4\pi t} + e^{-j4\pi t}] + \tfrac{1}{4}[e^{j6\pi t} + e^{-j6\pi t}] \\
&= \tfrac{1}{4}e^{-j6\pi t} + \tfrac{1}{2}e^{-j4\pi t} + e^{-j2\pi t} + 1 + e^{j2\pi t} + \tfrac{1}{2}e^{j4\pi t} + \tfrac{1}{4}e^{j6\pi t} \\
&= \underbrace{\tfrac{1}{4}e^{j(-3)(2\pi)t}}_{k=-3} + \underbrace{\tfrac{1}{2}e^{j(-2)(2\pi)t}}_{k=-2} + \underbrace{e^{j(-1)(2\pi)t}}_{k=-1} + \underbrace{e^{j(0)(2\pi)t}}_{k=0} + \underbrace{e^{j(1)(2\pi)t}}_{k=1} + \underbrace{\tfrac{1}{2}e^{j(2)(2\pi)t}}_{k=2} + \underbrace{\tfrac{1}{4}e^{j(3)(2\pi)t}}_{k=3}.
\end{aligned}$$

From the last line of the preceding equation, we deduce that $\omega_0 = 2\pi$, since a larger value for ω_0 would imply that some Fourier series coefficient indices are noninteger, which clearly makes no sense. Thus, we have that the Fourier series of x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1 & k = 0 \\ 1 & k \in \{-1, 1\} \\ \tfrac{1}{2} & k \in \{-2, 2\} \\ \tfrac{1}{4} & k \in \{-3, 3\} \\ 0 & \text{otherwise.} \end{cases}$$

(b) Since the system is LTI, we know that the output y has the form

(due to eigenfunction properties of LTI systems)

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t},$$

where

$$b_k = a_k H(k\omega_0).$$

Using the results from above, we can calculate the b_k as follows:

$H(k\omega_0) = 1$

$$\begin{aligned}
b_0 &= a_0 H([0][2\pi]) = 1(1) = 1, \\
b_1 &= a_1 H([1][2\pi]) = 1(1) = 1, \\
b_{-1} &= a_{-1} H([-1][2\pi]) = 1(1) = 1,
\end{aligned}$$

$H(k\omega_0) = 0$

$$\begin{aligned}
b_2 &= a_2 H([2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_{-2} &= a_{-2} H([-2][2\pi]) = \tfrac{1}{2}(0) = 0, \\
b_3 &= a_3 H([3][2\pi]) = \tfrac{1}{4}(0) = 0, \quad \text{and} \\
b_{-3} &= a_{-3} H([-3][2\pi]) = \tfrac{1}{4}(0) = 0.
\end{aligned}$$

we were given

$$H(w) = \begin{cases} 1 & w \in [-3\pi, 3\pi] \\ 0 & \text{otherwise} \end{cases}$$

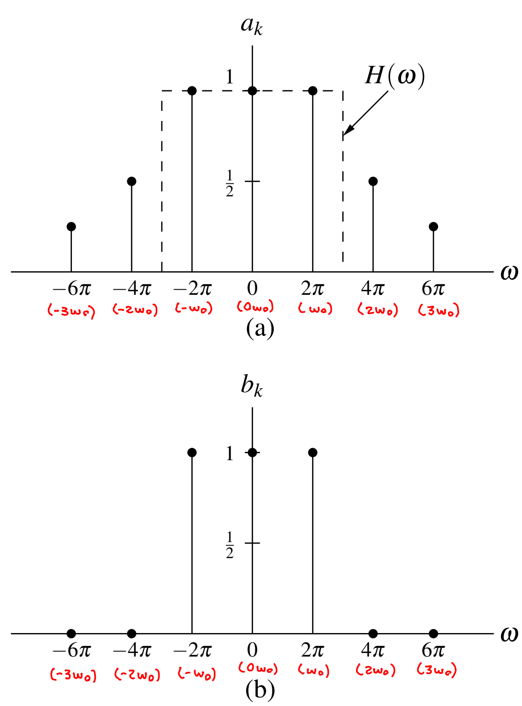
Thus, we have

$$b_k = \begin{cases} 1 & k \in \{-1,0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Lastly, we plot the frequency spectra of x and y in Figures 5.10(a) and (b), respectively. The frequency response H is superimposed on the plot of the frequency spectrum of x for illustrative purposes.

$b_K = a_K H(k\omega_0)$

$\omega_0 = 2\pi$



$\omega_0 = 2\pi$

Figure 5.10: Frequency spectra of the (a) input function x and (b) output function y.

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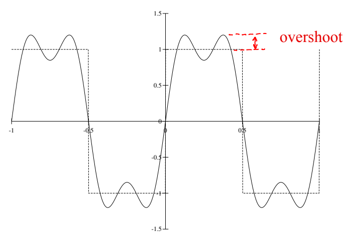
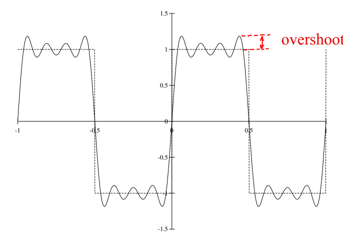
NOTE: THE APPROACH USED TO SOLVE THIS PROBLEM DID NOT INVOLVE CONVOLUTION!

THIS IS THE POWER OF EIGENFUNCTIONS!

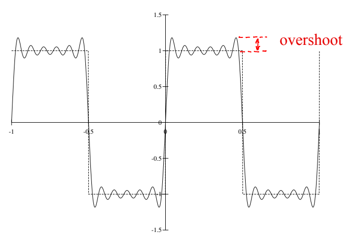
Gibbs Phenomenon: Periodic Square Wave Example



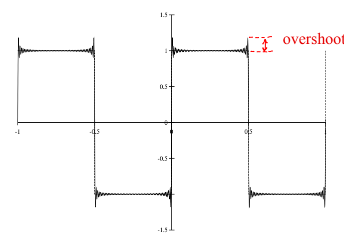
Fourier series truncated after the
3rd harmonic components



Fourier series truncated after the
7th harmonic components



Fourier series truncated after the
11th harmonic components



Fourier series truncated after the
101st harmonic components

Unit:
CT Fourier Transform

Example 6.1 (Fourier transform of the unit-impulse function)**.** Find the Fourier transform X of the function

$$x(t) = A\delta(t - t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

substitute given x into Fourier transform analysis equation

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\delta(t - t_0) e^{-j\omega t} dt \\ &= A \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt. \end{aligned}$$

pull constant A out of integral

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

sifting property

$= Ae^{-j\omega t}\big|_{t=t_0}$

$$X(\omega) = Ae^{-j\omega t_0}.$$

Thus, we have shown that

$$A\delta(t - t_0) \overset{\text{CTFT}}{\longleftrightarrow} Ae^{-j\omega t_0}.$$

From the Fourier transform analysis and synthesis equations, we have that the Fourier transform representation of x is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad \text{where} \quad X(\omega) = Ae^{-j\omega t_0}. \qquad \blacksquare$$

Example 6.3 (Fourier transform of the rectangular function)**.** Find the Fourier transform X of the function

$$x(t) = \operatorname{rect} t.$$

$\left[\operatorname{rect} t = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \right]$

Solution. From the definition of the Fourier transform, we can write

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t} dt.$$

substitute given function x into Fourier transform analysis equation

From the definition of the rectangular function, we can simplify this equation to obtain

change limits since $\operatorname{rect} t = 0$ for $|t| > \frac{1}{2}$

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t} dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t} dt. \end{aligned}$$

rect t = 1 for t in integration interval

Evaluating the integral and simplifying, we have

integrate

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t} \right] \Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega} \left(e^{j\omega/2} - e^{-j\omega/2} \right) \\ &= \tfrac{1}{j\omega} \left[2j \sin\left(\tfrac{\omega}{2}\right) \right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{\omega}{2}\right) \\ &= \left[\sin\left(\tfrac{\omega}{2}\right) \right] / \left(\tfrac{\omega}{2}\right) \\ &= \operatorname{sinc}\left(\tfrac{\omega}{2}\right). \end{aligned}$$

$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$

rewrite in form of sinc

definition of sinc function

Thus, we have shown that

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\tfrac{\omega}{2}\right).$$

Note: This is why the sinc function is of great importance.

■

Example 6.6. Consider the function x shown in Figure 6.5. Let $\hat{x}$ denote the Fourier transform representation of x (i.e., $\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$, where X denotes the Fourier transform of x). Determine the values $\hat{x}(-\frac{1}{2})$ and $\hat{x}(\frac{1}{2})$.

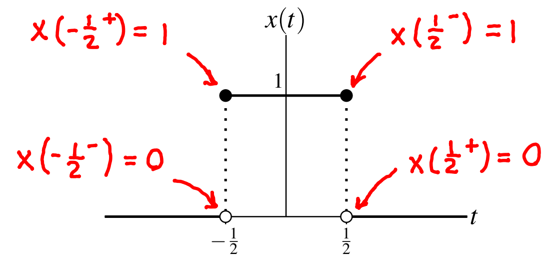


Figure 6.5: Function x.

At a point of discontinuity, the Fourier transform representation converges to the average of the left and right limits.

Solution. We begin by observing that x satisfies the Dirichlet conditions. Consequently, Theorem 6.3 applies. Thus, we have that

$$\begin{aligned}\hat{x}(-\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(-\tfrac{1}{2}^-) + x(-\tfrac{1}{2}^+)\right]\\ &= \tfrac{1}{2}(0+1)\\ &= \tfrac{1}{2} \quad \text{and}\end{aligned}$$

average of left and right limits

$$\begin{aligned}\hat{x}(\tfrac{1}{2}) &= \tfrac{1}{2}\left[x(\tfrac{1}{2}^-) + x(\tfrac{1}{2}^+)\right]\\ &= \tfrac{1}{2}(1+0)\\ &= \tfrac{1}{2}.\end{aligned}$$

average of left and right limits

■

Example 6.7 (Linearity property of the Fourier transform)**.** Using properties of the Fourier transform and the transform pair

$$e^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi\delta(\omega-\omega_0), \quad (1)$$

find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t),$$

where A and ω_0 are real constants.

Solution. We recall that (2) $\cos\alpha = \frac{1}{2}[e^{j\alpha}+e^{-j\alpha}]$ for any real α. Thus, we can write

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{A\cos(\omega_0 t)\})(\omega) \\ &= \left(\mathcal{F}\{\tfrac{A}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})\}\right)(\omega). \end{aligned}$$

from Euler (2)

Then, we use the linearity property of the Fourier transform to obtain

linearity

$$X(\omega) = \tfrac{A}{2}\mathcal{F}\{e^{j\omega_0 t}\}(\omega) + \tfrac{A}{2}\mathcal{F}\{e^{-j\omega_0 t}\}(\omega).$$

Using the given Fourier transform pair, we can further simplify the above expression for $X(\omega)$ as follows:

from given FT pair (1)

$$\begin{aligned} X(\omega) &= \tfrac{A}{2}[2\pi\delta(\omega+\omega_0)] + \tfrac{A}{2}[2\pi\delta(\omega-\omega_0)] \\ &= A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} A\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]. \quad \blacksquare$$

Example 6.9 (Time-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

$= v(t - [-\frac{\theta}{\omega_0}])$ table of FT pairs

Solution. Let 1 $v(t) = A\cos(\omega_0 t)$ so that 2 $x(t) = v(t + \frac{\theta}{\omega_0})$. Also, let $V = \mathcal{F}v$. From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad 3$$

Using this transform pair and the linearity property of the Fourier transform, we have that

linearity

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) && \text{from FT of 1} \\ 4 \quad &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{from FT pair 3 (and linearity)} \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) && \text{from FT of 2 using time-domain shifting property } [e^{-jw(-\theta/w_0)}] \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. && \text{substituting expression for } V(w) \text{ from 4} \end{aligned}$$

Thus, we have shown that

$$A\cos(\omega_0 t + \theta) \overset{\text{CTFT}}{\longleftrightarrow} A\pi e^{j\omega\theta/\omega_0}[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \qquad \blacksquare$$

Example 6.10 (Frequency-domain shifting property of the Fourier transform)**.** Find the Fourier transform X of the function

$$x(t) = \cos(\omega_0 t)\cos(20\pi t),$$

where ω_0 is a real constant.

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α. Using this relationship and the linearity property of the Fourier transform, we can write

$$\begin{aligned}
X(\omega) &= \left(\mathcal{F}\{\cos(\omega_0 t)\overbrace{(\tfrac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})}^{\cos(20\pi t)}\}\right)(\omega) \quad \text{distribute}\\
&= \left(\mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t) + \tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega) \quad \text{linearity property}\\
&= \tfrac{1}{2}\left(\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}\right)(\omega) + \tfrac{1}{2}\left(\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}\right)(\omega).
\end{aligned}$$

table of FT pairs

From Table 6.2, we have that

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]. \quad (1)$$

From this transform pair and the frequency-domain shifting property of the Fourier transform, we have

frequency domain shifting property

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega - 20\pi) + \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega + 20\pi) \quad \text{FT pair (1)}\\
&= \tfrac{1}{2}\left[\pi\left[\delta(v - \omega_0) + \delta(v + \omega_0)\right]\right]\Big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi\left[\delta(v - \omega_0) + \delta(v + \omega_0)\right]\right]\Big|_{v=\omega+20\pi} \quad \text{substitute}\\
&= \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]\right)\\
&= \tfrac{\pi}{2}\left[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)\right]. \quad \blacksquare
\end{aligned}$$

Example 6.11 (Time scaling property of the Fourier transform)**.** Using the Fourier transform pair

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right), \quad ①$$

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let ② $v(t) = \operatorname{rect} t$ so that ③ $x(t) = v(at)$. Also, let $V = \mathcal{F}v$. From the given transform pair, we know that

$$V(\omega) = (\mathcal{F}\{\operatorname{rect} t\})(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right). \tag{6.9}$$

← from FT of ② using FT pair ①

From the definition of v and the time-scaling property of the Fourier transform, we have

$$④ \rightarrow \quad X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

← from FT of ③ using time scaling property

Substituting the expression for V in (6.9) into the preceding equation, we have

$$X(\omega) = \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

substituting (6.9) into ④

Thus, we have shown that

$$\operatorname{rect}(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right). \qquad \blacksquare$$

Example 6.12 (Fourier transform of a real function)**.** Let X denote the Fourier transform of the function x. Show that, if x is real, then X is conjugate symmetric (i.e., $X(\omega) = X^*(-\omega)$ for all ω).

Solution. From the conjugation property of the Fourier transform, we have

$$\mathcal{F}\{x^*(t)\}(\omega) = X^*(-\omega).$$

from conjugation property

$x^* = x$ since x is real

Since x is real, we can replace x^* with x to yield

$$\mathcal{F}x(\omega) = X^*(-\omega),$$

$Fx = X$ (by definition)

or equivalently

$$X(\omega) = X^*(-\omega).$$

■

Example 6.13 (Fourier transform of the sinc function)**.** Using the transform pair

$$\underbrace{\operatorname{rect} t}_{v(t)} \overset{\text{CTFT}}{\longleftrightarrow} \underbrace{\operatorname{sinc}\left(\frac{\omega}{2}\right)}_{V(\omega)}, \quad ①$$

find the Fourier transform X of the function

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right).$$

Solution. From the given Fourier transform pair, we have

$$v(t) = \operatorname{rect} t \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad V(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right).$$

simply restating given FT pair ①

By duality, we have $\mathcal{F}V(\omega) = 2\pi v(-\omega)$

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) \underset{\text{duality}}{=} 2\pi v(-\omega) \underset{\text{given FT pair ①}}{=} 2\pi \operatorname{rect}(-\omega) \underset{\text{rect is even}}{=} 2\pi \operatorname{rect}\omega.$$

Thus, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi \operatorname{rect}\omega.$$

Observing that $V = x$ and $\mathcal{F}V = X$, we can rewrite the preceding relationship as

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X(\omega) = 2\pi \operatorname{rect}\omega.$$

Thus, we have shown that

$$X(\omega) = 2\pi \operatorname{rect}\omega.$$

■

table of FT pairs

Example 6.14 (Time-domain convolution property of the Fourier transform)**.** With the aid of Table 6.2, find the Fourier transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

Solution. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. From the time-domain convolution property of the Fourier transform, we know that

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\ &= X_1(\omega)X_2(\omega). \end{aligned} \tag{6.10}$$

time-domain convolution property

table of FT pairs

From Table 6.2, we know that

1

$$\begin{aligned} X_1(\omega) &= \left(\mathcal{F}\{e^{-2t}u(t)\}\right)(\omega) \\ &= \tfrac{1}{2+j\omega} \quad \text{and} \end{aligned}$$

table of FT pairs

2

$$\begin{aligned} X_2(\omega) &= \mathcal{F}u(\omega) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

table of FT pairs

Substituting these expressions for $X_1(\omega)$ and $X_2(\omega)$ into (6.10), we obtain

$X(\omega) = X_1(\omega)\, X_2(\omega)$ (6.10)

substituting 1 and 2 into (6.10)

$$\begin{aligned} X(\omega) &= [\tfrac{1}{2+j\omega}](\pi\delta(\omega) + \tfrac{1}{j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j\omega}(\tfrac{1}{2+j\omega}) \\ &= \tfrac{\pi}{2+j\omega}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2} \\ &= \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega-\omega^2}. \end{aligned}$$

$\left.\frac{\pi}{2+j\omega}\right|_{\omega=0}$

equivalence property of δ function

■

Example 6.15 (Frequency-domain convolution property)**.** Let x and y be functions related as

$$y(t) = x(t)\cos(\omega_c t),$$

where ω_c is a nonzero real constant. Let $Y = \mathcal{F}y$ and $X = \mathcal{F}x$. Find an expression for Y in terms of X.

Solution. To allow for simpler notation in what follows, we define

$$v(t) = \cos(\omega_c t) \quad (1)$$

table of FT pairs

and let V denote the Fourier transform of v. From Table 6.2, we have that

$$(2) \quad V(\omega) = \pi[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)]. \quad \leftarrow \text{from table of FT pairs}$$

From the definition of v, we have

$$(3) \quad y(t) = x(t)v(t). \quad \leftarrow \text{since } y(t) = x(t)\underbrace{\cos(\omega_c t)}_{v(t)}$$

Taking the Fourier transform of both sides of this equation, we have

$$Y(\omega) = \mathcal{F}\{x(t)v(t)\}(\omega). \quad \text{taking FT of both sides of (3)}$$

Using the frequency-domain convolution property of the Fourier transform, we obtain

frequency-domain convolution property

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi} X * V(\omega) \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. \quad && \text{definition of convolution}
\end{aligned}$$

Substituting the above expression for V, we obtain

substitute V from (2)

$$\begin{aligned}
Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)])d\lambda && \text{cancel } \pi\text{'s} \\
&= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-\omega_c)+\delta(\omega-\lambda+\omega_c)]d\lambda && \text{split into two integrals} \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\underbrace{\omega-\lambda-\omega_c})d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\underbrace{\omega-\lambda+\omega_c})d\lambda\right] && \delta \text{ is even} \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+\omega_c)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-\omega_c)d\lambda\right] && \text{regroup} \\
&= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-\omega_c)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+\omega_c)]d\lambda\right] && \text{sifting property} \\
&= \tfrac{1}{2}[X(\omega-\omega_c)+X(\omega+\omega_c)] \\
&= \tfrac{1}{2}X(\omega-\omega_c)+\tfrac{1}{2}X(\omega+\omega_c). && \text{expand}
\end{aligned}$$

■

Example 6.16 (Time-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. Taking the Fourier transform of both sides of the given equation for x yields

$$X(\omega) = \left(\mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}\right)(\omega).$$

Using the time-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \left(\mathcal{F}\left\{\tfrac{d}{dt}\delta(t)\right\}\right)(\omega) \\ &= j\omega\mathcal{F}\delta(\omega). \end{aligned}$$

from definition of x

time-domain differentiation property

Evaluating the Fourier transform of δ using Table 6.2, we obtain

$F\delta(\omega) = 1$

$$\begin{aligned} X(\omega) &= j\omega(1) \\ &= j\omega. \end{aligned}$$

■

Example 6.17 (Frequency-domain differentiation property)**.** Find the Fourier transform X of the function

$$x(t) = t\cos(\omega_0 t),$$

where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for x yields

$$X(\omega) = \mathcal{F}\{t\cos(\omega_0 t)\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

from definition of x

$$\begin{aligned} X(\omega) &= \mathcal{F}\{t\cos(\omega_0 t)\}(\omega) \\ &= j\left(\mathcal{D}\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega), \end{aligned}$$

frequency-domain differentiation property

where $\mathcal{D}$ denotes the derivative operator. Evaluating the Fourier transform on the right-hand side using Table 6.2, we obtain

table of FT pairs

from FT pair 1

$$\begin{aligned} X(\omega) &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). \end{aligned}$$

factor out π

derivative operator is linear

■

$$\cos(\omega_0 t) \overset{\text{FT}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] \quad (1)$$

Example 6.18 (Time-domain integration property of the Fourier transform)**.** Use the time-domain integration property of the Fourier transform in order to find the Fourier transform X of the function $x = u$.

Solution. We begin by observing that x can be expressed in terms of an integral as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \quad 1$$

Now, we consider the Fourier transform of x. We have

from 1

$$X(\omega) = \left(\mathcal{F} \left\{ \int_{-\infty}^{t} \delta(\tau) d\tau \right\} \right) (\omega).$$

time-domain integration property

From the time-domain integration property, we can write

$$X(\omega) = \tfrac{1}{j\omega} \mathcal{F}\delta(\omega) + \pi \mathcal{F}\delta(0)\delta(\omega).$$

Evaluating the two Fourier transforms on the right-hand side using Table 6.2, we obtain

$\mathcal{F}\delta(\omega) = 1$

$$\begin{aligned} X(\omega) &= \tfrac{1}{j\omega}(1) + \pi(1)\delta(\omega) \\ &= \tfrac{1}{j\omega} + \pi\delta(\omega). \end{aligned}$$

drop 1's

Thus, we have shown that $u(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega)$. ■

Example 6.19 (Energy of the sinc function)**.** Consider the function $x(t) = \operatorname{sinc}\left(\frac{1}{2}t\right)$, which has the Fourier transform X given by $X(\omega) = 2\pi \operatorname{rect} \omega$. Compute the energy of x.

①

Solution. We could directly compute the energy of x as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt. = \int_{-\infty}^{\infty} \left|\frac{\sin t/2}{t/2}\right|^2 dt \to ☹ \end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega \\ &= 2\pi \int_{-1/2}^{1/2} d\omega \\ &= 2\pi [\omega]|_{-1/2}^{1/2} \\ &= 2\pi [\tfrac{1}{2} + \tfrac{1}{2}] \\ &= 2\pi. \end{aligned}$$

from given X in ①

rect $t = 1$ for $t \in [-\frac{1}{2}, \frac{1}{2}]$ and zero otherwise

cancel one 2π factor

integrate

Thus, we have

$$E = \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt = 2\pi. \quad \blacksquare$$

Answer (g).

We are asked to find the Fourier transform Y of

$$y(t) = \left[te^{-j5t}x(t)\right]^*.$$

In what follows, we use the prime symbol to denote the derivative (i.e., f' denotes the derivative of f). To begin, we have

$$\begin{aligned} y(t) &= \left[te^{-j5t}x(t)\right]^* \\ &= \left[e^{-j5t}\underbrace{tx(t)}\right]^*. \end{aligned}$$

$v_1(t) = tx(t)$ 1

Letting $v_1(t) = tx(t)$, we have

$$y(t) = \left[\underbrace{e^{-j5t}v_1(t)}\right]^*.$$

$v_2(t) = e^{-j5t}\, v_1(t)$ 2

Letting $v_2(t) = e^{-j5t}v_1(t)$, we have

$$y(t) = v_2^*(t). \quad 3$$

Thus, we have written $y(t)$ as

$$3 \longrightarrow \quad y(t) = v_2^*(t)$$

where

$$\begin{aligned} 1 \longrightarrow \quad & v_1(t) = tx(t) \quad \text{and} \\ 2 \longrightarrow \quad & v_2(t) = e^{-j5t}v_1(t). \end{aligned}$$

Taking the Fourier transforms of the preceding equations, we obtain

$$\begin{aligned} 4 \quad V_1(\omega) &= jX'(\omega), && \longleftarrow \text{FT of 1 using frequency-domain differentiation property} \\ 5 \quad V_2(\omega) &= V_1(\omega+5), \quad \text{and} && \longleftarrow \text{FT of 2 using frequency-domain shifting property} \\ 6 \quad Y(\omega) &= V_2^*(-\omega). && \longleftarrow \text{FT of 3 using conjugation property} \end{aligned}$$

Combining the above equations, we have

$$\begin{aligned} Y(\omega) &= V_2^*(-\omega) && \longleftarrow 6 \\ &= [V_1(-\omega+5)]^* && \text{substitute 5} \\ &= \left[jX'(-\omega+5)\right]^* && \text{substitute 4} \\ &= -jX'^*(-\omega+5). && (ab)^* = a^*b^* \end{aligned}$$

Example 6.26. Let X and Y denote the Fourier transforms of x and y, respectively. Suppose that $y(t) = x(t)\cos(at)$, where a is a nonzero real constant. Find an expression for Y in terms of X.

Solution. Essentially, we need to take the Fourier transform of both sides of the given equation. There are two obvious ways in which to do this. One is to use the time-domain multiplication property of the Fourier transform, and another is to use the frequency-domain shifting property. We will solve this problem using each method in turn in order to show that the two approaches do not involve an equal amount of effort.

You really don't want to do this!

FIRST SOLUTION (USING AN UNENLIGHTENED APPROACH). We use the time-domain multiplication property. To allow for simpler notation in what follows, we define

$$v(t) = \cos(at)$$

table of FT pairs

and let V denote the Fourier transform of v. From Table 6.2, we have that

from table of FT pairs

$$V(\omega) = \pi[\delta(\omega - a) + \delta(\omega + a)].$$

Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t)v(t)\})(\omega) && \text{from definition of } y \\ &= \tfrac{1}{2\pi} X * V(\omega) && \text{time-domain multiplication property} \\ &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)V(\omega-\lambda)d\lambda. && \text{definition of convolution} \end{aligned}$$

Substituting the above expression for V, we obtain

$$\begin{aligned} Y(\omega) &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)(\pi[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)])d\lambda && \text{cancel } \pi\text{'s} \\ &= \tfrac{1}{2}\int_{-\infty}^{\infty} X(\lambda)[\delta(\omega-\lambda-a)+\delta(\omega-\lambda+a)]d\lambda && \text{split into two integrals} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda-a})d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\underline{\omega-\lambda+a})d\lambda\right] && \delta \text{ is even} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega+a)d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta(\lambda-\omega-a)d\lambda\right] && \text{make shifts explicit} \\ &= \tfrac{1}{2}\left[\int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega-a)]d\lambda + \int_{-\infty}^{\infty} X(\lambda)\delta[\lambda-(\omega+a)]d\lambda\right] && \text{sifting property} \\ &= \tfrac{1}{2}[X(\omega-a)+X(\omega+a)] \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

Note that the above solution is essentially identical to the one appearing earlier in Example 6.15 on page 1.

This approach is much simpler

SECOND SOLUTION (USING AN ENLIGHTENED APPROACH). We use the frequency-domain shifting property. Taking the Fourier transform of both sides of the given equation, we obtain

$$\begin{aligned} Y(\omega) &= (\mathcal{F}\{x(t)\cos(at)\})(\omega) && \text{from definition of } y \\ &= \left(\mathcal{F}\{\tfrac{1}{2}(e^{jat}+e^{-jat})x(t)\}\right)(\omega) && \text{Euler; linearity property} \\ &= \tfrac{1}{2}\left(\mathcal{F}\{e^{jat}x(t)\}\right)(\omega) + \tfrac{1}{2}\left(\mathcal{F}\{e^{-jat}x(t)\}\right)(\omega) && \text{frequency-domain shifting property} \\ &= \tfrac{1}{2}X(\omega-a)+\tfrac{1}{2}X(\omega+a). \end{aligned}$$

COMMENTARY. Clearly, of the above two solution methods, the second approach is simpler and much less error prone. Generally, the use of the time-domain multiplication property tends to lead to less clean solutions, as this forces a convolution to be performed in the frequency domain and convolution is often best avoided if possible. ■

THE TAKEAWAY: Only use the time-domain multiplication property when absolutely necessary, since its use will result in the appearance of a convolution operation.

Answer (j).

We are asked to find the Fourier transform X of

$$x(t) = \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1) d\tau.$$

$$\begin{aligned} x(t) &= \int_{-\infty}^{5t} e^{-\tau-1} u(\tau-1)\, d\tau \\ &= \int_{-\infty}^{5t} e^{-2} \underbrace{e^{-(\tau-1)} u(\tau-1)}_{v_1(\tau-1)}\, d\tau \quad \text{where } v_1(t) = e^{-t}u(t) \quad 1 \\ &= \int_{-\infty}^{5t} e^{-2} \underbrace{v_1(\tau-1)}_{v_2(\tau)}\, d\tau \quad \text{where } v_2(t) = v_1(t-1) \quad 2 \\ &= \underbrace{e^{-2} \int_{-\infty}^{5t} v_2(\tau)\, d\tau}_{v_3(5t)} \quad \text{where } v_3(t) = e^{-2}\int_{-\infty}^{t} v_2(\tau)\, d\tau \quad 3 \\ &= v_3(5t) \quad 4 \end{aligned}$$

We begin by rewriting $x(t)$ as

4 $\rightarrow$ $$x(t) = v_3(5t),$$

where

1 $\rightarrow$ $$v_1(t) = e^{-t}u(t),$$
2 $\rightarrow$ $$v_2(t) = v_1(t-1), \quad \text{and}$$
3 $\rightarrow$ $$v_3(t) = \int_{-\infty}^{t} e^{-2} v_2(\tau) d\tau. = e^{-2}\int_{-\infty}^{t} v_2(\tau)\, d\tau$$

Taking the Fourier transform of both sides of each of the above equations yields

5 $$V_1(\omega) = \frac{1}{1+j\omega}, \leftarrow \text{FT of 1 using FT table}$$
6 $$V_2(\omega) = e^{-j\omega} V_1(\omega), \leftarrow \text{FT of 2 using time shifting property}$$
7 $$V_3(\omega) = e^{-2}\left[\frac{1}{j\omega} V_2(\omega) + \pi V_2(0)\delta(\omega)\right], \text{ and} \leftarrow \text{FT of 3 using integration property}$$
8 $$X(\omega) = \tfrac{1}{5} V_3(\omega/5). \leftarrow \text{FT of 4 using time scaling property}$$

Combining the above results, we have

8 $\rightarrow$
$$\begin{aligned} X(\omega) &= \tfrac{1}{5} V_3(\omega/5) \quad \text{substitute } 7 \\ &= \tfrac{1}{5} e^{-2}\left[\left(\frac{1}{j(\omega/5)}\right) V_2(\omega/5) + \pi V_2(0)\delta(\omega/5)\right] \\ &= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right) V_2(\omega/5) + \pi V_2(0)\delta(\omega/5)\right] \quad \text{substitute } 6 \\ &= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right) e^{-j\omega/5} V_1(\omega/5) + \pi V_1(0)\delta(\omega/5)\right] \quad \text{substitute } 5 \\ &= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right) e^{-j\omega/5}\left(\frac{1}{1+j(\omega/5)}\right) + \pi\delta(\omega/5)\right] \\ &= \tfrac{1}{5e^2}\left[\left(\frac{5}{j\omega}\right)\left(\frac{5}{5+j\omega}\right) e^{-j\omega/5} + \pi\delta(\omega/5)\right] \quad \text{simplify} \\ &= \tfrac{1}{5e^2}\left[\left(\frac{25}{j5\omega-\omega^2}\right) e^{-j\omega/5} + \pi\delta(\omega/5)\right]. \end{aligned}$$

Example 6.20. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively. Suppose that X_1 and X_2 are as shown in Figures 6.6(a) and (b). Determine whether x_1 and x_2 are periodic.

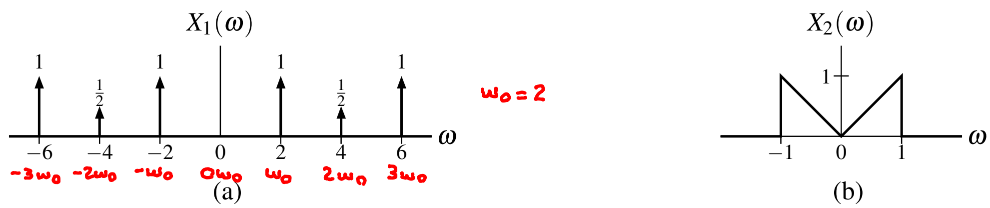


Figure 6.6: Frequency spectra. The frequency spectra (a) X_1 and (b) X_2.

Solution. We know that the Fourier transform X of a T-periodic function x must be of the form

$$X(\omega) = \sum_{k=-\infty}^{\infty} \alpha_k \delta(\omega - k\omega_0),$$

where $\omega_0 = \frac{2\pi}{T}$ and the $\{\alpha_k\}$ are complex constants. The spectrum X_1 does have this form, with $\omega_0 = 2$ and $T = \frac{2\pi}{2} = \pi$. Therefore, x_1 must be π-periodic. The spectrum X_2 does not have this form. Therefore, x_2 must not be periodic. ■

Example 6.21. Consider the periodic function x with fundamental period $T = 2$ as shown in Figure 6.7. Using the Fourier transform, find the Fourier series representation of x.

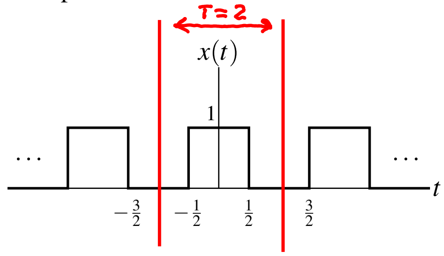


Figure 6.7: Periodic function x.

since T=2

Solution. Let ω_0 denote the fundamental frequency of x. We have that $\omega_0 = \frac{2\pi}{T} = \pi$. Let $y(t) = \operatorname{rect} t$ (i.e., y corresponds to a single period of the periodic function x). Thus, we have that

$$x(t) = \sum_{k=-\infty}^{\infty} y(t-2k).$$

Let Y denote the Fourier transform of y. Taking the Fourier transform of y, we obtain

$$Y(\omega) = (\mathcal{F}\{\operatorname{rect} t\})(\omega) \overset{\text{FT table}}{=} \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \quad (1)$$

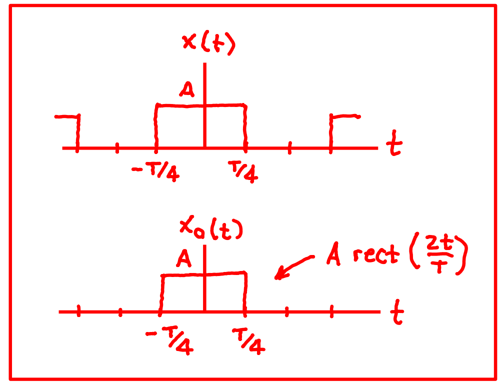
Now, we seek to find the Fourier series representation of x, which has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

Using the Fourier transform, we have

$$\begin{aligned} c_k &= \tfrac{1}{T} Y(k\omega_0) && \text{sample FT of } y \text{ at } k\omega_0 \text{ for } k^{th} \text{ FS coefficient} \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\omega_0}{2}k\right) && \text{substitute (1)} \\ &= \tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\pi}{2}k\right). && \omega_0 = \pi \end{aligned}$$

■



Example 6.24. Consider the periodic function x given by

$$x(t) = \sum_{k=-\infty}^{\infty} x_0(t-kT),$$

where a single period of x is given by

$$x_0(t) = A\,\mathrm{rect}\left(\tfrac{2t}{T}\right)$$

and A is a real constant. Find the Fourier transform X of the function x.

Solution. From (6.16b), we know that

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\,\delta(\omega - k\omega_0)$$

$$\begin{aligned} X(\omega) &= \mathcal{F}\left\{\sum_{k=-\infty}^{\infty} x_0(t-kT)\right\}(\omega) \\ &= \sum_{k=-\infty}^{\infty} \omega_0 X_0(k\omega_0)\,\delta(\omega - k\omega_0). \end{aligned}$$

using (6.16)

table of FT pairs

So, we need to find X_0. Using the linearity property of the Fourier transform and Table 6.2, we have

$$\begin{aligned} X_0(\omega) &= \mathcal{F}\left\{A\,\mathrm{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= A\mathcal{F}\left\{\mathrm{rect}\left(\tfrac{2t}{T}\right)\right\}(\omega) \\ &= \tfrac{AT}{2}\,\mathrm{sinc}\left(\tfrac{\omega T}{4}\right). \end{aligned}$$

from definition of x

linearity

FT table

Thus, we have that

$$\begin{aligned} X(\omega) &= \sum_{k=-\infty}^{\infty} \omega_0 \left(\tfrac{AT}{2}\right) \mathrm{sinc}\left(\frac{k\omega_0 T}{4}\right) \delta(\omega - k\omega_0) \\ &= \sum_{k=-\infty}^{\infty} \pi A\,\mathrm{sinc}\left(\tfrac{\pi k}{2}\right) \delta(\omega - k\omega_0). \end{aligned}$$

$\omega_0 = \frac{2\pi}{T}$

■

Example 6.30 (Frequency spectrum of a time-shifted signum function)**.** The function

$$x(t) = \operatorname{sgn}(t-1)$$

has the Fourier transform

$$X(\omega) = \tfrac{2}{j\omega}e^{-j\omega}. \qquad (1)$$

(a) Find and plot the magnitude and phase spectra of x. (b) Determine at what frequency (or frequencies) x has the most information.

Solution. (a) First, we find the magnitude spectrum $|X(\omega)|$. From the expression for $X(\omega)$, we can write

take magnitude of both sides of (1)

$$\begin{aligned} |X(\omega)| &= \left|\tfrac{2}{j\omega}e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right|\left|e^{-j\omega}\right| \\ &= \left|\tfrac{2}{j\omega}\right| \\ &= \tfrac{2}{|\omega|}. \end{aligned}$$

$|ab| = |a||b|$

$|e^{j\theta}| = 1$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Next, we find the phase spectrum $\arg X(\omega)$. First, we observe that $\arg X(\omega)$ is not well defined if $\omega = 0$. So, we assume that $\omega \neq 0$. From the expression for $X(\omega)$, we can write (for $\omega \neq 0$)

take argument of both sides of (1)

$$\begin{aligned} \arg X(\omega) &= \arg\left\{\tfrac{2}{j\omega}e^{-j\omega}\right\} \\ &= \arg e^{-j\omega} + \arg \tfrac{2}{j\omega} \\ &= -\omega + \arg\tfrac{2}{j\omega} \\ &= -\omega + \arg(-\tfrac{j2}{\omega}) \\ &= \begin{cases} -\frac{\pi}{2} - \omega & \omega > 0 \\ \frac{\pi}{2} - \omega & \omega < 0 \end{cases} \\ &= -\tfrac{\pi}{2}\operatorname{sgn}\omega - \omega. \end{aligned}$$

$\arg(ab) = \arg a + \arg b$

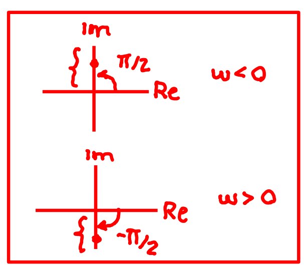
$\arg(e^{j\theta}) = \theta$

$\frac{1}{j} = -j$

(*)

definition of signum function

In the above simplification, we used the fact that



(*) $$\arg\tfrac{2}{j\omega} = \arg(-\tfrac{j2}{\omega}) = \begin{cases} -\frac{\pi}{2} & \omega > 0 \\ \frac{\pi}{2} & \omega < 0. \end{cases}$$

Finally, using numerical calculation, we can plot the graphs of $|X(\omega)|$ and $\arg X(\omega)$ to obtain the results shown in Figures 6.10(a) and (b).

(b) Since $|X(\omega)|$ is largest for $\omega = 0$, x has the most information at the frequency 0.

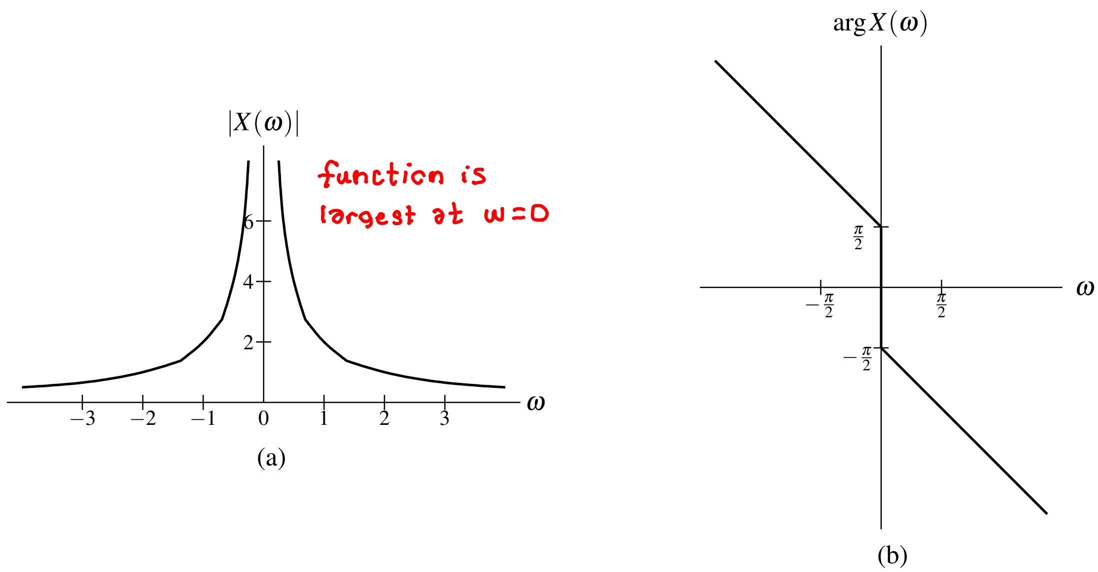


Figure 6.10: Frequency spectrum of the time-shifted signum function. (a) Magnitude spectrum and (b) phase spectrum of x.

■

Example 6.34 (Differential equation to frequency response)**.** A LTI system with input x and output y is characterized by the differential equation

$$7y''(t) + 11y'(t) + 13y(t) = 5x'(t) + 3x(t),$$

where x', y', and y'' denote the first derivative of x, the first derivative of y, and the second derivative of y, respectively. Find the frequency response H of this system.

Solution. Taking the Fourier transform of the given differential equation, we obtain

$\left(\frac{d}{dt}\right)^n x(t) \xleftrightarrow{FT} (jw)^n X(w)$

$$7(j\omega)^2 Y(\omega) + 11 j\omega Y(\omega) + 13Y(\omega) = 5 j\omega X(\omega) + 3X(\omega).$$

move terms containing Y and X to the left- and right-hand sides, respectively and factor

Rearranging the terms and factoring, we have

$$(-7\omega^2 + 11 j\omega + 13)Y(\omega) = (5 j\omega + 3)X(\omega).$$

divide both sides by $(-7w^2 + 11jw + 13)\,X(w)$

Thus, H is given by

① $\dfrac{Y(w)}{X(w)} = \dfrac{5jw+3}{-7w^2+11jw+13}$

$$H(\omega) \overset{*}{=} \frac{Y(\omega)}{X(\omega)} = \underbrace{\frac{5 j\omega + 3}{-7\omega^2 + 11 j\omega + 13}}_{①}.$$ ■

$*$ Since system is LTI, $Y(w) = X(w)\,H(w) \implies H(w) = \dfrac{Y(w)}{X(w)}$

Example 6.35 (Frequency response to differential equation)**.** A LTI system with input x and output y has the frequency response

$$H(\omega) = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Since system is LTI,
$Y(\omega) = X(\omega)H(\omega) \Rightarrow$
$H(\omega) = \frac{Y(\omega)}{X(\omega)}$

Find the differential equation that characterizes this system.

Solution. From the given frequency response H, we have

$$\frac{Y(\omega)}{X(\omega)} = \frac{-7\omega^2 + 11j\omega + 3}{-5\omega^2 + 2}.$$

Multiplying both sides by $(-5\omega^2 + 2)X(\omega)$, we have

$$-5\omega^2 Y(\omega) + 2Y(\omega) = -7\omega^2 X(\omega) + 11j\omega X(\omega) + 3X(\omega).$$

write with powers of $j\omega$

Applying some simple algebraic manipulation yields

$$5(j\omega)^2 Y(\omega) + 2Y(\omega) = 7(j\omega)^2 X(\omega) + 11(j\omega)X(\omega) + 3X(\omega).$$

Taking the inverse Fourier transform of the preceding equation, we obtain

$$\left(\frac{d}{dt}\right)^n x(t) \xleftrightarrow{FT} (j\omega)^n X(\omega)$$

$$5y''(t) + 2y(t) = 7x''(t) + 11x'(t) + 3x(t).$$ ■

Example 6.38 (Bandpass filtering)**.** Consider a LTI system with the impulse response

$$h(t) = \tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t).$$

from FT table:

$1 \overset{\text{FT}}{\longleftrightarrow} 2\pi\delta(\omega)$

$\cos(\omega_0 t) \overset{\text{FT}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

Using frequency-domain methods, find the response y of the system to the input

$$x(t) = -1 + 2\cos(2t) + \cos(4t) - \cos(6t).$$

taking FT

Solution. Taking the Fourier transform of x, we have

$$\begin{aligned} X(\omega) &= -2\pi\delta(\omega) + 2(\pi[\delta(\omega-2)+\delta(\omega+2)]) + \pi[\delta(\omega-4)+\delta(\omega+4)] - \pi[\delta(\omega-6)+\delta(\omega+6)] \\ &= -\pi\delta(\omega+6) + \pi\delta(\omega+4) + 2\pi\delta(\omega+2) - 2\pi\delta(\omega) + 2\pi\delta(\omega-2) + \pi\delta(\omega-4) - \pi\delta(\omega-6). \end{aligned}$$

The frequency spectrum X is shown in Figure 6.22(a). Now, we compute the frequency response H of the system. Using the results of Example 6.36, we can determine H to be

Example 6.36 found the FT pair

$\frac{2\omega_b}{\pi}\operatorname{sinc}(\omega_b t)\cos(\omega_a t) \overset{\text{FT}}{\longleftrightarrow} \operatorname{rect}\left(\frac{\omega-\omega_a}{2\omega_b}\right) + \operatorname{rect}\left(\frac{\omega+\omega_a}{2\omega_b}\right)$

$$\begin{aligned} H(\omega) &= \mathcal{F}\{\tfrac{2}{\pi}\operatorname{sinc}(t)\cos(4t)\}(\omega) \\ &= \operatorname{rect}\left(\tfrac{\omega-4}{2}\right) + \operatorname{rect}\left(\tfrac{\omega+4}{2}\right) \\ &= \begin{cases} 1 & 3 \le |\omega| \le 5 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

using result from Example 6.36 with $\omega_b = 1$, $\omega_a = 4$

definition of rect function

The frequency response H is shown in Figure 6.22(b). The frequency spectrum Y of the output is given by

$$\begin{aligned} Y(\omega) &= H(\omega)X(\omega) \\ &= \pi\delta(\omega+4) + \pi\delta(\omega-4). \end{aligned}$$

only two shifted delta functions are nonzero when $H(\omega) \neq 0$ [see Figures 6.22(a) and (b).]

taking inverse FT

Taking the inverse Fourier transform, we obtain

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\pi\delta(\omega+4) + \pi\delta(\omega-4)\}(t) \\ &= \mathcal{F}^{-1}\{\pi[\delta(\omega+4) + \delta(\omega-4)]\}(t) \\ &= \cos(4t). \end{aligned}$$

from table of FT pairs

$\cos(\omega_0 t) \overset{\text{FT}}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

■

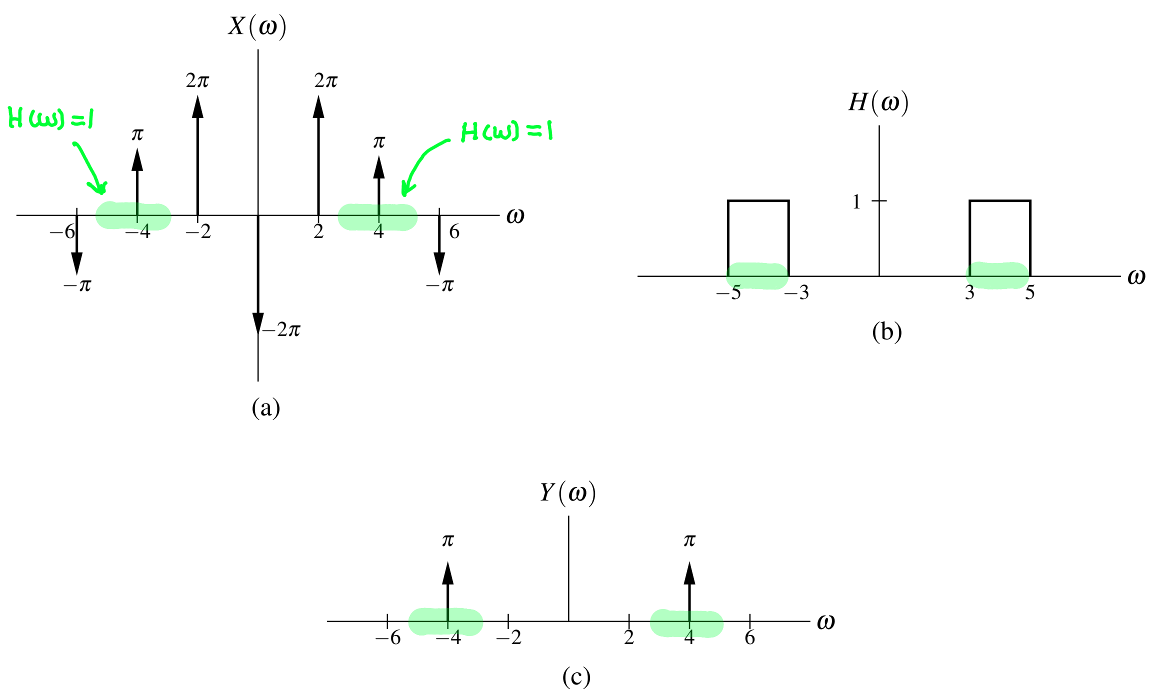


Figure 6.22: Frequency spectra for bandpass filtering example. (a) Frequency spectrum of the input x. (b) Frequency response of the system. (c) Frequency spectrum of the output y.

Example 6.40 (Simple RL network)**.** Consider the resistor-inductor (RL) network shown in Figure 6.26 with input v_1 and output v_2. This system is LTI, since it can be characterized by a linear differential equation with constant coefficients. (a) Find the frequency response H of the system. (b) Find the response v_2 of the system to the input $v_1(t) = \operatorname{sgn} t$.

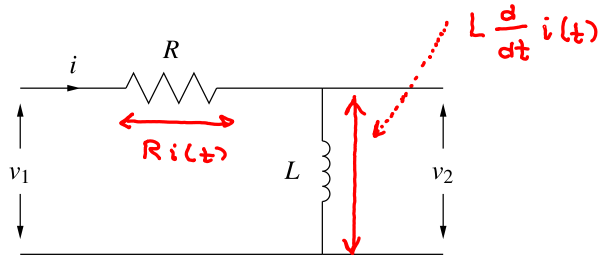


Figure 6.26: Simple RL network.

Solution. (a) From basic circuit analysis, we can write

$$v_1(t) = Ri(t) + L\tfrac{d}{dt}i(t) \quad \text{and} \tag{6.35}$$

$$v_2(t) = L\tfrac{d}{dt}i(t). \tag{6.36}$$

(Recall that the voltage v across an inductor L is related to the current i through the inductor as $v(t) = L\frac{d}{dt}i(t)$.) Taking the Fourier transform of (6.35) and (6.36) yields

using time-domain differentiation property

$\frac{d}{dt}x(t) \overset{FT}{\longleftrightarrow} j\omega X(\omega)$

$$\begin{aligned} V_1(\omega) &= RI(\omega) + j\omega LI(\omega) \\ &= (R + j\omega L)I(\omega) \quad \text{and} \end{aligned} \tag{6.37}$$

$$V_2(\omega) = j\omega LI(\omega). \tag{6.38}$$

From (6.37) and (6.38), we have

(*) Since System is LTI, $V_2(\omega) = V_1(\omega)\,H(\omega) \Rightarrow H(\omega) = \frac{V_2(\omega)}{V_1(\omega)}$

(*)

$$\begin{aligned} H(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{j\omega LI(\omega)}{(R + j\omega L)I(\omega)} \\ &= \frac{j\omega L}{R + j\omega L}. \end{aligned} \tag{6.39}$$

substitute (6.38) in numerator and (6.37) in denominator

cancel I's

Thus, we have found the frequency response of the system.

(b) Now, suppose that $v_1(t) = \operatorname{sgn} t$ (as given). Taking the Fourier transform of the input v_1 (with the aid of Table 6.2), we have

$= F\{\operatorname{sgn} t\}(\omega)$ from FT table

$$V_1(\omega) = \frac{2}{j\omega}. \tag{6.40}$$

From the definition of the system, we know

from (*)

$$V_2(\omega) = H(\omega)V_1(\omega). \tag{6.41}$$

Substituting (6.40) and (6.39) into (6.41), we obtain

$V_1(\omega)$ $H(\omega)$

substitute

$$\begin{aligned} V_2(\omega) &= \left(\frac{j\omega L}{R + j\omega L}\right)\left(\frac{2}{j\omega}\right) \\ &= \frac{2L}{R + j\omega L}. \end{aligned}$$

cancel factors of $j\omega$

⋮

Taking the inverse Fourier transform of both sides of this equation, we obtain

taking inverse FT

$$\begin{aligned} v_2(t) &= \mathcal{F}^{-1}\left\{\frac{2L}{R+j\omega L}\right\}(t) \\ &= \mathcal{F}^{-1}\left\{\frac{2}{R/L+j\omega}\right\}(t) \\ &= 2\mathcal{F}^{-1}\left\{\frac{1}{R/L+j\omega}\right\}(t). \end{aligned}$$

divide numerator and denominator by L

linearity

Using Table 6.2, we can simplify to obtain

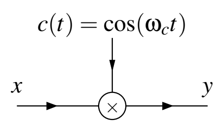
from FT table

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

$$v_2(t) = 2e^{-(R/L)t}u(t).$$

Thus, we have found the response v_2 to the input $v_1(t) = \operatorname{sgn} t$. ■

DSB-SC AM: Transmitter



$$y(t) = \cos(\omega_c t)x(t)$$

take FT

$$X = \mathcal{F}x, \quad Y = \mathcal{F}y$$

use modulation property
not multiplication property!

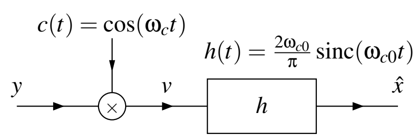
$$
\begin{aligned}
Y(\omega) &= \mathcal{F}\{\cos(\omega_c t)x(t)\}(\omega) \\
&= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)x(t)\right\}(\omega) \\
&= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)\right] \\
&= \tfrac{1}{2}\left[X(\omega - \omega_c) + X(\omega + \omega_c)\right]
\end{aligned}
$$

Euler

linearity property

modulation property

DSB-SC AM: Receiver



$$v(t) = \cos(\omega_c t)y(t), \quad h(t) = \frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0} t), \quad \hat{x}(t) = v * h(t)$$

(1) (2) (3)

use modulation property, not multiplication property!!!

$$Y = \mathcal{F}y, \quad V = \mathcal{F}v, \quad H = \mathcal{F}h, \quad \hat{X} = \mathcal{F}\hat{x}$$

$$\begin{aligned} V(\omega) &= \mathcal{F}\{\cos(\omega_c t)y(t)\}(\omega) \\ &= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t} + e^{-j\omega_c t}\right)y(t)\right\}(\omega) \\ &= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t}y(t)\right\}(\omega) + \mathcal{F}\left\{e^{-j\omega_c t}y(t)\right\}(\omega)\right] \\ &= \tfrac{1}{2}\left[Y(\omega - \omega_c) + Y(\omega + \omega_c)\right] \end{aligned}$$

FT of (1)

Euler

linearity property

modulation property

linearity

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\frac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0} t)\right\}(\omega) \\ &= 2\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right) \end{aligned}$$

FT of (2)

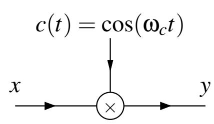
from FT table

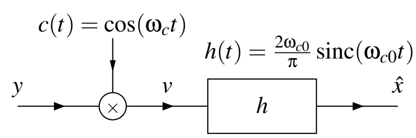
$\frac{B}{\pi}\operatorname{sinc}(Bt) \overset{FT}{\longleftrightarrow} \operatorname{rect}\left(\frac{w}{2B}\right)$

$$\hat{X}(\omega) = H(\omega)V(\omega)$$

FT of (3) using convolution property

DSB-SC AM: Complete System





① $Y(\omega) = \frac{1}{2}\left[X(\omega-\omega_c)+X(\omega+\omega_c)\right]$ from result for transmitter

② $V(\omega) = \frac{1}{2}\left[Y(\omega-\omega_c)+Y(\omega+\omega_c)\right]$ from result for receiver

substitute ①

$$\begin{aligned}
&= \tfrac{1}{2}\left[\tfrac{1}{2}\left[X([\omega-\omega_c]-\omega_c)+X([\omega-\omega_c]+\omega_c)\right]+\right.\\
&\qquad \left.\tfrac{1}{2}\left[X([\omega+\omega_c]-\omega_c)+X([\omega+\omega_c]+\omega_c)\right]\right]\\
&= \tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c)
\end{aligned}$$

simplify

③ $\hat{X}(\omega) = H(\omega)V(\omega)$ from result for receiver

substitute ②

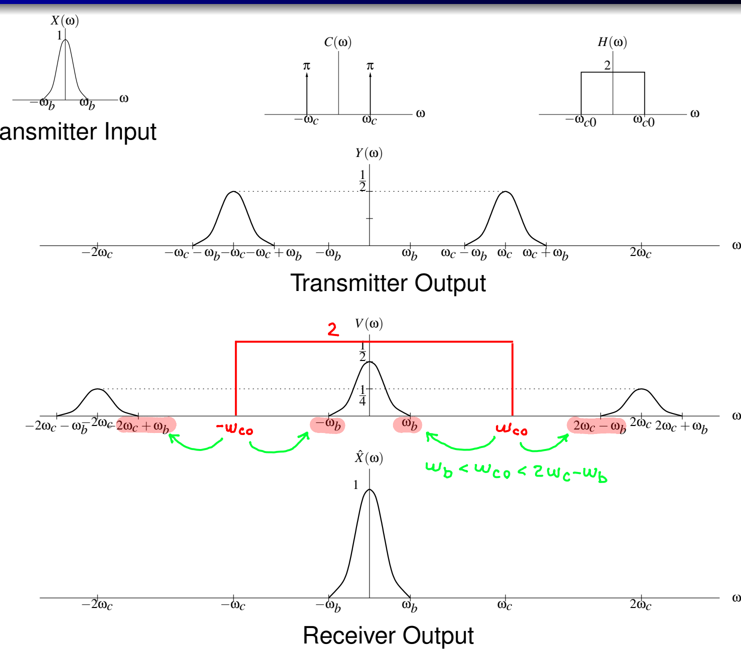
$$\begin{aligned}
&= H(\omega)\left[\tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c)\right]\\
&= \tfrac{1}{2}H(\omega)X(\omega)+\tfrac{1}{4}H(\omega)X(\omega-2\omega_c)+\tfrac{1}{4}H(\omega)X(\omega+2\omega_c)\\
&= \tfrac{1}{2}\left[2X(\omega)\right]+\tfrac{1}{4}(0)+\tfrac{1}{4}(0)\\
&= X(\omega)
\end{aligned}$$

multiply

$X(\omega-2\omega_c)=0$ and $X(\omega+2\omega_c)=0$ when $H(\omega)\neq 0$ (since $\omega_b < \omega_{c0} < 2\omega_c - \omega_b$)

simplify

DSB-SC AM: Spectra



Analysis of Double Side-Band Suppressed-Carrier Amplitude Modulation (DSB/SC AM)

Now, let us consider the communication system shown in Figure 6.29. This system is known as a double-side-band/suppressed-carrier (DSB/SC) amplitude modulation (AM) system. The receiver in Figure 6.29(b) contains a LTI subsystem that is labelled with its impulse response h. The DSB/SC AM system is very similar to the one considered earlier in Figure 6.27. In the new system, however, multiplication by a complex sinusoid has been replaced by multiplication by a real sinusoid. The new system also requires that the input signal x be bandlimited to frequencies in the interval $[-\omega_b, \omega_b]$ and that

$$\omega_b < \omega_{c0} < 2\omega_c - \omega_b. \tag{6.45}$$

The reasons for this restriction will become clear after having studied this system in more detail.

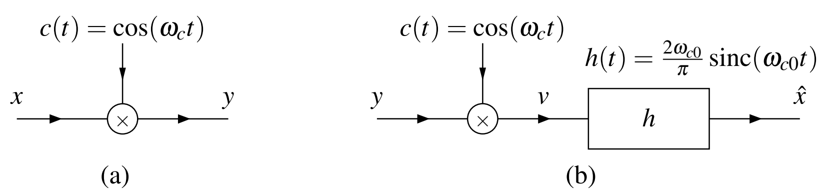


Figure 6.29: DSB/SC amplitude modulation system. (a) Transmitter and (b) receiver.

Consider the transmitter shown in Figure 6.29(a). The transmitter is a system with input x and output y that is characterized by the equation

$$y(t) = c(t)x(t),$$

where

$$c(t) = \cos(\omega_c t).$$

Taking the Fourier transform of both sides of the preceding equation, we obtain

$$\begin{aligned} Y(\omega) &= \mathcal{F}\{cx\}(\omega) \\ &= \mathcal{F}\{\cos(\omega_c t)x(t)\}(\omega) \qquad \text{Euler} \\ &= \mathcal{F}\left\{\tfrac{1}{2}[e^{j\omega_c t} + e^{-j\omega_c t}]x(t)\right\}(\omega) \\ &= \tfrac{1}{2}\left[\mathcal{F}\{e^{j\omega_c t}x(t)\}(\omega) + \mathcal{F}\{e^{-j\omega_c t}x(t)\}(\omega)\right] \qquad \text{linearity} \\ &= \tfrac{1}{2}\left[X(\omega - \omega_c) + X(\omega + \omega_c)\right]. \qquad \text{frequency-domain shifting property} \end{aligned} \tag{6.46}$$

(Note that, above, we used the fact that $\cos(\omega_c t) = \frac{1}{2}(e^{j\omega_c t} + e^{-j\omega_c t})$.) Thus, the frequency spectrum of the (transmitter) output is the average of two shifted versions of the frequency spectrum of the (transmitter) input. The relationship between the frequency spectra of the input and output can be seen through Figures 6.30(a) and (d). Observe that we have managed to shift the frequency spectrum of the input signal into a different range of frequencies for transmission as desired. Next, we must determine whether the receiver can recover the original signal x.

Consider the receiver shown in Figure 6.29(b). The receiver is a system with input y and output $\hat{x}$ that is characterized by the equations

$$v(t) = c(t)y(t) \quad \text{and} \qquad = \cos(\omega_c t)\, y(t) \tag{6.47a}$$

$$\hat{x}(t) = v * h(t), \tag{6.47b}$$

where c is as defined earlier and

$$h(t) = \tfrac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0} t). \tag{6.47c}$$

Let H, Y, V, and $\hat{X}$ denote the Fourier transforms of h, y, v and $\hat{x}$, respectively. Taking the Fourier transform of $\hat{X}$ (in (6.47b)), we have

$\hat{x}(t) = v * h(t)$

$$\hat{X}(\omega) = H(\omega)V(\omega). \tag{6.48}$$

table of FT pairs

Taking the Fourier transform of h (in (6.47c)) with the assistance of Table 6.2, we have

$h(t) = \frac{2\omega_{c0}}{\pi} \operatorname{sinc}(\omega_{c0} t)$

$$\begin{aligned} H(\omega) &= \mathcal{F}\left\{\tfrac{2\omega_{c0}}{\pi}\operatorname{sinc}(\omega_{c0}t)\right\}(\omega) \\ &= 2\operatorname{rect}\tfrac{\omega}{2\omega_{c0}} \\ &= \begin{cases} 2 & |\omega| \le \omega_{c0} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

from FT table

$\frac{B}{\pi}\operatorname{sinc}(Bt) \xleftrightarrow{FT} \operatorname{rect}\left(\frac{\omega}{2B}\right)$

definition of rect function

Taking the Fourier transform of v (in (6.47a)) yields

$v(t) = c(t)\,y(t) = \cos(\omega_c t)\,y(t)$

$$\begin{aligned} V(\omega) &= \mathcal{F}\{cy\}(\omega) \\ &= \mathcal{F}\{\cos(\omega_c t)y(t)\}(\omega) \\ &= \mathcal{F}\left\{\tfrac{1}{2}\left(e^{j\omega_c t}+e^{-j\omega_c t}\right)y(t)\right\}(\omega) \\ &= \tfrac{1}{2}\left[\mathcal{F}\left\{e^{j\omega_c t}y(t)\right\}(\omega)+\mathcal{F}\left\{e^{-j\omega_c t}y(t)\right\}(\omega)\right] \\ &= \tfrac{1}{2}[Y(\omega-\omega_c)+Y(\omega+\omega_c)]. \end{aligned}$$

Euler

linearity

frequency-domain shifting property

Substituting the expression for Y in (6.46) into this equation, we obtain

$Y(\omega) = \frac{1}{2}\left[X(\omega-\omega_c) + X(\omega+\omega_c)\right]$

$$\begin{aligned} V(\omega) &= \tfrac{1}{2}\left[\tfrac{1}{2}[X([\omega-\omega_c]-\omega_c)+X([\omega-\omega_c]+\omega_c)]+\tfrac{1}{2}[X([\omega+\omega_c]-\omega_c)+X([\omega+\omega_c]+\omega_c)]\right] \\ &= \tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c). \end{aligned} \tag{6.49}$$

simplify

The relationship between V and X can be seen via Figures 6.30(a) and (e). Substituting the above expression for V into (~~[illegible]~~) and simplifying, we obtain

typo (6.48) ← $\hat{X}(\omega) = H(\omega)V(\omega)$

$$\begin{aligned} \hat{X}(\omega) &= H(\omega)V(\omega) \\ &= H(\omega)\left[\tfrac{1}{2}X(\omega)+\tfrac{1}{4}X(\omega-2\omega_c)+\tfrac{1}{4}X(\omega+2\omega_c)\right] \\ &= \tfrac{1}{2}H(\omega)X(\omega)+\tfrac{1}{4}H(\omega)X(\omega-2\omega_c)+\tfrac{1}{4}H(\omega)X(\omega+2\omega_c) \\ &= \tfrac{1}{2}[2X(\omega)]+\tfrac{1}{4}(0)+\tfrac{1}{4}(0) \\ &= X(\omega). \end{aligned}$$

substitute V from (6.49)

distribute

*

$\operatorname{rect}\left(\frac{\omega}{2\omega_{c0}}\right) = 0$ for $|\omega| > \omega_{c0}$

$\omega_b < \omega_{c0} < 2\omega_c - \omega_b$

* In the above simplification, since $H(\omega) = 2\operatorname{rect}\frac{\omega}{2\omega_{c0}}$ and condition (6.45) holds, we were able to deduce that $H(\omega)X(\omega) = 2X(\omega)$, $H(\omega)X(\omega-2\omega_c) = 0$, and $H(\omega)X(\omega+2\omega_c) = 0$. The relationship between $\hat{X}$ and X can be seen from Figures 6.30(a) and (f). Thus, we have that $\hat{X} = X$, implying $\hat{x} = x$. So, we have recovered the original signal x at the receiver. This system has managed to shift x into a different frequency range before transmission and then recover x at the receiver. This is exactly what we wanted to accomplish.

The reason that this simplification is valid is probably more easily seen by looking at Figure 6.30(e).

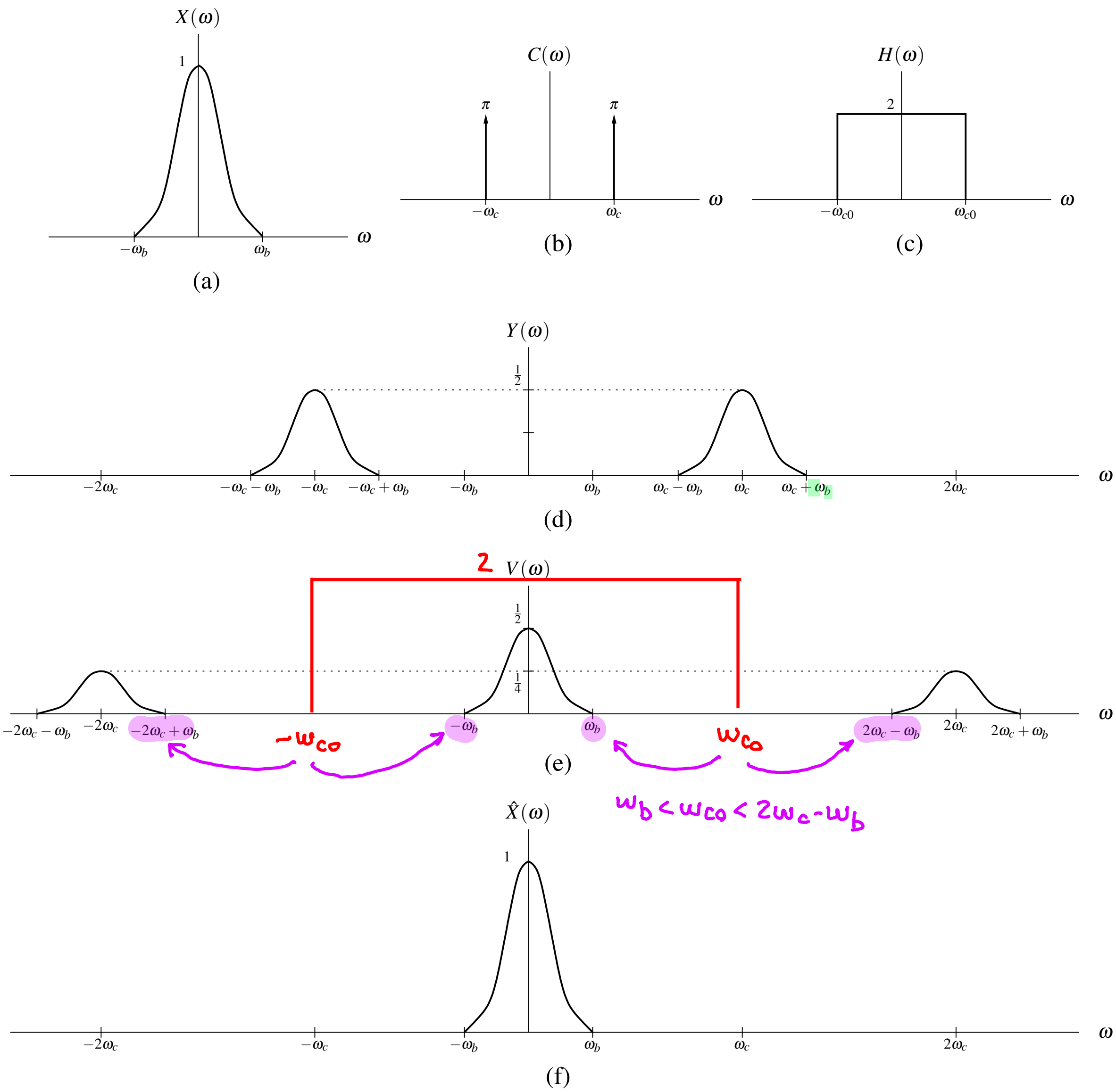
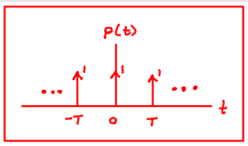


Figure 6.30: Signal spectra for DSB/SC amplitude modulation. (a) Spectrum of the transmitter input. (b) Spectrum of the sinusoidal function used in the transmitter and receiver. (c) Frequency response of the filter in the receiver. (d) Spectrum of the transmitted signal. (e) Spectrum of the multiplier output in the receiver. (f) Spectrum of the receiver output.

Sampling: Fourier Series for a Periodic Impulse Train



⊛

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

① $$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$ p has Fourier series representation, since p is periodic

② $$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t}\,dt$$ Fourier series analysis equation

see plot of p in figure ⊛

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t}\,dt$$

integrand is zero everywhere outside integration interval

$$= \frac{1}{T}\int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t}\,dt$$

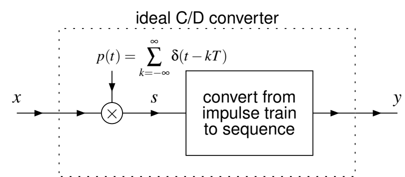
sifting property

$$= \frac{1}{T}$$

$$= \frac{\omega_s}{2\pi}$$ $T = \frac{2\pi}{\omega_s}$ by definition

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$ substitute ② into ①

Sampling: Multiplication by a Periodic Impulse Train



use modulation property, not multiplication property !!!

$$s(t) = p(t)x(t), \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT), \quad \omega_s = \frac{2\pi}{T}$$

1 2

3 $$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

result of finding Fourier series representation of p in 2

$$s(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} x(t)$$

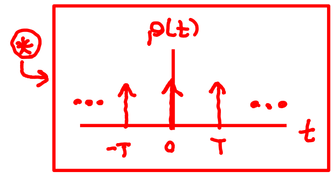
substitute Fourier series representation of p in 3 into 1

$$X = \mathcal{F}x, \quad S = \mathcal{F}s$$

take FT using modulation property

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

Analysis of Sampling System



s(t) = x(t) p(t)   (6.51)

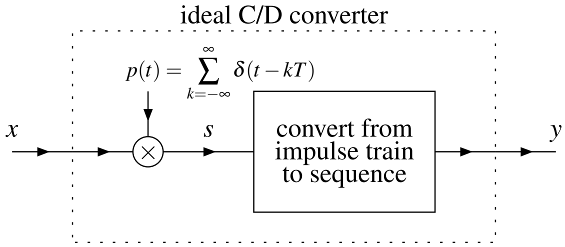


Figure 6.36: Model of ideal C/D converter with input function x and output sequence y.

Now, let us consider the above model of sampling in more detail. In particular, we would like to find the relationship between the frequency spectra of the original function x and its impulse-train sampled version s. In what follows, let X, Y, P, and S denote the Fourier transforms of x, y, p, and s, respectively. Since p is T-periodic, it can be represented in terms of a Fourier series as

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}. \tag{6.52}$$

from definition of Fourier series

Using the Fourier series analysis equation, we calculate the coefficients c_k to be

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T} \int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \\ &= \frac{\omega_s}{2\pi}. \end{aligned} \tag{6.53}$$

Fourier series analysis equation

(*)

Sifting property

$T = \frac{2\pi}{\omega_s}$

Substituting (6.52) and (6.53) into (6.51), we obtain

$s(t) = x(t)\,p(t)$

$$\begin{aligned} s(t) &= x(t) \sum_{k=-\infty}^{\infty} \frac{\omega_s}{2\pi} e^{jk\omega_s t} \\ &= \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}. \end{aligned}$$

replace p(t) by its Fourier series representation

rearrange

Taking the Fourier transform of s yields

take FT using frequency-domain shifting property

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s). \tag{6.54}$$

Thus, the spectrum of the impulse-train sampled function s is a scaled sum of an infinite number of shifted copies of the spectrum of the original function x.

Example 6.41. Let x denote a continuous-time audio signal with Fourier transform X. Suppose that $|X(\omega)| = 0$ for all $|\omega| \geq 44100\pi$. Determine the largest period T with which x can be sampled that will allow x to be exactly recovered from its samples.

44100π rad/s = 22.05 kHz

Solution. The function x is bandlimited to frequencies in the range $(-\omega_m, \omega_m)$, where $\omega_m = 44100\pi$. From the sampling theorem, we know that the minimum sampling rate required is given by

from sampling theorem

$$\begin{aligned}\omega_s &= 2\omega_m \\ &= 2(44100\pi) \\ &= 88200\pi.\end{aligned}$$

$\omega_m = 44100\pi$

88200π rad/s = 44.1 kHz

take reciprocal for corresponding sampling period

Thus, the largest permissible sampling period is given by

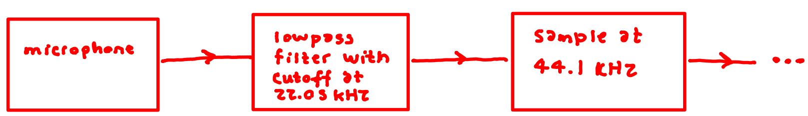
$$\begin{aligned}T &= \frac{2\pi}{\omega_s} \\ &= \frac{2\pi}{88200\pi} \\ &= \frac{1}{44100}.\end{aligned}$$

■

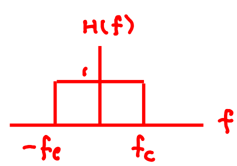
$\boxed{\text{Why does CD-quality audio use a sampling rate of 44.1 kHz?}}$

In practice, how do we ensure the audio signal to be sampled is sufficiently bandlimited?

The human auditory system (assuming pristine hearing) can sense frequencies up to about 22.05 kHz.



- filter prevents aliasing
- removed frequencies cannot be detected by humans

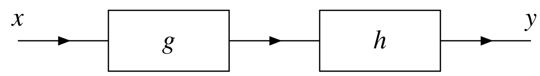


$f_c = 22.05$ kHz

Example 6.39 (Communication channel equalization)**.** Consider a LTI communication channel with frequency response

$$H(\omega) = \tfrac{1}{3+j\omega}.$$

Unfortunately, this channel has the undesirable effect of attenuating higher frequencies. Find the frequency response G of an equalizer that when connected in series with the communication channel yields an ideal (i.e., distortionless) channel. The new system with equalization is shown in Figure 6.24, where g and h denote the inverse Fourier transforms of G and H, respectively.



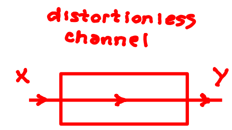


Figure 6.24: System from example that employs equalization.

Solution. An ideal communication channel has a frequency response equal to one for all frequencies. Consequently, we want $H(\omega)G(\omega) = 1$ or equivalently $G(\omega) = 1/H(\omega)$. Thus, we conclude that

rearrange

$$\begin{aligned} G(\omega) &= \frac{1}{H(\omega)} && \text{substitute given } H \\ &= \frac{1}{\left(\frac{1}{3+j\omega}\right)} && \text{simplify} \\ &= 3 + j\omega. \end{aligned}$$

■

Unit:
Partial Fraction Expansions

Example B.1 (Simple pole)**.** Find the partial fraction expansion of the function

$$f(z) = \frac{3}{z^2+3z+2}.$$

strictly proper

Solution. First, we rewrite f with the denominator polynomial factored to obtain

$$f(z) = \frac{3}{(z+1)(z+2)}.$$

simple (i.e., 1st order) poles at -1 and -2

From this, we know that f has a partial fraction expansion of the form

$$f(z) = \frac{A_1}{z+1} + \frac{A_2}{z+2}, \quad (1)$$

where A_1 and A_2 are constants to be determined. Now, we calculate A_1 and A_2 as follows:

$$\left.\begin{aligned}
A_1 &= (z+1)f(z)|_{z=-1} \\
&= \left.\frac{3}{z+2}\right|_{z=-1} \\
&= 3 \quad \text{and} \\
A_2 &= (z+2)f(z)|_{z=-2} \\
&= \left.\frac{3}{z+1}\right|_{z=-2} \\
&= -3.
\end{aligned}\right\} (2)$$

Thus, the partial fraction expansion of f is given by

from (1) and (2)

$$f(z) = \frac{3}{z+1} - \frac{3}{z+2}.$$

■

Example B.2 (Repeated pole). Find the partial fraction expansion of the function

$$f(z) = \frac{4z+8}{(z+1)^2(z+3)}.$$

strictly proper with 2nd order pole at −1 and 1st order pole at −3

Solution. Since f has a repeated pole, we know that f has a partial fraction expansion of the form

terms contributed by pole at −1

$$f(z) = \frac{A_{1,1}}{z+1} + \frac{A_{1,2}}{(z+1)^2} + \frac{A_{2,1}}{z+3}. \quad ①$$

term contributed by pole at −3

where $A_{1,1}$, $A_{1,2}$, and $A_{2,1}$ are constants to be determined. To calculate these constants, we proceed as follows:

coefficient number

pole order

$$\begin{aligned}
A_{1,1} &= \frac{1}{(2-1)!}\left[\left(\tfrac{d}{dz}\right)^{2-1}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{1!}\left[\tfrac{d}{dz}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \left[\tfrac{d}{dz}\left(\frac{4z+8}{z+3}\right)\right]\Big|_{z=-1} \\
&= \left[4(z+3)^{-1} + (-1)(z+3)^{-2}(4z+8)\right]\Big|_{z=-1} \\
&= \left[\frac{4}{(z+3)^2}\right]\Big|_{z=-1} \\
&= \tfrac{4}{4} \\
&= 1,
\end{aligned}$$

formula for case of repeated pole

substitute for f

differentiate

$$\begin{aligned}
A_{1,2} &= \frac{1}{(2-2)!}\left[\left(\tfrac{d}{dz}\right)^{2-2}\left[(z+1)^2 f(z)\right]\right]\Big|_{z=-1} \\
&= \frac{1}{0!}\left[(z+1)^2 f(z)\right]\Big|_{z=-1} \\
&= \left[\frac{4z+8}{z+3}\right]\Big|_{z=-1} \\
&= \tfrac{4}{2} \\
&= 2, \quad \text{and}
\end{aligned}$$

formula for case of repeated pole

$$\begin{aligned}
A_{2,1} &= (z+3)f(z)\big|_{z=-3} \\
&= \frac{4z+8}{(z+1)^2}\Big|_{z=-3} \\
&= \tfrac{-4}{4} \\
&= -1.
\end{aligned}$$

formula for case of simple pole

substitute for f

Thus, the partial fraction expansion of f is given by

$$f(z) = \frac{1}{z+1} + \frac{2}{(z+1)^2} - \frac{1}{z+3}.$$

substitute computed coefficients into ① ■

Unit:
Laplace Transform

Relationship Between the Laplace and Fourier Transforms

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Recall the definition of the Laplace transform in (7.2). Consider now the special case of (7.2) where $s = j\omega$ and ω is real (i.e., $\operatorname{Re}(s) = 0$). In this case, (7.2) becomes

$$\begin{aligned} X(j\omega) &= \left[\int_{-\infty}^{\infty} x(t) e^{-st} dt \right] \Bigg|_{s=j\omega} && \text{from definition of LT} \\ &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt && \text{substitute jw for s} \\ &= \mathcal{F}x(\omega). && \text{from definition of FT} \end{aligned}$$

Thus, the Fourier transform is simply the Laplace transform evaluated at $s = j\omega$, assuming that this quantity is well defined (i.e., converges). In other words,

$$X(j\omega) = \mathcal{F}x(\omega). \tag{7.4}$$

Incidentally, it is due to the preceding relationship that the Fourier transform of x is sometimes written as $X(j\omega)$. When this notation is used, the function X actually corresponds to the Laplace transform of x rather than its Fourier transform (i.e., the expression $X(j\omega)$ corresponds to the Laplace transform evaluated at points on the imaginary axis).

Relationship Between the Laplace and Fourier Transforms (General Case)

$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

Now, consider the general case of an arbitrary complex value for s in (7.2). Let us express s in Cartesian form as $s = \sigma + j\omega$ where σ and ω are real. Substituting $s = \sigma + j\omega$ into (7.2), we obtain

$$\begin{aligned} X(\sigma + j\omega) &= \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt \\ &= \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt \\ &= \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \end{aligned}$$

substituting $\sigma + j\omega$ for s in LT definition

split exponential in two

definition of FT

Thus, we have shown

$$X(\sigma + j\omega) = \mathcal{F}\{e^{-\sigma t} x(t)\}(\omega). \tag{7.5}$$

Thus, the Laplace transform of x can be viewed as the (CT) Fourier transform of $x'(t) = e^{-\sigma t} x(t)$ (i.e., x weighted by a real exponential function).

Example 7.3. Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned}
X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \\
&= \int_{0}^{\infty} e^{-(s+a)t}\,dt \\
&= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_0^{\infty}.
\end{aligned}$$

definition of LT

combine exponentials and use u to change limits

integrate

At this point, we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges to a finite value. This yields

real exponential $\begin{cases} 0 & \sigma+a>0 \\ \infty & \sigma+a<0 \end{cases}$

complex sinusoid finite but limit does not exist

$$\begin{aligned}
X(s) &= \left[\left(-\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_0^{\infty} \\
&= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_0^{\infty} \\
&= \left(\tfrac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)\infty}e^{-j\omega\infty} - 1\right].
\end{aligned}$$

factor and split exponentials

take difference

Thus, we can see that the above expression only converges for $\sigma + a > 0$ (i.e., $\mathrm{Re}(s) > -a$). In this case, we have that

$$\begin{aligned}
X(s) &= \left(\tfrac{-1}{\sigma+a+j\omega}\right)[0-1] \\
&= \left(\tfrac{-1}{s+a}\right)(-1) \\
&= \frac{1}{s+a}.
\end{aligned}$$

if Re(s) > -a

rewrite in terms of s ($s=\sigma+j\omega$)

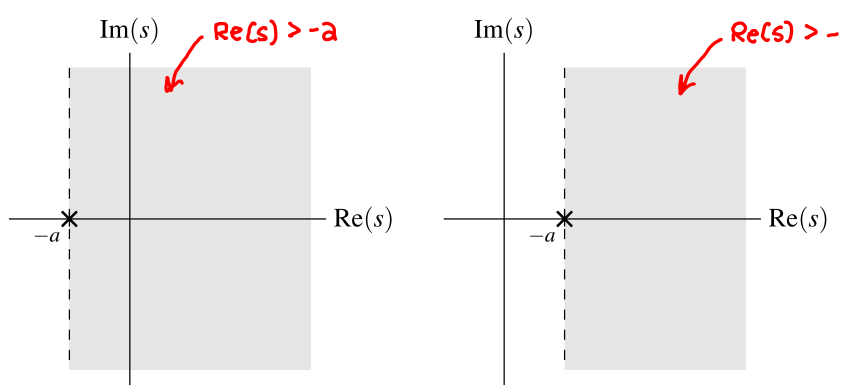
simplify

Thus, we have that

$$e^{-at}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s) > -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.2(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.



(a) (b)

Figure 7.2: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

■

Example 7.4. Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we can write

$$\begin{aligned} X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\ &= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt \\ &= \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt \\ &= \int_{-\infty}^{0} -e^{-(s+a)t}\,dt \\ &= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}. \end{aligned}$$

definition of LT

use u to change limits

combine exponentials

integrate

In order to more easily determine when the above expression converges to a finite value, we substitute $s = \sigma + j\omega$. This yields

real exponential

$\begin{cases} 0 & \sigma+a<0 \\ \infty & \sigma+a>0 \end{cases}$

complex sinusoid

finite but limit not well defined

$$\begin{aligned} X(s) &= \left[\left(\tfrac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} \\ &= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Big|_{-\infty}^{0} \\ &= \left(\tfrac{1}{\sigma+a+j\omega}\right)\left[1 - \underbrace{e^{(\sigma+a)\infty}e^{j\omega\infty}}\right]. \end{aligned}$$

split exponential

take difference

Thus, we can see that the above expression only converges for $\sigma + a < 0$ (i.e., $\operatorname{Re}(s) < -a$). In this case, we have

$$\begin{aligned} X(s) &= \left(\tfrac{1}{\sigma+a+j\omega}\right)[1-0] \\ &= \frac{1}{s+a}. \end{aligned}$$

if $\operatorname{Re}(s) < -a$

rewrite in terms of s ($s = \sigma + j\omega$)

Thus, we have that

$$-e^{-at}u(-t) \xleftrightarrow{\text{LT}} \frac{1}{s+a} \quad \text{for } \operatorname{Re}(s) < -a.$$

Note: We must specify this region of convergence since $\frac{1}{s+a}$ is not correct for all $s \in \mathbb{C}$

The region of convergence for X is illustrated in Figures 7.3(a) and (b) for the cases of $a > 0$ and $a < 0$, respectively.

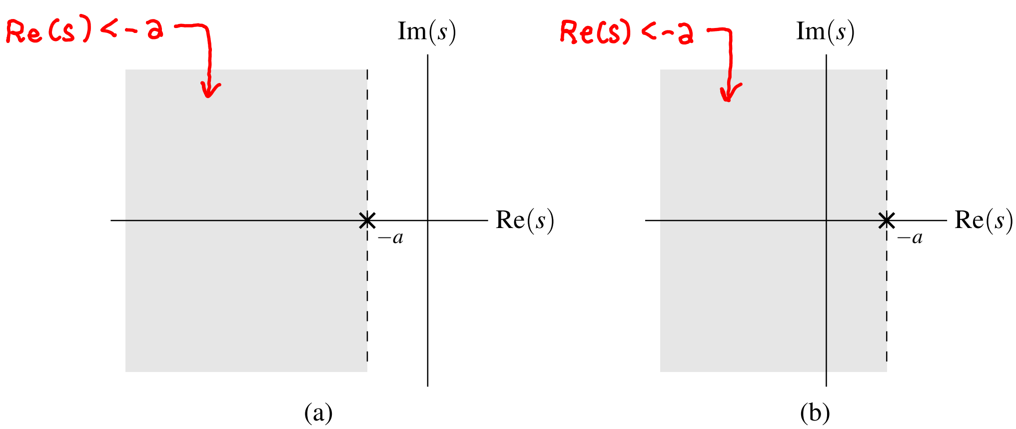


Figure 7.3: Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

different (and this is critical for invertibility of LT)

NOTE:

	different		same		
Example 7.3	$e^{-at}u(t)$	$\xleftrightarrow{\text{LT}}$	$\frac{1}{s+a}$	for	$\operatorname{Re}(s) > -a$
Example 7.4	$-e^{-at}u(-t)$	$\xleftrightarrow{\text{LT}}$	$\frac{1}{s+a}$	for	$\operatorname{Re}(s) < -a$

Example 7.7. The Laplace transform X of the function x has the algebraic expression

$$X(s) = \frac{s+\frac{1}{2}}{(s^2+2s+2)(s^2+s-2)}.$$ ← rational function

Identify all of the possible ROCs of X. For each ROC, indicate whether the corresponding function x is left sided, right sided, two sided, or finite duration.

Solution. The possible ROCs associated with X are determined by the poles of this function. So, we must find the poles of X. Factoring the denominator of X, we obtain

$$X(s) = \frac{s+\frac{1}{2}}{(s+1-j)(s+1+j)(s+2)(s-1)}.$$

these factors obtained by using quadratic formula

Thus, X has poles at -2, $-1-j$, $-1+j$, and 1. Since these poles only have three distinct real parts (namely, -2, -1, and 1), there are four possible ROCs:

i) $\mathrm{Re}(s) < -2$,
ii) $-2 < \mathrm{Re}(s) < -1$,
iii) $-1 < \mathrm{Re}(s) < 1$, and
iv) $\mathrm{Re}(s) > 1$.

These ROCs are plotted in Figures 7.8(a), (b), (c), and (d), respectively. The first ROC is a left-half plane, so the corresponding x must be left sided. The second ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding x must be two sided. The third ROC is a vertical strip (i.e., neither a left- nor right-half plane), so the corresponding x must be two sided. The fourth ROC is a right-half plane, so the corresponding x must be right sided.

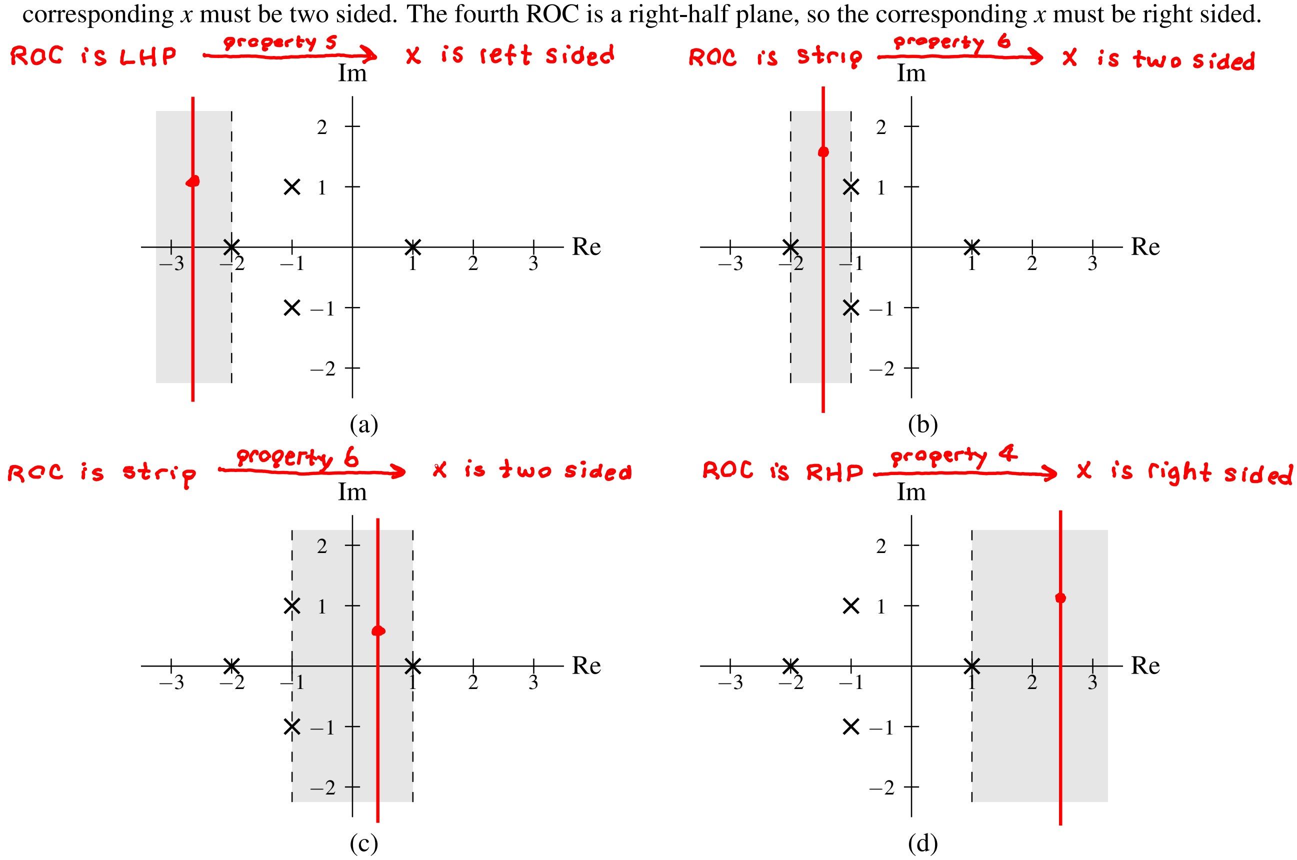


Figure 7.8: ROCs for example. The (a) first, (b) second, (c) third, and (d) fourth possible ROCs for X.

■

Example 7.8 (Linearity property of the Laplace transform)**.** Find the Laplace transform of the function

$$x = x_1 + x_2,$$

where

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

Solution. Using Laplace transform pairs from Table 7.2, we have

$$\begin{aligned}
① \quad X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) && \text{from LT table}\\
&= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and}\\
② \quad X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) && \text{linearity}\\
&= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) && \text{from LT table and } ⊛\\
&= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 && \text{common denominator}\\
&= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.
\end{aligned}$$

$⊛ \; [\operatorname{Re}(s) > -2] \cap [\operatorname{Re}(s) > -1] = \operatorname{Re}(s) > -1$

So, from the definition of X, we can write

$$\begin{aligned}
X(s) &= \mathcal{L}\{x_1 + x_2\}(s) && \text{linearity}\\
&= X_1(s) + X_2(s) && \text{substitute expressions for } X_1 \text{ and } X_2 \text{ in ① and ②}\\
&= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} && \text{common denominator}\\
&= \frac{s+2+1}{(s+1)(s+2)} && \text{simplify}\\
&= \frac{s+3}{(s+1)(s+2)}.
\end{aligned}$$

but is it larger than the intersection?

Now, we must determine the ROC of X. We know that the ROC of X must contain the intersection of the ROCs of X_1 and X_2. So, the ROC must contain $\operatorname{Re}(s) > -1$. Furthermore, the ROC cannot be larger than this intersection, since X has a pole at -1. Therefore, the ROC of X is $\operatorname{Re}(s) > -1$. The various ROCs are illustrated in Figure 7.9. So, in conclusion, we have

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \qquad \blacksquare$$

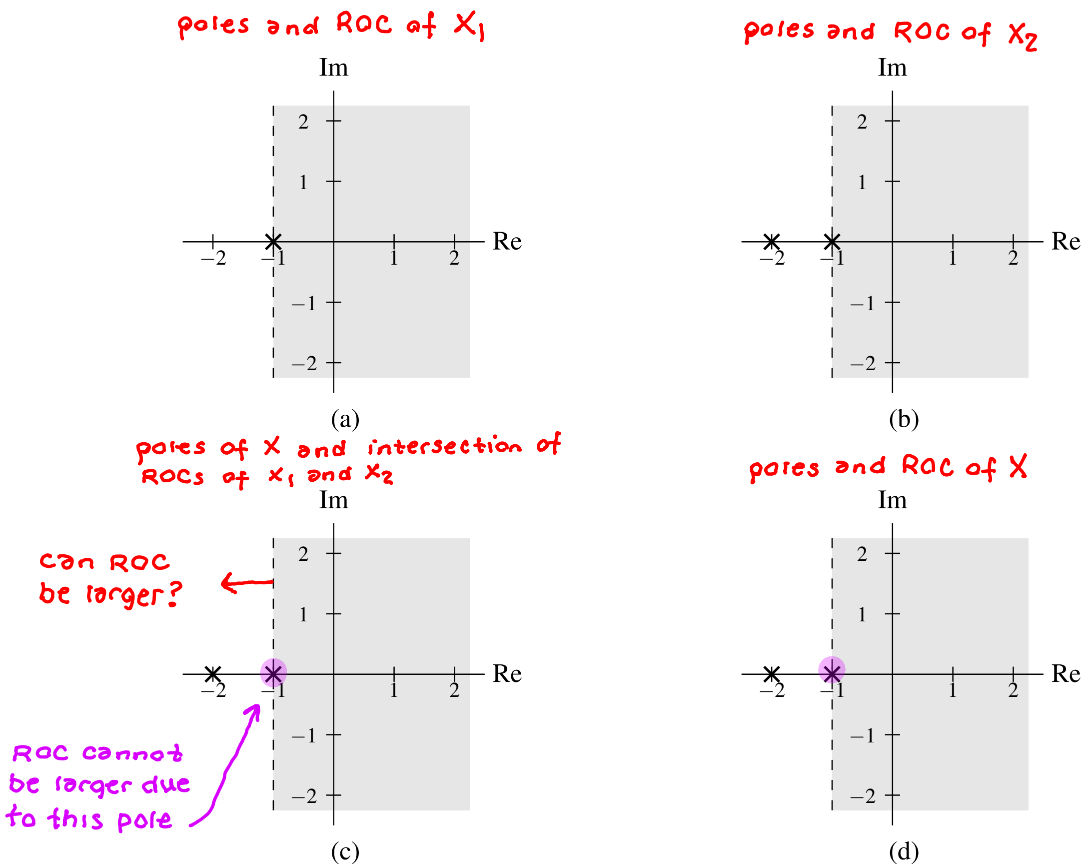


Figure 7.9: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Example 7.9 (Linearity property of the Laplace transform and pole-zero cancellation)**.** Find the Laplace transform X of the function

$$x = x_1 - x_2,$$

$$\begin{cases} x_1(t) = e^{-t}u(t) \\ x_2(t) = e^{-t}u(t) - e^{-2t}u(t) \end{cases}$$

where x_1 and x_2 are as defined in the previous example.

Solution. From the previous example, we know that

$$\left.\begin{aligned} &(1)\quad X_1(s) = \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\ &(2)\quad X_2(s) = \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}\right\} \text{from LT table}$$

From the definition of X, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 - x_2\}(s) && \text{linearity} \\ &= X_1(s) - X_2(s) && \text{substituting expressions for } X_1 \text{ and } X_2 \text{ in (1) and (2)} \\ &= \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} && \text{common denominator} \\ &= \frac{s+2-1}{(s+1)(s+2)} && \text{simplify numerator} \\ &= \frac{s+1}{(s+1)(s+2)} && \text{pole-zero cancellation; cancel common factor of } s+1 \\ &= \frac{1}{s+2}. \end{aligned}$$

Now, we must determine the ROC of X. We know that the ROC of X must at least contain the intersection of the ROCs of X_1 and X_2. Therefore, the ROC must contain $\operatorname{Re}(s) > -1$. Since X is rational, we also know that the ROC must be bounded by poles or extend to infinity. Since X has only one pole and this pole is at -2, the ROC must also include $-2 < \operatorname{Re}(s) < -1$. Therefore, the ROC of X is $\operatorname{Re}(s) > -2$. In effect, the pole at -1 has been cancelled by a zero at the same location. As a result, the ROC of X is larger than the intersection of the ROCs of X_1 and X_2. The various ROCs are illustrated in Figure 7.10. So, in conclusion, we have

$$X(s) = \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$

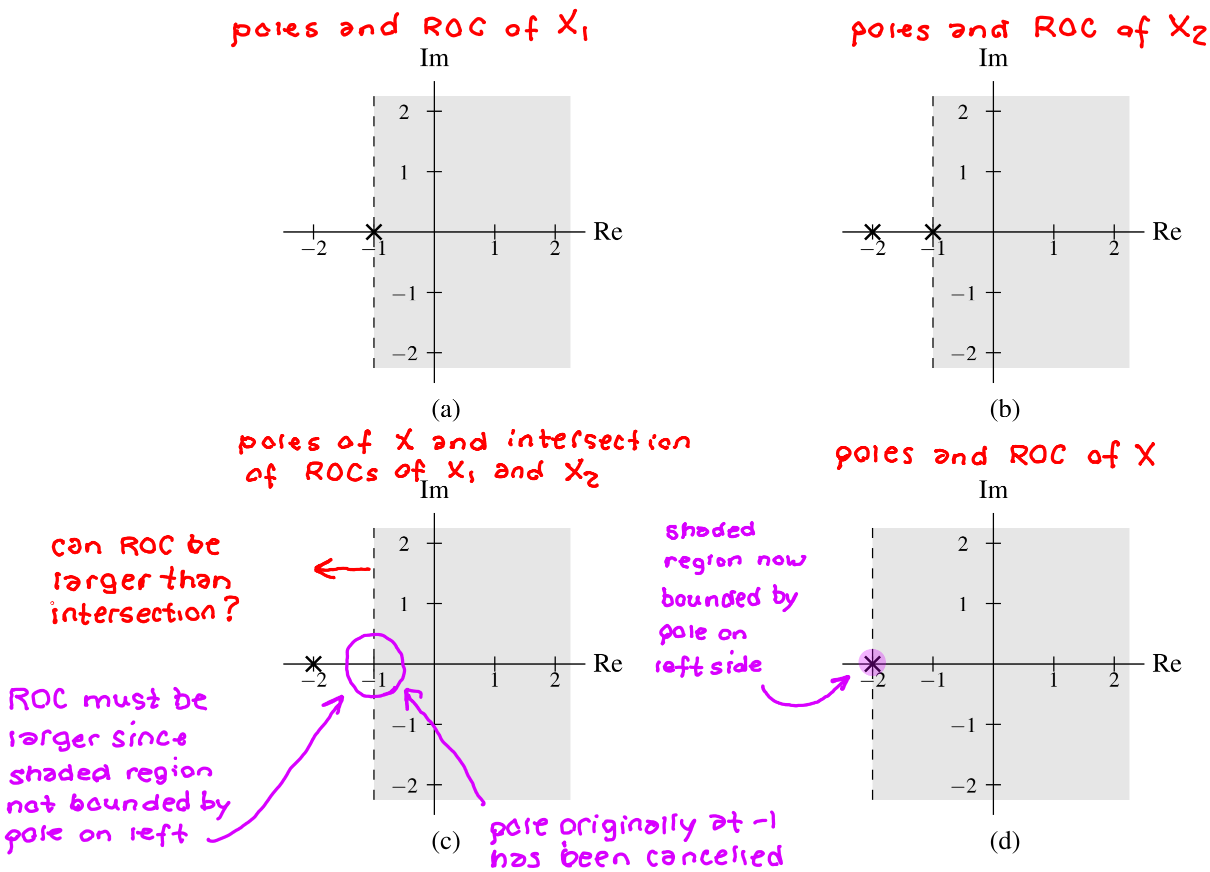


Figure 7.10: ROCs for the linearity example. The (a) ROC of X_1, (b) ROC of X_2, (c) ROC associated with the intersection of the ROCs of X_1 and X_2, and (d) ROC of X.

Example 7.10 (Time-domain shifting property)**.** Find the Laplace transform X of

table of LT pairs

$$x(t) = u(t-1).$$

Solution. From Table 7.2, we know that

$$u(t) \overset{\text{LT}}{\longleftrightarrow} 1/s \ \text{ for } \operatorname{Re}(s) > 0.$$

from LT table

Using the time-domain shifting property, we can deduce

multiply by e^{-s}

ROC unchanged

shift by 1

$$x(t) = u(t-1) \overset{\text{LT}}{\longleftrightarrow} X(s) = e^{-s}\left(\tfrac{1}{s}\right) \ \text{ for } \operatorname{Re}(s) > 0.$$

Therefore, we have

$$X(s) = \frac{e^{-s}}{s} \ \text{ for } \operatorname{Re}(s) > 0. \qquad \blacksquare$$

Example 7.11 (Laplace-domain shifting property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of

$$x(t) = e^{5t}e^{-|t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1.$$

shift ROC by 5

Using the Laplace-domain shifting property, we can deduce

shift s by 5

multiply by e^{5t}

$$x(t) = e^{5t}e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{2}{1-(s-5)^2} \quad \text{for } \underbrace{-1+5}_{4} < \operatorname{Re}(s) < \underbrace{1+5}_{6},$$

Thus, we have

$$X(s) = \frac{2}{1-(s-5)^2} \quad \text{for } 4 < \operatorname{Re}(s) < 6.$$

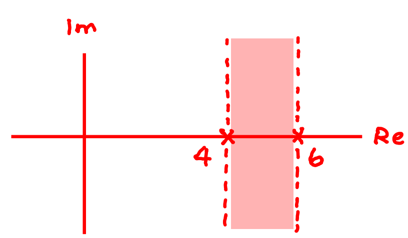
not strictly necessary except to check answer

Rewriting X in factored form, we have

$$X(s) = \frac{2}{1-(s-5)^2} = \frac{2}{1-(s^2-10s+25)} = \frac{2}{-s^2+10s-24} = \frac{-2}{s^2-10s+24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

$$X(s) = \frac{-2}{(s-4)(s-6)} \quad \text{for } 4 < \operatorname{Re}(s) < 6. \qquad \blacksquare$$



sanity check:

are stated algebraic expression and stated ROC self consistent?

yes, ROC bounded by poles

Example 7.12 (Time-domain scaling property). Using only properties of the Laplace transform and the transform pair

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform of the function

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \text{ for } -1 < \text{Re}(s) < 1.$$

ROC scales by 3

time and amplitude scale

Using the time-domain scaling property, we can deduce

time scale by 3

$$x(t) = e^{-|3t|} \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{|3|}\frac{2}{1-(\frac{s}{3})^2} \quad \text{for } \underbrace{3(-1)}_{-3} < \text{Re}(s) < \underbrace{3(1)}_{3}.$$

Thus, we have

$$X(s) = \frac{2}{3\left[1-(\frac{s}{3})^2\right]} \quad \text{for } -3 < \text{Re}(s) < 3.$$

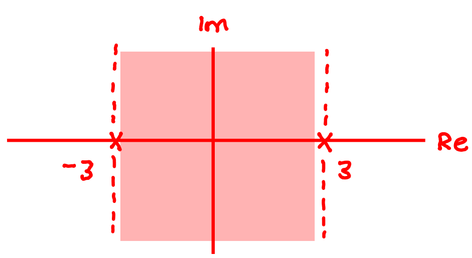
not strictly necessary except as sanity check

Simplifying, we have

$$X(s) = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \text{Re}(s) < 3. \qquad \blacksquare$$



sanity check: are stated algebraic expression and stated ROC self consistent?

yes, ROC is bounded by poles

Example 7.19. Using properties of the Laplace transform and the Laplace transform pair

$$e^{-a|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{-2a}{(s+a)(s-a)} \quad \text{for } -a < \text{Re}(s) < a,$$

find the Laplace transform X of the function

$$x(t) = e^{-5|3t-7|}.$$

Solution. We begin by re-expressing x in terms of the following equations:

Sanity check:
$x(t) = v_2(3t)$
$= v_1(3t-7)$
$= e^{-5|3t-7|}$

$$1 \qquad v_1(t) = e^{-5|t|},$$
$$2 \qquad v_2(t) = v_1(t-7), \quad \text{and}$$
$$3 \qquad x(t) = v_2(3t).$$

In what follows, let R_{V_1}, R_{V_2}, and R_X denote the ROCs of V_1, V_2, and X, respectively. Taking the Laplace transform of the above three equations, we obtain

algebraic expression | ROC

$$4 \qquad V_1(s) = \frac{-10}{(s+5)(s-5)}, \quad R_{V_1} = (-5 < \text{Re}(s) < 5),$$
from LT of 1 using given LT pair

$$5 \qquad V_2(s) = e^{-7s}V_1(s), \quad R_{V_2} = R_{V_1},$$
from LT of 2 using time-domain shifting property

$$6 \qquad X(s) = \tfrac{1}{3}V_2(s/3), \quad \text{and} \quad R_X = 3R_{V_2}.$$
from LT of 3 using time-scaling property

Combining the above equations, we have

$$6 \longrightarrow X(s) = \tfrac{1}{3}V_2(s/3)$$
substituting 5 for V_2
$$= \tfrac{1}{3}e^{-7(s/3)}V_1(s/3)$$
multiply
$$= \tfrac{1}{3}e^{-7s/3}V_1(s/3)$$
substituting 4 for V_1
$$= \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{and}$$

$$6 \longrightarrow R_X = 3R_{V_2}$$
substituting 5 for R_{V2}
$$= 3R_{V_1}$$
substituting 4 for R_{V1}
$$= 3(-5 < \text{Re}(s) < 5)$$
multiply
$$= -15 < \text{Re}(s) < 15.$$

Thus, we have shown that

$$X(s) = \tfrac{1}{3}e^{-7s/3}\frac{-10}{(s/3+5)(s/3-5)} \quad \text{for } -15 < \text{Re}(s) < 15. \qquad \blacksquare$$

Example 7.13 (Conjugation property)**.** Using only properties of the Laplace transform and the transform pair

$$\underbrace{e^{(-1-j)t}u(t)}_{v(t)} \overset{\text{LT}}{\longleftrightarrow} \underbrace{\frac{1}{s+1+j}}_{V(s)} \quad \text{for } \operatorname{Re}(s) > -1,$$

find the Laplace transform of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. To begin, let $v(t) = e^{(-1-j)t}u(t)$ (i.e., v is the function whose Laplace transform is given in the Laplace-transform pair above) and let V denote the Laplace transform of v. First, we determine the relationship between x and v. We have

$$\begin{aligned} x(t) &= \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^* \\ &= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^* \\ &= \left[e^{(-1-j)t}u(t)\right]^* \\ &= v^*(t). \end{aligned}$$

$z^{**} = z$

$(z_1 z_2)^* = z_1^* z_2^*$

u is real

from definition of v

Thus, $x = v^*$. Next, we find the Laplace transform of x. We are given

$$v(t) = e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} V(s) = \frac{1}{s+1+j} \quad \text{for } \operatorname{Re}(s) > -1.$$

Using the conjugation property, we can deduce

conjugate

conjugate s and overall

ROC unchanged

$$x(t) = e^{(-1+j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{1}{s^*+1+j}\right)^* \quad \text{for } \operatorname{Re}(s) > -1.$$

Simplifying the algebraic expression for X, we have

$$X(s) = \left(\frac{1}{s^*+1+j}\right)^* = \frac{1^*}{[s^*+1+j]^*} = \frac{1}{s+1-j}.$$

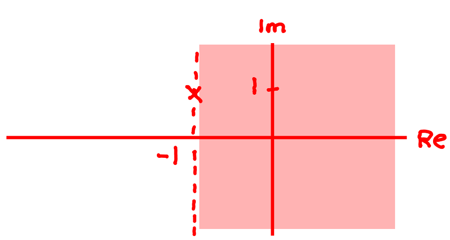
$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$

$(z_1+z_2)^* = z_1^* + z_2^*$

Therefore, we can conclude

$$X(s) = \frac{1}{s+1-j} \quad \text{for } \operatorname{Re}(s) > -1.$$

■



sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, the ROC is bounded by poles or extends to ±∞

Example 7.14 (Time-domain convolution property). Find the Laplace transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

LT table

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

Solution. From Table 7.2, we have that

$$\left.\begin{array}{l} x_1(t) = \sin(3t)u(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s) = \dfrac{3}{s^2+9} \text{ for } \operatorname{Re}(s) > 0 \quad \text{and} \\ x_2(t) = tu(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s) = \dfrac{1}{s^2} \text{ for } \operatorname{Re}(s) > 0. \end{array}\right\} \text{ from LT table}$$

Using the time-domain convolution property, we have

ROC equals intersection since no pole-zero cancellation

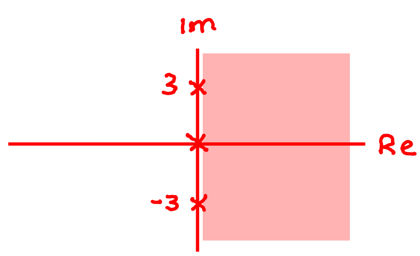
$$x_1 * x_2(t) = x(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) \text{ for } \{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}.$$

convolve

multiply

The ROC of X is $\{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation occurs. Simplifying the expression for X, we conclude

$$X(s) = \frac{3}{s^2\underbrace{(s^2+9)}_{(s+3j)(s-3j)}} \text{ for } \operatorname{Re}(s) > 0. \qquad A \cap A = A \qquad \blacksquare$$



sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, the ROC is bounded by poles or extends to $\pm\infty$

Example 7.15 (Time-domain differentiation property)**.** Find the Laplace transform X of the function

LT table

$$x(t) = \tfrac{d}{dt}\delta(t).$$

Solution. From Table 7.2, we have that

$$\delta(t) \overset{\text{LT}}{\longleftrightarrow} 1 \text{ for all } s.$$

ROC contains original ROC

Using the time-domain differentiation property, we can deduce

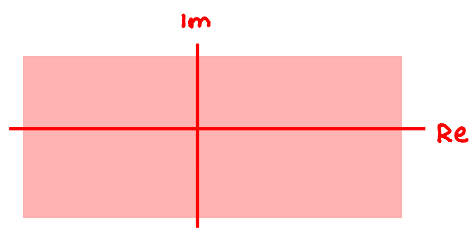
differentiate multiply by s

$$x(t) = \tfrac{d}{dt}\delta(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \underbrace{s(1)}_{s} \text{ for all } s.$$

Therefore, we have

$$X(s) = s \quad \text{for all } s.$$ ■

obviously, ROC cannot be larger



Sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, since no poles, ROC fills entire plane

Example 7.16 (Laplace-domain differentiation property)**.** Using only the properties of the Laplace transform and the transform pair

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform of the function

$$x(t) = te^{-2t}u(t).$$

Solution. We are given

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

multiply by t   -d/ds   ROC unchanged

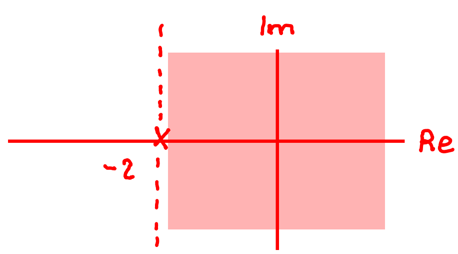
$$x(t) = te^{-2t}u(t) \quad \overset{\text{LT}}{\longleftrightarrow} \quad X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) \quad \text{for } \operatorname{Re}(s) > -2.$$

Simplifying the algebraic expression for X, we have

$$X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) = -\tfrac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \operatorname{Re}(s) > -2. \qquad \blacksquare$$



Sanity check: are the stated algebraic expression and stated ROC self consistent?

yes, the ROC is bounded by poles or extends to $\pm\infty$

Example 7.17 (Time-domain integration property)**.** Find the Laplace transform of the function

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau.$$

LT table

Solution. From Table 7.2, we have that

$$e^{-2t}\sin(t)u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{(s+2)^2+1} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the time-domain integration property, we can deduce

integrate

multiply by 1/s

ROC is intersected with Re(s) > 0 (cannot be larger since no poles cancelled)

$$x(t) = \int_{-\infty}^{t} e^{-2\tau}\sin(\tau)u(\tau)d\tau \overset{\text{LT}}{\longleftrightarrow} X(s) = \underbrace{\frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right)}_{\text{simplify}} \quad \text{for } \underbrace{\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}}_{\operatorname{Re}(s) > 0}.$$

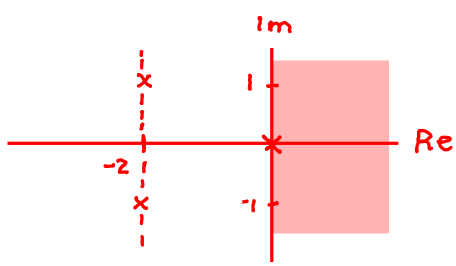
The ROC of X is $\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for X, we have

$$X(s) = \frac{1}{s}\left(\frac{1}{(s+2)^2+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+4+1}\right) = \frac{1}{s}\left(\frac{1}{s^2+4s+5}\right).$$

Therefore, we have

$$X(s) = \frac{1}{s(s^2+4s+5)} \quad \text{for } \operatorname{Re}(s) > 0.$$

[Note: $s^2+4s+5 = (s+2-j)(s+2+j)$.] (s+2-j)(s+2+j) ■



sanity check:
are the stated algebraic expression and stated ROC self consistent?
yes, the ROC is bounded by poles or extends to ±∞

Example 7.18 (Initial and final value theorems)**.** A bounded causal function x with a (finite) limit at infinity has the Laplace transform

$$X(s) = \frac{2s^2+3s+2}{s^3+2s^2+2s} \quad \text{for } \operatorname{Re}(s) > 0.$$

Determine $x(0^+)$ and $\lim_{t\to\infty} x(t)$.

Solution. Since x is causal (i.e., $x(t) = 0$ for all $t < 0$) and does not have any singularities at the origin, the initial value theorem can be applied. From this theorem, we have

$$\begin{aligned}
x(0^+) &= \lim_{s\to\infty} sX(s) && \text{substitute given } X \\
&= \lim_{s\to\infty} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] && \text{multiply} \\
&= \lim_{s\to\infty} \frac{2s^2+3s+2}{s^2+2s+2} && \text{take limit (highest power terms dominate)} \\
&= 2.
\end{aligned}$$

Since x is bounded and causal and has well-defined limit at infinity, we can apply the final value theorem. From this theorem, we have

$$\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) && \text{substitute given } X \\
&= \lim_{s\to 0} s\left[\frac{2s^2+3s+2}{s^3+2s^2+2s}\right] && \text{multiply} \\
&= \left.\frac{2s^2+3s+2}{s^2+2s+2}\right|_{s=0} && \text{evaluate at } s=0 \\
&= 1.
\end{aligned}$$

In passing, we note that the inverse Laplace transform x of X can be shown to be

$$x(t) = [1 + e^{-t}\cos t]u(t).$$

As we would expect, the values calculated above for $x(0^+)$ and $\lim_{t\to\infty} x(t)$ are consistent with this formula for x. ■

Example 7.25. Using a Laplace transform table and properties of the Laplace transform, find the Laplace transform X of the function x shown in Figure 7.13.

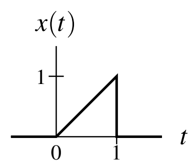


Figure 7.13: Function for the Laplace transform example.

Second solution (which incurs less work by avoiding differentiation). First, we express x using unit-step functions to yield

$$\begin{aligned} x(t) &= t[u(t) - u(t-1)] \\ &= tu(t) - tu(t-1). \end{aligned}$$

add and subtract $u(t-1)$

To simplify the subsequent Laplace transform calculation, we choose to rewrite x as

$$\begin{aligned} x(t) &= tu(t) - tu(t-1) + u(t-1) - u(t-1) \\ &= tu(t) - (t-1)u(t-1) - u(t-1). \end{aligned}$$

group two middle terms together

taking LT

(This is motivated by a preference to compute the Laplace transform of $(t-1)u(t-1)$ instead of $tu(t-1)$.) Taking the Laplace transform of both sides of the preceding equation, we obtain

tu(t) time shifted by 1

u(t) time shifted by 1 and then multiplied by t (requires differentiation)

$$X(s) = \underbrace{\mathcal{L}\{tu(t)\}(s)}_{1} - \underbrace{\mathcal{L}\{(t-1)u(t-1)\}(s)}_{2} - \underbrace{\mathcal{L}\{u(t-1)\}(s)}_{3}. \quad (*)$$

We have

$$1 \quad \mathcal{L}\{tu(t)\}(s) = \frac{1}{s^2}, \quad \text{from LT table}$$

time shifting

$$\begin{aligned} 2 \quad \mathcal{L}\{(t-1)u(t-1)\}(s) &= e^{-s}\mathcal{L}\{tu(t)\}(s) \\ &= e^{-s}\left(\frac{1}{s^2}\right) \quad \text{LT table} \\ &= \frac{e^{-s}}{s^2}, \quad \text{and} \quad \text{multiply} \end{aligned}$$

time shifting

$$\begin{aligned} 3 \quad \mathcal{L}\{u(t-1)\}(s) &= e^{-s}\mathcal{L}\{u(t)\}(s) \\ &= e^{-s}\left(\frac{1}{s}\right) \quad \text{LT table} \\ &= \frac{e^{-s}}{s}. \quad \text{multiply} \end{aligned}$$

Combining the above results, we have

Substituting 1, 2, and 3 into (*)

$$\begin{aligned} X(s) &= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} \\ &= \frac{1 - e^{-s} - se^{-s}}{s^2}. \end{aligned}$$

Since x is finite duration, the ROC of X is the entire complex plane. ■

Example 7.27. Find the inverse Laplace transform x of

$$X(s) = \frac{2}{s^2 - s - 2} \quad \text{for } -1 < \operatorname{Re}(s) < 2.$$

Solution. We begin by rewriting X in the factored form

$$X(s) = \frac{2}{(s+1)(s-2)}.$$

strictly proper with
1st order poles at −1 and 2

Then, we find a partial fraction expansion of X. We know that X has an expansion of the form

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2}.$$

Calculating the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= (s+1)X(s)|_{s=-1} \\ &= \left.\frac{2}{s-2}\right|_{s=-1} \\ &= -\frac{2}{3} \quad \text{and} \\ A_2 &= (s-2)X(s)|_{s=2} \\ &= \left.\frac{2}{s+1}\right|_{s=2} \\ &= \frac{2}{3}. \end{aligned}$$

So, X has the expansion

−1 < Re(s) < 2

$$X(s) = \frac{2}{3}\left(\frac{1}{s-2}\right) - \frac{2}{3}\left(\frac{1}{s+1}\right).$$

Taking the inverse Laplace transform of both sides of this equation, we have

$$x(t) = \frac{2}{3}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t)}_{1} - \frac{2}{3}\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t)}_{2}. \tag{7.6}$$

LT table

Using Table 7.2 and the given ROC, we have

−1 < Re(s) < 2

$$\begin{aligned} &1 \quad -e^{2t}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s-2} \quad \text{for } \operatorname{Re}(s) < 2 \quad \text{and} \\ &2 \quad e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

ROC must contain
−1 < Re(s) < 2
(see A and B)

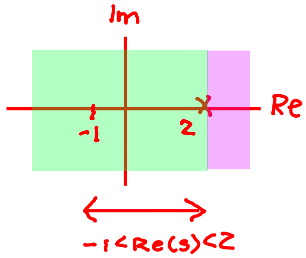
Substituting these results into (7.6), we obtain

$$\begin{aligned} x(t) &= \tfrac{2}{3}[-e^{2t}u(-t)] - \tfrac{2}{3}[e^{-t}u(t)] \\ &= -\tfrac{2}{3}e^{2t}u(-t) - \tfrac{2}{3}e^{-t}u(t). \end{aligned}$$

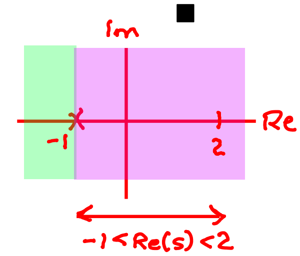
substituting the inverse LTs
from 1 and 2

■

A
possible ROCs
for $\frac{1}{s-2}$



B
possible ROCs
for $\frac{1}{s+1}$



Example 7.28 (Rational function with a repeated pole). Find the inverse Laplace transform x of

$$X(s) = \frac{2s+1}{(s+1)^2(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$

strictly proper with 1st order pole at −2 and 2nd order pole at −1

Solution. To begin, we find a partial fraction expansion of X. We know that X has an expansion of the form

terms from pole at −1

$$X(s) = \boxed{\frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2}} + \boxed{\frac{A_{21}}{s+2}}.$$

term from pole at −2

Calculating the coefficients of the expansion, we obtain

formula for repeated pole case

coefficient index

pole order

$$\begin{aligned}
A_{11} &= \frac{1}{(2-1)!}\left[\left(\frac{d}{ds}\right)^{2-1}[(s+1)^2X(s)]\right]\bigg|_{s=-1} = \frac{1}{1!}\left[\tfrac{d}{ds}[(s+1)^2X(s)]\right]\big|_{s=-1} = \left[\frac{d}{ds}\left(\frac{2s+1}{s+2}\right)\right]\bigg|_{s=-1} \\
&= \left[\frac{(s+2)(2)-(2s+1)(1)}{(s+2)^2}\right]\bigg|_{s=-1} = \left[\frac{2s+4-2s-1}{(s+2)^2}\right]\bigg|_{s=-1} = \left[\frac{3}{(s+2)^2}\right]\bigg|_{s=-1} = 3, \\
A_{12} &= \frac{1}{(2-2)!}\left[\left(\frac{d}{ds}\right)^{2-2}[(s+1)^2X(s)]\right]\bigg|_{s=-1} = \frac{1}{0!}\left[(s+1)^2X(s)\right]\big|_{s=-1} = \frac{2s+1}{s+2}\bigg|_{s=-1} = \frac{-1}{1} = -1, \quad \text{and}
\end{aligned}$$

formula for simple pole case

$$A_{21} = (s+2)X(s)|_{s=-2} = \frac{2s+1}{(s+1)^2}\bigg|_{s=-2} = \frac{-3}{1} = -3.$$

Thus, X has the expansion

Re(s) > −1

$$X(s) = \frac{3}{s+1} - \frac{1}{(s+1)^2} - \frac{3}{s+2}.$$

Taking the inverse Laplace transform of both sides of this equation yields

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t). \tag{7.7}$$

Re(s) > −1 Re(s) > −1 Re(s) > −2

At this point, it is important to remember that every Laplace transform has an associated ROC, which is an essential component of the Laplace transform. So, when computing the inverse Laplace transform of a function, we must be careful to use the correct ROC for the function. Thus, in order to compute the three inverse Laplace transforms appearing in (7.7), we must associate a ROC with each of the three expressions $\frac{1}{s+1}$, $\frac{1}{(s+1)^2}$, and $\frac{1}{s+2}$. Some care must be exercised in doing so, since each of these expressions has more than one possible ROC and only one is correct. The possible ROCs for each of these expressions is shown in Figure 7.16. In the case of each of these expressions, the correct ROC to use is the one that contains the ROC of X (i.e., $\operatorname{Re}(s) > -1$).

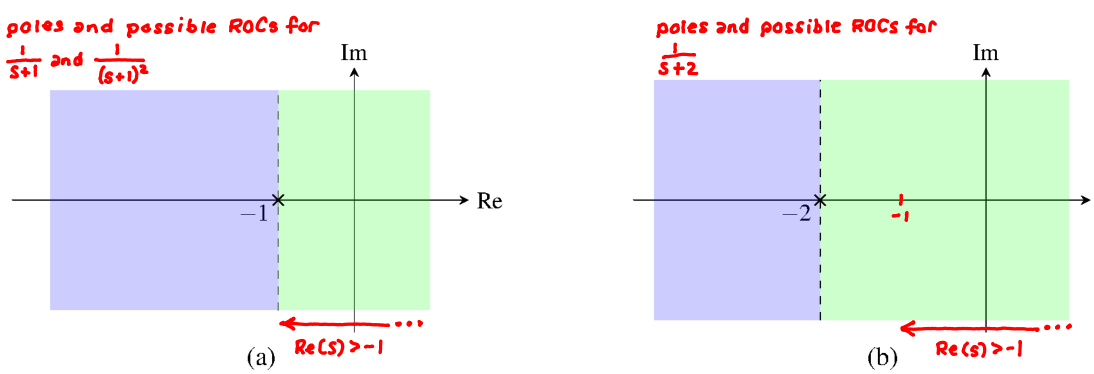


Figure 7.16: The poles and possible ROCs for the rational expressions (a) $\frac{1}{s+1}$ and $\frac{1}{(s+1)^2}$; and (b) $\frac{1}{s+2}$.

LT table

From Table 7.2, we have

$$x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}(t) - 3\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}(t) \quad (7.7)$$

$\operatorname{Re}(s) > -1$ $\quad$ $\operatorname{Re}(s) > -1$ $\quad$ $\operatorname{Re}(s) > -2$

$$\begin{aligned}
&① \quad e^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1,\\
&② \quad te^{-t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{(s+1)^2} \quad \text{for } \operatorname{Re}(s) > -1, \quad \text{and}\\
&③ \quad e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \tfrac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.
\end{aligned}$$

Substituting these results into (7.7), we obtain

Substituting ①, ②, and ③ into (7.7)

$$\begin{aligned}
x(t) &= 3e^{-t}u(t) - te^{-t}u(t) - 3e^{-2t}u(t)\\
&= \left(3e^{-t} - te^{-t} - 3e^{-2t}\right)u(t).
\end{aligned}$$

■

Example 7.31. For the LTI system with each system function H below, determine whether the system is causal.

rational

(a) $H(s) = \frac{1}{s+1}$ for $\operatorname{Re}(s) > -1$;

(b) $H(s) = \frac{1}{s^2-1}$ for $-1 < \operatorname{Re}(s) < 1$;

not rational

(c) $H(s) = \frac{e^s}{s+1}$ for $\operatorname{Re}(s) < -1$; and

(d) $H(s) = \frac{e^s}{s+1}$ for $\operatorname{Re}(s) > -1$.

causal $\Rightarrow$ ROC is RHP

if rational: causal $\Leftrightarrow$ ROC is RHP

Solution. (a) The poles of H are plotted in Figure 7.19(a) and the ROC is indicated by the shaded area. The system function H is rational and the ROC is the right-half plane to the right of the rightmost pole. Therefore, the system is causal.

(b) The poles of H are plotted in Figure 7.19(b) and the ROC is indicated by the shaded area. The system function is rational but the ROC is not a right-half plane. Therefore, the system is not causal.

(c) The system function H has a left-half plane ROC. Therefore, h is a left-sided signal. Thus, the system is not causal.

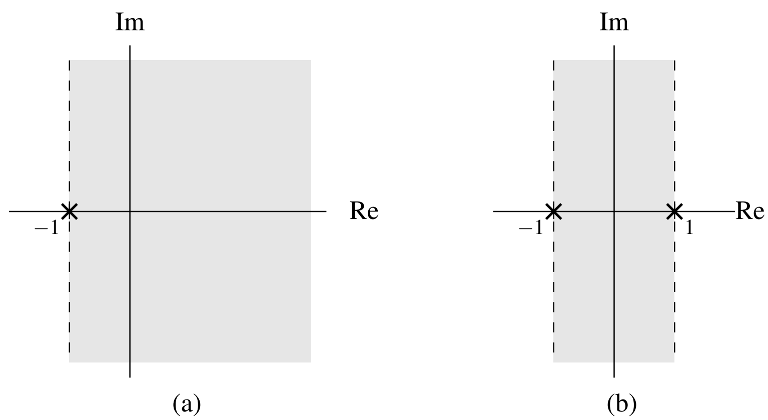
(d) The system function H has a right-half plane ROC but is not rational. Thus, we cannot make any conclusion directly from the system function. Instead, we draw our conclusion from the impulse response h. Taking the inverse Laplace transform of H, we obtain

$$h(t) = e^{-(t+1)}u(t+1).$$

not causal function since $h(t) \neq 0$ for $t \in (-1, 0)$

Thus, the impulse response h is not causal. Therefore, the system is not causal.

H is rational and ROC is RHP $\Rightarrow$ h is causal



H is rational and ROC is not RHP $\Rightarrow$ h is not causal

Figure 7.19: Pole and ROCs of the rational system functions in the causality example. The cases of the (a) first (b) second system functions.

■

Example 7.32. A LTI system has the system function

$$H(s) = \frac{1}{(s+1)(s+2)}.$$

Given that the system is BIBO stable, determine the ROC of H.

Solution. Clearly, the system function H is rational with poles at -1 and -2. Therefore, only three possibilities exist for the ROC:

i) $\mathrm{Re}(s) < -2$,
ii) $-2 < \mathrm{Re}(s) < -1$, and
iii) $\mathrm{Re}(s) > -1$.

In order for the system to be stable, however, the ROC of H must include the entire imaginary axis. Therefore, the ROC must be $\mathrm{Re}(s) > -1$. This ROC is illustrated in Figure 7.20.

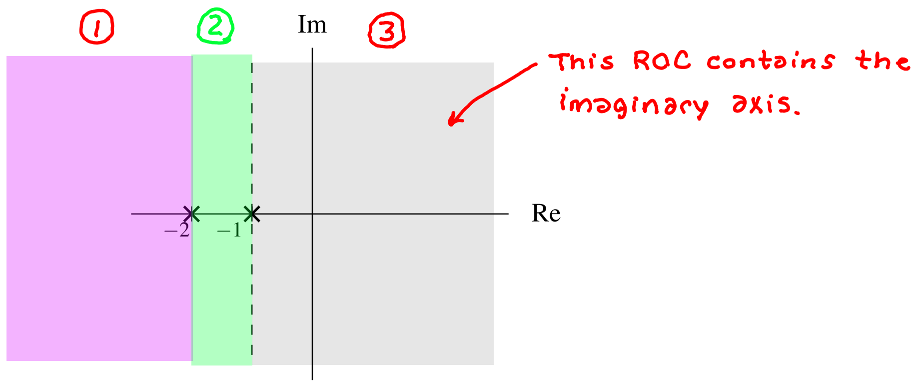


Figure 7.20: ROC for example.

■

Example 7.33. A LTI system is causal and has the system function

$$H(s) = \frac{1}{(s+2)(s^2+2s+2)}.$$

Determine whether this system is BIBO stable.

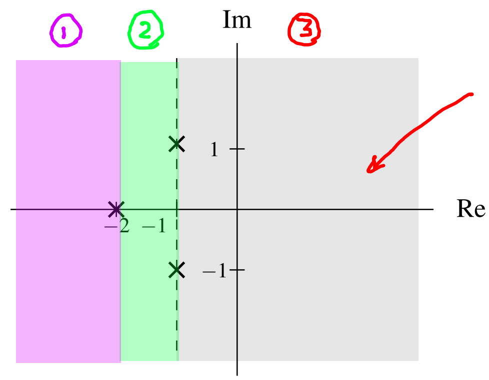
Solution. We begin by factoring H to obtain

$$H(s) = \frac{1}{(s+2)(s+1-j)(s+1+j)}.$$

(Using the quadratic formula, one can confirm that $s^2+2s+2=0$ has roots at $s=-1\pm j$.) Thus, H has poles at -2, $-1+j$, and $-1-j$. The poles are plotted in Figure 7.21. Since the system is causal and all of the poles of H are in the left half of the plane, the system is stable.

Since causal, ROC of H is RHP

Three possibilities exist for the ROC of H as shown.



This ROC is RHP. This ROC contains imaginary axis.

Figure 7.21: Poles of the system function.

■

Example 7.34. For each LTI system with system function H given below, determine the ROC of H that corresponds to a BIBO stable system.

(a) $H(s) = \dfrac{s(s-1)}{(s+2)(s+1+j)(s+1-j)}$;

(b) $H(s) = \dfrac{s}{(s+1)(s-1)(s-1-j)(s-1+j)}$;

(c) $H(s) = \dfrac{(s+j)(s-j)}{(s+2-j)(s+2+j)}$; and

(d) $H(s) = \dfrac{s-1}{s}$.

all rational functions

Solution. (a) The function H has poles at -2, $-1+j$, and $-1-j$. The poles are shown in Figure 7.22(a). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

i) $\mathrm{Re}(s) < -2$,
ii) $-2 < \mathrm{Re}(s) < -1$, and
iii) $\mathrm{Re}(s) > -1$.

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\mathrm{Re}(s) > -1$. This is the shaded region in the Figure 7.22(a).

(b) The function H has poles at -1, 1, $1+j$, and $1-j$. The poles are shown in Figure 7.22(b). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only three distinct ROCs are possible:

i) $\mathrm{Re}(s) < -1$,
ii) $-1 < \mathrm{Re}(s) < 1$, and
iii) $\mathrm{Re}(s) > 1$.

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $-1 < \mathrm{Re}(s) < 1$. This is the shaded region in Figure 7.22(b).

(c) The function H has poles at $-2+j$ and $-2-j$. The poles are shown in Figure 7.22(c). Since H is rational, the ROC must be bounded by poles or extend to infinity. Consequently, only two distinct ROCs are possible:

i) $\mathrm{Re}(s) < -2$ and
ii) $\mathrm{Re}(s) > -2$.

Since we want a stable system, the ROC must include the entire imaginary axis. Therefore, the ROC must be $\mathrm{Re}(s) > -2$. This is the shaded region in Figure 7.22(c).

(d) The function H has a pole at 0. The pole is shown in Figure 7.22(d). Since H is rational, it cannot converge at 0 (which is a pole of H). Consequently, the ROC can never include the entire imaginary axis. Therefore, the system function H can never be associated with a stable system. ■

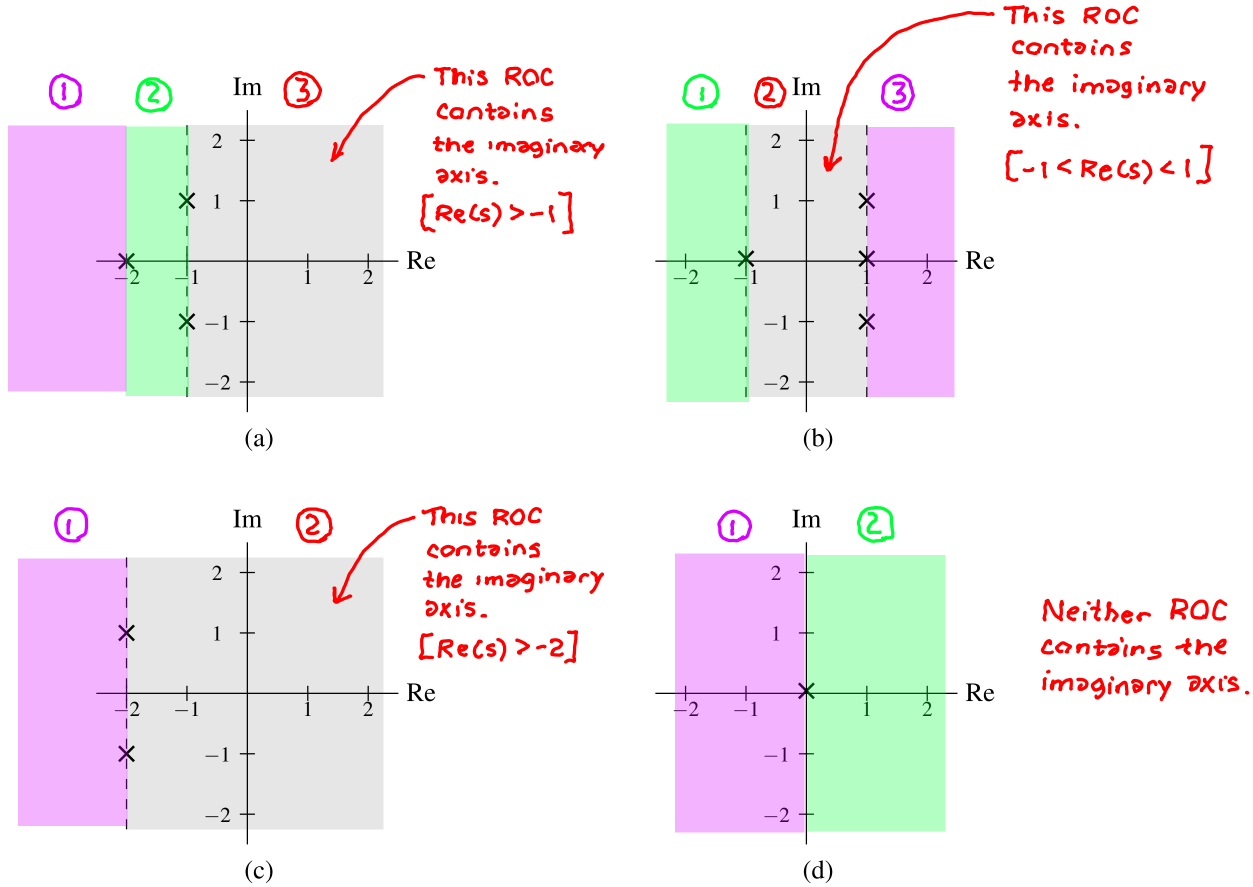


Figure 7.22: Poles and ROCs of the system function H in the (a) first, (b) second, (c) third, and (d) fourth parts of the example.

Example 7.35. Consider the LTI system with system function

$$H(s) = \frac{s+1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

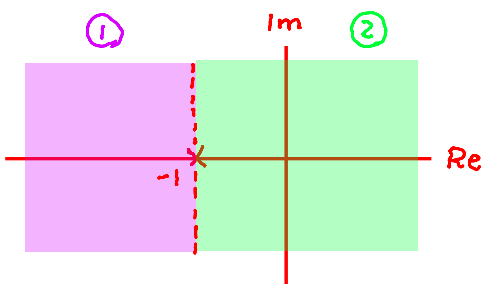
Determine all possible inverses of this system. Comment on the stability of each of these inverse systems.

Solution. The system function H_{inv} of the inverse system is given by

$$H_{\mathsf{inv}}(s) = \frac{1}{H(s)} = \frac{s+2}{s+1}.$$

Two ROCs are possible for H_{inv}:
  i) $\operatorname{Re}(s) < -1$ and
  ii) $\operatorname{Re}(s) > -1$.

Each ROC is associated with a distinct inverse system. The first ROC is associated with an unstable system since this ROC does not include the imaginary axis. The second ROC is associated with a stable system, since this ROC includes the entire imaginary axis. ■



region 1 does not contain the imaginary axis and therefore corresponds to an unstable system

region 2 contains the imaginary axis and therefore corresponds to a stable system

Example 7.36 (Differential equation to system function)**.** A LTI system with input x and output y is characterized by the differential equation

$$y''(t) + \tfrac{D}{M}y'(t) + \tfrac{K}{M}y(t) = x(t),$$

where D, K, and M are positive real constants, and the prime symbol is used to denote derivative. Find the system function H of this system.

Solution. Taking the Laplace transform of the given differential equation, we obtain

$$s^2Y(s) + \tfrac{D}{M}sY(s) + \tfrac{K}{M}Y(s) = X(s).$$

taking LT using time-domain differentiation property

Rearranging the terms and factoring, we have

$$\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)Y(s) = X(s).$$

rearrange terms and factor

Dividing both sides by $\left(s^2 + \tfrac{D}{M}s + \tfrac{K}{M}\right)X(s)$, we obtain

$$\frac{Y(s)}{X(s)} = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}.$$

divide both sides by $(s^2 + \frac{D}{M}s + \frac{K}{M})X(s)$

Thus, H is given by

$$H(s) = \frac{1}{s^2 + \frac{D}{M}s + \frac{K}{M}}.$$

$Y(s) = X(s)H(s) \Rightarrow$
$H(s) = \frac{Y(s)}{X(s)}$

■

Example 7.37 (System function to differential equation)**.** A LTI system with input x and output y has the system function

$$H(s) = \frac{s}{s+R/L},$$

where L and R are positive real constants. Find the differential equation that characterizes this system.

Solution. Let X and Y denote the Laplace transforms of x and y, respectively. To begin, we have

$$\begin{aligned} Y(s) &= H(s)X(s) \\ &= \left(\frac{s}{s+R/L}\right)X(s). \end{aligned}$$

system is LTI

substitute given H

multiply both sides by (s+R/L)

Rearranging this equation, we obtain

$$\begin{aligned} (s+\tfrac{R}{L})Y(s) &= sX(s) \\ \Rightarrow \quad sY(s)+\tfrac{R}{L}Y(s) &= sX(s). \end{aligned}$$

simplify

take inverse LT (and use linearity)

Taking the inverse Laplace transform of both sides of this equation (by using the linearity and time-differentiation properties of the Laplace transform), we have

$$\begin{aligned} \mathcal{L}^{-1}\{sY(s)\}(t)+\tfrac{R}{L}\mathcal{L}^{-1}Y(t) &= \mathcal{L}^{-1}\{sX(s)\}(t) \\ \Rightarrow \quad \tfrac{d}{dt}y(t)+\tfrac{R}{L}y(t) &= \tfrac{d}{dt}x(t). \end{aligned}$$

time-domain differentiation property

Therefore, the system is characterized by the differential equation

$$\tfrac{d}{dt}y(t)+\tfrac{R}{L}y(t) = \tfrac{d}{dt}x(t). \qquad \blacksquare$$

Example 7.38 (Simple RC network)**.** Consider the resistor-capacitor (RC) network shown in Figure 7.24 with input v_1 and output v_2. This system is LTI and can be characterized by a linear differential equation with constant coefficients. (a) Find the system function H of this system. (b) Determine whether the system is BIBO stable. (c) Determine the step response of the system.

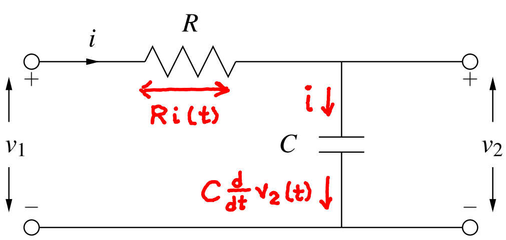


Figure 7.24: Simple RC network.

Solution. (a) From basic circuit analysis, we have

$$v_1(t) = Ri(t) + v_2(t) \quad \text{and} \tag{7.14a}$$
$$i(t) = C\tfrac{d}{dt}v_2(t). \tag{7.14b}$$

taking LT

Taking the Laplace transform of (7.14) yields

$$V_1(s) = RI(s) + V_2(s) \quad \text{and} \tag{7.15a}$$
$$I(s) = CsV_2(s). \tag{7.15b}$$

substitute (7.15b) into (7.15a) to eliminate I

Substituting (7.15b) into (7.15a) and rearranging, we obtain

$$\begin{aligned} & V_1(s) = R[CsV_2(s)] + V_2(s) \\ \Rightarrow\ & V_1(s) = RCsV_2(s) + V_2(s) \\ \Rightarrow\ & V_1(s) = [1+RCs]V_2(s) \\ \Rightarrow\ & \frac{V_2(s)}{V_1(s)} = \frac{1}{1+RCs}. \end{aligned}$$

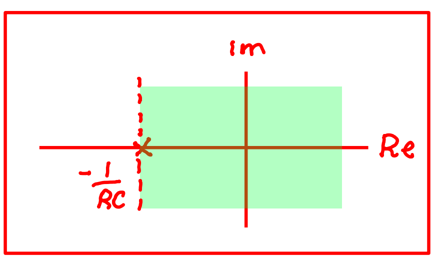
factor

rearrange

Thus, we have that the system function H is given by

$V_2(s) = V_1(s)H(s) \Longrightarrow H(s) = \dfrac{V_2(s)}{V_1(s)}$

⊛



$$\begin{aligned} H(s) &= \frac{1}{1+RCs} \\ &= \frac{\frac{1}{RC}}{s+\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{s-(-\frac{1}{RC})}. \end{aligned}$$

divide numerator and denominator by RC

explicity show poles

Since the system can be physically realized, it must be causal. Therefore, the ROC of H must be a right-half plane. Thus, we may infer that the ROC of H is $\operatorname{Re}(s) > -\frac{1}{RC}$. So, we have

see ⊛

$$H(s) = \frac{1}{1+RCs} \quad \text{for } \operatorname{Re}(s) > -\tfrac{1}{RC}.$$

(b) Since resistance and capacitance are (strictly) positive quantities, $R > 0$ and $C > 0$. Thus, $-\frac{1}{RC} < 0$. Consequently, the ROC contains the imaginary axis and the system is stable.

(c) Now, let us calculate the step response of the system. We know that the system input-output behavior is characterized by the equation

$$\begin{aligned} V_2(s) &= H(s)V_1(s) \\ &= \left(\frac{1}{1+RCs}\right)V_1(s). \end{aligned}$$

Since system is LTI

substitute for H

To compute the step response, we need to consider an input equal to the unit-step function. So, $v_1 = u$, implying that $V_1(s) = \frac{1}{s}$. Substituting this expression for V_1 into the above expression for V_2, we have

$$\begin{aligned} V_2(s) &= \left(\frac{1}{1+RCs}\right)\left(\frac{1}{s}\right) \\ &= \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}. \end{aligned}$$

$v_1(t) = u(t) \xleftrightarrow{\text{LT}} V_1(s) = \frac{1}{s}$

divide numerator and denominator by RC

Now, we need to compute the inverse Laplace transform of V_2 in order to determine v_2. To simplify this task, we find the partial fraction expansion for V_2. We know that this expansion is of the form

$$V_2(s) = \frac{A_1}{s} + \frac{A_2}{s+\frac{1}{RC}}.$$

Solving for the coefficients of the expansion, we obtain

$$\begin{aligned} A_1 &= sV_2(s)|_{s=0} \\ &= 1 \quad \text{and} \\ A_2 &= (s+\tfrac{1}{RC})V_2(s)|_{s=-\frac{1}{RC}} \\ &= \frac{\frac{1}{RC}}{-\frac{1}{RC}} \\ &= -1. \end{aligned}$$

Thus, we have that V_2 has the partial fraction expansion given by

$$V_2(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}.$$

Taking the inverse Laplace transform of both sides of the equation, we obtain

$$v_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}(t) - \mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{RC}}\right\}(t).$$

LT table

RHP

Using Table 7.2 and the fact that the system is causal (which implies the necessary ROC), we obtain

inverse LT

$$\begin{aligned} v_2(t) &= u(t) - e^{-t/(RC)}u(t) \\ &= \left(1 - e^{-t/(RC)}\right)u(t). \end{aligned}$$

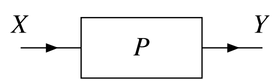
■

$u(t) \xleftrightarrow{\text{LT}} \frac{1}{s}$ for $\operatorname{Re}(s) > 0$

$e^{-at}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+a}$ for $\operatorname{Re}(s) > -a$

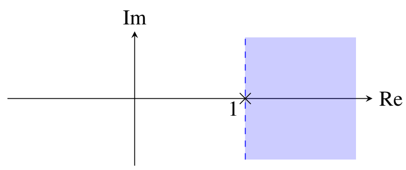
Stabilization Example: Unstable Plant

- causal LTI plant:



$$P(s) = \frac{10}{s-1}$$ has pole at 1

- ROC of P:



ROC is RHP since system is causal

system is not BIBO stable since ROC does not contain imaginary axis

- system is not BIBO stable

Stabilization Example: Using Pole-Zero Cancellation

- system formed by series interconnection of plant and causal LTI compensator:

$$X \longrightarrow \boxed{W} \longrightarrow \boxed{P} \longrightarrow Y$$

$$P(s) = \frac{10}{s-1}, \quad W(s) = \frac{s-1}{10(s+1)}$$

- system function H of overall system:

pole-zero cancellation

$$H(s) = W(s)P(s) = \left(\frac{s-1}{10(s+1)}\right)\left(\frac{10}{s-1}\right) = \frac{1}{s+1}$$

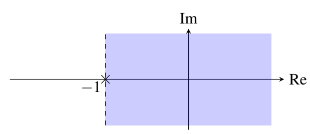
has pole at −1

connecting systems in series multiplies system functions

substitute given W and P

multiply

- ROC of H:



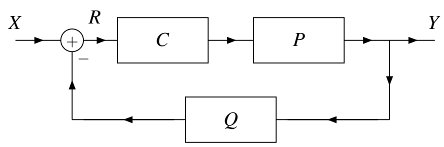
ROC is RHP since system is causal

system is BIBO stable since ROC contains imaginary axis

- overall system is BIBO stable

Stabilization Example: Using Feedback (1)

- feedback system (with causal LTI compensator and sensor):



$$P(s) = \frac{10}{s-1}, \quad C(s) = \beta, \quad Q(s) = 1$$

- system function H of feedback system:

substitute given C, P, and Q and simplify

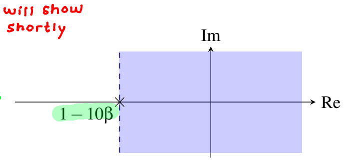
$$H(s) = \frac{C(s)P(s)}{1+C(s)P(s)Q(s)} = \frac{10\beta}{s-(1-10\beta)}$$

we will show this shortly

has pole at $1-10\beta$

- ROC of H:

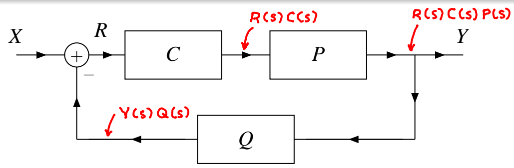
ROC contains imaginary axis if $1-10\beta<0$



ROC is RHP since system is causal

- feedback system is BIBO stable if and only if $1 - 10\beta < 0$ or equivalently $\beta > \frac{1}{10}$

Stabilization Example: Using Feedback (2)



① $R(s) = X(s) - Q(s)Y(s)$ ← equation for adder

② $Y(s) = C(s)P(s)R(s)$ ← equation for output

$$
\begin{aligned}
Y(s) &= C(s)P(s)R(s) && \text{from ②} \\
&= C(s)P(s)[X(s) - Q(s)Y(s)] && \text{substituting formula for R from ①} \\
&= C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) && \text{multiply}
\end{aligned}
$$

move terms containing Y to the left-hand side and factor

$$[1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s)$$

divide both sides by X(s)[1+C(s)P(s)Q(s)]

$$H(s) = \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}$$

Y(s) = X(s)H(s)

Stabilization Example: Using Feedback (3)

$$P(s) = \tfrac{10}{s-1}, \quad C(s) = \beta, \quad Q(s) = 1$$ given

result from previous slide

$$\begin{aligned}
H(s) &= \frac{C(s)P(s)}{1+C(s)P(s)Q(s)} \\
&= \frac{\beta(\frac{10}{s-1})}{1+\beta(\frac{10}{s-1})(1)} \\
&= \frac{10\beta}{s-1+10\beta} \\
&= \frac{10\beta}{s-(1-10\beta)}
\end{aligned}$$

substitute given C, P, and Q

multiply by $\frac{s-1}{s-1}$

rewrite to explicitly show pole

pole at $1-10\beta$

Remarks on Stabilization Via Pole-Zero Cancellation

- Pole-zero cancellation is not achievable in practice, and therefore it cannot be used to stabilize real-world systems.
- The theoretical models used to represent real-world systems are only approximations due to many factors, including the following:
  - Determining the system function of a system involves measurement, which always has some error.
  - A system cannot be built with such precision that it will have exactly some prescribed system function.
  - The system function of most systems will vary at least slightly with changes in the physical environment.
  - Although a LTI model is used to represent a system, the likely reality is that the system is not exactly LTI, which introduces error.
- Due to approximation error, the effective poles and zeros of the system function will only be approximately where they are expected to be.
- Since pole-zero cancellation requires that a pole and zero be placed at exactly the same location, any error will prevent this cancellation from being achieved.

Example 7.40 (Stabilization of unstable plant)**.** Consider the causal LTI system with input Laplace transform X, output Laplace transform Y, and system function

$$P(s) = \frac{10}{s-1},$$

as depicted in Figure 7.27. One can easily confirm that this system is not BIBO stable, due to the pole of P at 1. (Since the system is causal, the ROC of P is the RHP given by $\operatorname{Re}(s) > 1$. Clearly, this ROC does not include the imaginary axis. Therefore, the system is not stable.) In what follows, we consider two different strategies for stabilizing this unstable system as well as their suitability for use in practice.

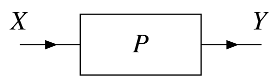


Figure 7.27: Plant.

(a) Stabilization of unstable plant via pole-zero cancellation. Suppose that the system in Figure 7.27 is connected in series with another causal LTI system with system function

$$W(s) = \frac{s-1}{10(s+1)},$$

in order to yield a new system with input Laplace transform X and output Laplace transform Y, as shown in Figure 7.28(a). Show that this new system is BIBO stable.

(b) Stabilization of unstable plant via feedback. Suppose now that the system in Figure 7.27 is interconnected with two other causal LTI systems with system functions C and Q, as shown in Figure 7.28(b), in order to yield a new system with input Laplace transform X, output Laplace transform Y, and system function H. Moreover, suppose that

$$C(s) = \beta \quad \text{and} \quad Q(s) = 1,$$

where β is a real constant. Show that, with an appropriate choice of β, the resulting system is BIBO stable.

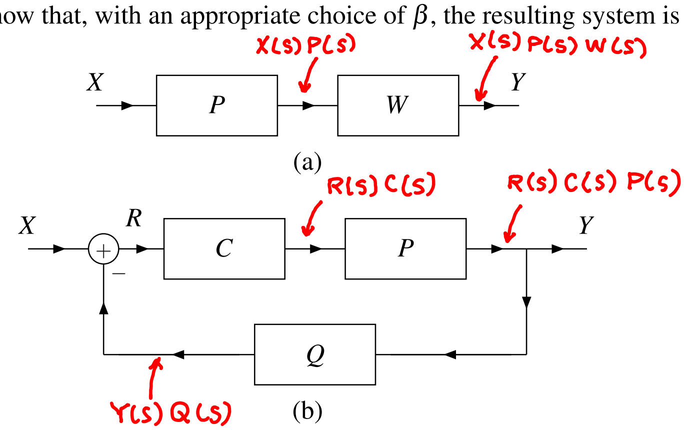


Figure 7.28: Two configurations for stabilizing the unstable plant. (a) Simple cascade system and (b) feedback control system.

(c) Practical issues. Parts (a) and (b) of this example consider two different schemes for stabilizing the unstable system in Figure 7.27. As it turns out, a scheme like the one in part (a) is not useful in practice. Identify the practical problems associated with this approach. Indicate whether the scheme in part (b) suffers from the same shortcomings.

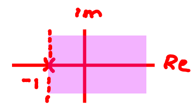
Solution. (a) From the block diagram in Figure 7.28(a), the system function H of the overall system is

$\underbrace{P(s)W(s)}_{H(s)}$ $Y(s) = \underbrace{P(s)W(s)}_{H(s)} X(s)$

$$
\begin{aligned}
H(s) &= P(s)W(s) \\
&= \left(\frac{10}{s-1}\right)\left(\frac{s-1}{10(s+1)}\right) \\
&= \frac{1}{s+1}.
\end{aligned}
$$

substitute given P and W

Simplify



Since the system is causal and H is rational, the ROC of H is $\operatorname{Re}(s) > -1$. Since the ROC includes the imaginary axis, the system is BIBO stable.

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a somewhat reasonable step response. Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

System has reasonable step response

$$
\begin{aligned}
s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}(t) \\
&= (1-e^{-t})u(t).
\end{aligned}
$$

Evidently, s is a somewhat crude approximation of the desired response u.

(b) From the block diagram in Figure 7.28(b), we can write

$$
R(s) = X(s) - Q(s)Y(s) \quad \text{and} \quad (B1)
$$
$$
Y(s) = C(s)P(s)R(s). \quad (B2)
$$

Combining these equations (by substituting the expression for R in the first equation into the second equation), we obtain

Substituting B1 into B2

$$
\begin{aligned}
& Y(s) = C(s)P(s)[X(s) - Q(s)Y(s)] \\
\Rightarrow\ & Y(s) = C(s)P(s)X(s) - C(s)P(s)Q(s)Y(s) \\
\Rightarrow\ & [1 + C(s)P(s)Q(s)]Y(s) = C(s)P(s)X(s) \\
\Rightarrow\ & \frac{Y(s)}{X(s)} = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.
\end{aligned}
$$

rearrange and factor

divide both sides by $[1 + C(s)P(s)Q(s)]X(s)$

Since $H(s) = \frac{Y(s)}{X(s)}$, we have

Since $H(s) = \frac{Y(s)}{X(s)}$

$$
H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)}.
$$

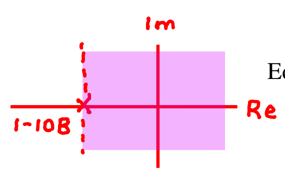
Substituting the given expressions for P, C, and Q, we have

substituting given P, C, and Q

$$
\begin{aligned}
H(s) &= \frac{\beta(\frac{10}{s-1})}{1 + \beta(\frac{10}{s-1})(1)} \\
&= \frac{10\beta}{s - 1 + 10\beta} \\
&= \frac{10\beta}{s - \underbrace{(1 - 10\beta)}_{\text{pole}}}.
\end{aligned}
$$

Simplify

explicitly show poles



The system function H is rational and has a single pole at $1-10\beta$. Since the system is causal, the ROC must be the RHP given by $\operatorname{Re}(s) > 1-10\beta$. For the system to be stable, we require that the ROC includes the imaginary axis. Thus, the system is stable if $1-10\beta < 0$ which implies $10\beta > 1$, or equivalently $\beta > \frac{1}{10}$.

system has reasonable step response

Although our only objective in this example is to stabilize the unstable plant, we note that, as it turns out, the system also has a reasonable step response. (This is not by chance, however. Some care had to be exercised in the choice of the form of the compensator system function C. The process involved in making this choice requires knowledge of control systems beyond the scope of this book, however.) Recall that, for a control system, the output should track the input. Since, in the case of the step response, the input is u, we would like the output to at least approximate u. The step response s is given by

$$\begin{aligned}
s(t) &= \mathcal{L}^{-1}\{U(s)H(s)\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{s(s-[1-10\beta])}\right\}(t) \\
&= \mathcal{L}^{-1}\left\{\frac{10\beta}{10\beta-1}\left(\frac{1}{s}-\frac{1}{s-(1-10\beta)}\right)\right\}(t) \\
&= \frac{10\beta}{10\beta-1}\left(1-e^{-(10\beta-1)t}\right)u(t) \\
&\approx u(t) \quad \text{for large } \beta.
\end{aligned}$$

Clearly, as β increases, s better approximates the desired response u.

(c) The scheme in part (a) for stabilizing the unstable plant relies on pole-zero cancellation. Unfortunately, in practice, it is not possible to achieve pole-zero cancellation. In short, the issue is one of approximation. Our analysis of systems is based on theoretical models specified in terms of equations. These theoretical models, however, are only approximations of real-world systems. This approximate nature is due to many factors, including (but not limited to) the following:

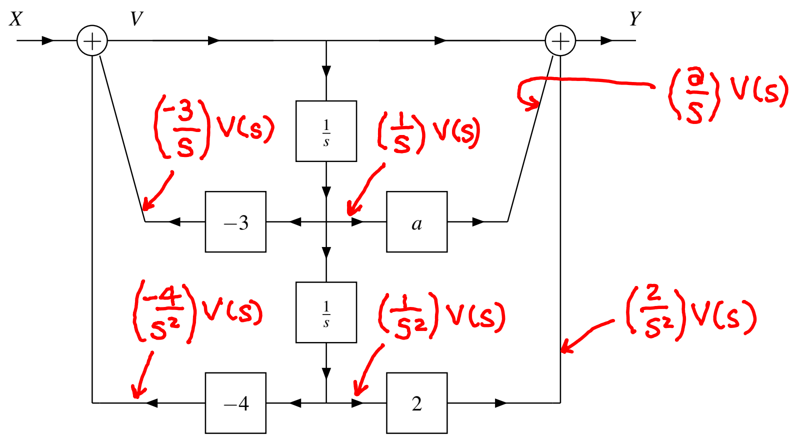
1. We cannot determine the system function of a system exactly, since this involves measurement, which always has some error.
2. We cannot build a system with such precision that it will have exactly some prescribed system function. The system function will only be close to the desired one.
3. The system function of most systems will vary at least slightly with changes in the physical environment (e.g., changes in temperature and pressure, or changes in gravitational forces due to changes in the phase of the moon, and so on).
4. Although a system may be represented by a LTI model, the likely reality is that the system is not exactly LTI, which introduces error.

For reasons such as these, the effective poles and zeros of the system function will only be approximately where we expect them to be. Pole-zero cancellation, however, requires a pole and zero to be placed at exactly the same location. So, any error will prevent the pole-zero cancellation from occurring. Since at least some small error is unavoidable in practice, the desired pole-zero cancellation will not be achieved.

The scheme in part (b) for stabilizing the unstable plant is based on feedback. With the feedback approach, the poles of the system function are not cancelled with zeros. Instead, the poles are completely changed/relocated. For this reason, we can place the poles such that, even if the poles are displaced slightly (due to approximation error), the stability of the system will not be compromised. Therefore, this second scheme does not suffer from the same practical problem that the first one does. ■

7.30 Consider the system $\mathcal{H}$ with input Laplace transform X and output Laplace transform Y as shown in the figure. In the figure, each subsystem is LTI and causal and labelled with its system function, and a is a real constant. (a) Find the system function H of the system $\mathcal{H}$. (b) Determine whether the system $\mathcal{H}$ is BIBO stable.

systematic approach to obtaining system function:
1) label system input and system output
2) label each adder output
3) write equation for each adder output and system output
4) combine equations to obtain system function



Short Answer. (a) $H(s) = \frac{s^2+as+2}{s^2+3s+4}$ for $\operatorname{Re}(s) > -\frac{3}{2}$; (b) system is BIBO stable.

Answer (a,b).

From the system block diagram, we have:

$$Y(s) = V(s) + \left(\frac{a}{s}\right)V(s) + \left(\frac{2}{s^2}\right)V(s) \quad \text{and} \quad 1$$

$$V(s) = X(s) + \left(-\frac{3}{s}\right)V(s) + \left(-\frac{4}{s^2}\right)V(s). \quad 2$$

The preceding two equations can be rearranged to yield

$$3 \qquad Y(s) = \left(1 + \frac{a}{s} + \frac{2}{s^2}\right)V(s) \quad \text{and} \qquad \text{rearrange } 1$$

$$4 \qquad X(s) = \left(1 + \frac{3}{s} + \frac{4}{s^2}\right)V(s). \qquad \text{rearrange } 2$$

divide 3 by 4

Thus, $H(s)$ is given by

* $Y(s) = X(s)H(s) \Rightarrow$
$H(s) = \frac{Y(s)}{X(s)}$

$$H(s) \overset{*}{=} \frac{Y(s)}{X(s)} = \frac{1+a/s+2/s^2}{1+3/s+4/s^2} \underset{\text{rearrange}}{=} \frac{s^2+as+2}{s^2+3s+4}$$

Solving for the poles of $H(s)$, we obtain

$$\frac{-3 \pm \sqrt{9-4(1)(4)}}{2(1)} = -\tfrac{3}{2} \pm \tfrac{j\sqrt{7}}{2}.$$

Since the poles have negative real parts, the system is BIBO stable.

Example 7.42 (Unilateral Laplace transform of second-order derivative)**.** Find the unilateral Laplace transform Y of y in terms of the unilateral Laplace transform X of x, where

$$y(t) = x''(t)$$

and the prime symbol denotes derivative (e.g., x'' is the second derivative of x)

Solution. Define the function

$$v(t) = x'(t) \tag{7.17}$$

so that

$$y(t) = v'(t). \tag{7.18}$$

Let V denote the unilateral Laplace transform of v. Taking the unilateral Laplace transform of (7.17) (using the time-domain differentiation property), we have

$$\begin{aligned} V(s) &= \mathcal{L}_{\mathrm{u}}\{x'\}(s) \\ &= sX(s) - x(0^-). \end{aligned} \tag{7.19}$$

time-domain differentiation property

Taking the unilateral Laplace transform of (7.18) (using the time-domain differentiation property), we have

$$\begin{aligned} Y(s) &= \mathcal{L}_{\mathrm{u}}\{v'\}(s) \\ &= sV(s) - v(0^-). \end{aligned} \tag{7.20}$$

time-domain differentiation property

Substituting (7.19) into (7.20), we have

$$\begin{aligned} Y(s) &= s\left[sX(s) - x(0^-)\right] - v(0^-) \\ &= s^2X(s) - sx(0^-) - x'(0^-). \end{aligned}$$

substituting (7.19) into (7.20)

$v = x'$ and multiply

Thus, we have that

$$Y(s) = s^2X(s) - sx(0^-) - x'(0^-). \qquad \blacksquare$$

Example 7.43. Consider the causal incrementally-linear TI system with input x and output y characterized by the differential equation

$$y''(t) + 3y'(t) + 2y(t) = x(t),$$

where the prime symbol denotes derivative. If $x(t) = 5u(t)$, $y(0^-) = 1$, and $y'(0^-) = -1$, find y.

Solution. We begin by taking the unilateral Laplace transform of both sides of the given differential equation. This yields

$$\begin{aligned}
& \mathcal{L}_{\mathrm{u}}\left\{y'' + 3y' + 2y\right\}(s) = \mathcal{L}_{\mathrm{u}}x(s) && \text{linearity} \\
\Rightarrow\quad & \mathcal{L}_{\mathrm{u}}\left\{y''\right\}(s) + 3\mathcal{L}_{\mathrm{u}}\left\{y'\right\}(s) + 2\mathcal{L}_{\mathrm{u}}y(s) = \mathcal{L}_{\mathrm{u}}x(s) && \text{take ULT} \\
\Rightarrow\quad & \left[s^2Y(s) - sy(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2Y(s) = X(s) && \text{multiply} \\
\Rightarrow\quad & s^2Y(s) - sy(0^-) - y'(0^-) + 3sY(s) - 3y(0^-) + 2Y(s) = X(s) && \text{move terms not containing Y to right-hand side and factor out Y on left-hand side} \\
\Rightarrow\quad & \left[s^2 + 3s + 2\right]Y(s) = X(s) + sy(0^-) + y'(0^-) + 3y(0^-) && \text{divide both sides by } s^2+3s+2 \\
\Rightarrow\quad & Y(s) = \frac{X(s) + sy(0^-) + y'(0^-) + 3y(0^-)}{s^2 + 3s + 2}. \quad 1
\end{aligned}$$

Since $x(t) = 5u(t)$ 3, we have

$$\underset{\text{take ULT of 3}}{X(s)} = \mathcal{L}_{\mathrm{u}}\{5u(t)\}(s) \overset{\text{ULT table}}{=} \tfrac{5}{s}. \quad 2$$

Substituting this expression for X and the given initial conditions into the above equation yields

$$Y(s) = \frac{\left(\frac{5}{s}\right) + s - 1 + 3}{s^2 + 3s + 2} = \frac{s^2 + 2s + 5}{s(s+1)(s+2)}. \quad \text{substituting 2 into 1}$$

Now, we must find a partial fraction expansion of Y. Such an expansion is of the form

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{A_3}{s+2}.$$

Calculating the expansion coefficients, we obtain

$$\begin{aligned}
A_1 &= sY(s)\big|_{s=0} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{(s+1)(s+2)}\right|_{s=0} \\
&= \tfrac{5}{2}, \\
A_2 &= (s+1)Y(s)\big|_{s=-1} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{s(s+2)}\right|_{s=-1} \\
&= -4, \quad \text{and} \\
A_3 &= (s+2)Y(s)\big|_{s=-2} && \text{from formula for simple pole case} \\
&= \left.\frac{s^2 + 2s + 5}{s(s+1)}\right|_{s=-2} \\
&= \tfrac{5}{2}.
\end{aligned}$$

So, we can rewrite Y as

$$Y(s) = \frac{5/2}{s} - \frac{4}{s+1} + \frac{5/2}{s+2}.$$

recall:

$Y(s) = \frac{5}{2}\left(\frac{1}{s}\right) - 4\left(\frac{1}{s+1}\right) + \frac{5}{2}\left(\frac{1}{s+2}\right)$

Taking the inverse unilateral Laplace transform of Y yields

taking inverse ULT

$$
\begin{aligned}
y(t) &= \mathcal{L}_u^{-1}Y(t) \\
&= \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s}\right\}(t) - 4\mathcal{L}_u^{-1}\left\{\frac{1}{s+1}\right\}(t) + \tfrac{5}{2}\mathcal{L}_u^{-1}\left\{\frac{1}{s+2}\right\}(t) \\
&= \tfrac{5}{2} - 4e^{-t} + \tfrac{5}{2}e^{-2t} \quad \text{for } t \geq 0. \qquad \blacksquare
\end{aligned}
$$

linearity

from ULT table

$1 \overset{\text{ULT}}{\longleftrightarrow} \frac{1}{s}$; $e^{-at} \overset{\text{ULT}}{\longleftrightarrow} \frac{1}{s+a}$