Symmetric Extension for Quincunx Filter Banks

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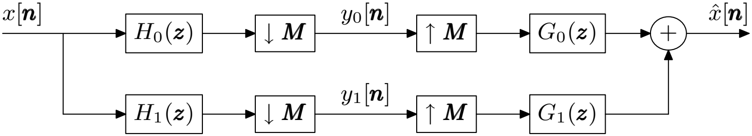
Outline

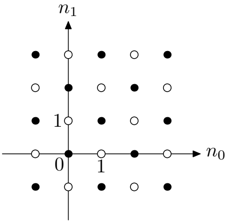
Two-channel filter banks



- Boundary problem
- Nonexpansive transform
- Symmetric extension in one-dimensional case

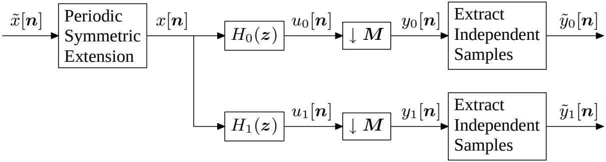
Quincunx filter banks

- Two-dimensional two-channel nonseparable filter banks
- Quincunx lattice



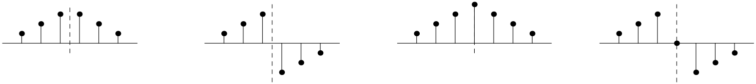
- Downsampling matrix $\boldsymbol{M} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- Coset vectors $\boldsymbol{k}_0 = [0 \quad 0]^T$ and $\boldsymbol{k}_1 = [1 \quad 0]^T$

- Structure of the analysis side with symmetric extension



Symmetries in 1-D and 2-D cases

- One-dimensional case
  - Half-sample and whole-sample
  - Symmetry and antisymmetry



- Two-dimensional case
  - Centrosymmetry (linear phase)

$$x[\boldsymbol{n}] = Sx[2\boldsymbol{c} - \boldsymbol{n}] \quad \text{for all } \boldsymbol{n} \in \mathbb{Z}^2. \tag{1}$$

  - Four-fold symmetries
    - ★ Quadrantal centrosymmetry
    - ★ Rotated quadrantal centrosymmetry

Quadrantal centrosymmetry

- Definition

$$
\begin{aligned}
x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\
&= Sx[2c_0 - n_0, n_1] \\
&= Tx[n_0, 2c_1 - n_1]
\end{aligned} \tag{2}
$$

- Four types of quadrantal centrosymmetry

even-even, $S = 1$, $T = 1$

d	b	b	d
c	a	a	c
c	a	a	c
d	b	b	d

odd-odd, $S = -1$, $T = -1$

-d	-b	b	d
-c	-a	a	c
c	a	-a	-c
d	b	-b	-d

even-odd, $S = 1$, $T = -1$

d	b	b	d
c	a	a	c
-c	-a	-a	-c
-d	-b	-b	-d

odd-even, $S = -1$, $T = 1$

d	b	-b	-d
c	a	-a	-c
c	a	-a	-c
d	b	-b	-d

Rotated quadrantal centrosymmetry

- Definition

$$\begin{aligned} x[n_0, n_1] &= STx[2c_0 - n_0, 2c_1 - n_1] \\ &= Sx[c_0 - c_1 + n_1, c_1 - c_0 + n_0] \\ &= Tx[c_0 + c_1 - n_1, c_0 + c_1 - n_0] \end{aligned} \quad (3)$$

- Examples

f	g	i	h	e
g	c	d	b	h
i	d	a	d	i
h	b	d	c	g
e	h	i	g	f

Symmetry center $\boldsymbol{c} \in \mathbb{Z}^2$

e	g	h	i	f	e
g	c	b	d	c	f
h	b	a	a	d	i
i	d	a	a	b	h
f	c	d	b	c	g
e	f	i	h	g	e

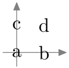
Symmetry center $\boldsymbol{c} \notin \mathbb{Z}^2$

Symmetric extension of sequences

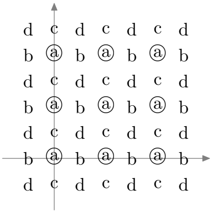
- Original sequence $\tilde{x}$: finite extent on the rectangular region $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$
- Rows and columns are extended separately to have half-sample or whole-sample symmetry and corresponding periodicity
- Extended 2-D sequence x: quadrantal centrosymmetric about $\boldsymbol{c}_x$ and $\boldsymbol{P}$-periodic
  - Type-1: $\boldsymbol{c}_x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\boldsymbol{P} = \begin{bmatrix} 2L_0-2 & 0 \\ 0 & 2L_1-2 \end{bmatrix}$
  - Type-2: $\boldsymbol{c}_x = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$, $\boldsymbol{P} = \begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1-2 \end{bmatrix}$
  - Type-3: $\boldsymbol{c}_x = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}^T$, $\boldsymbol{P} = \begin{bmatrix} 2L_0-2 & 0 \\ 0 & 2L_1 \end{bmatrix}$
  - Type-4: $\boldsymbol{c}_x = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T$, $\boldsymbol{P} = \begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1 \end{bmatrix}$
- Other symmetry centers: $\boldsymbol{P}\boldsymbol{k} + \boldsymbol{c}_x$ for $\boldsymbol{k} \in \frac{1}{2}\mathbb{Z}^2$

Symmetric extension examples

Input sequence $\tilde{x}$

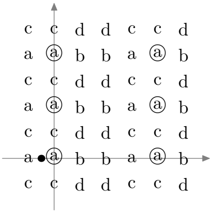


Type-1



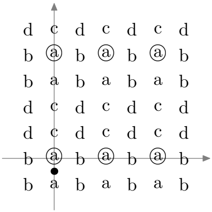
$$\boldsymbol{c}_x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \quad \boldsymbol{P} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Type-2



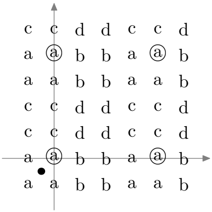
$$\boldsymbol{c}_x = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T \quad \boldsymbol{P} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$$

Type-3



$$\boldsymbol{c}_x = \begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}^T \quad \boldsymbol{P} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Type-4

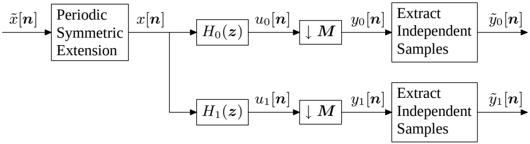


$$\boldsymbol{c}_x = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}^T \quad \boldsymbol{P} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Preservation of the properties

- Convolution: $y = x * h$
  - Symmetry
    x and h: centrosymmetric/quadrantally centrosymmetric
    $\Rightarrow y$: centrosymmetric/quadrantally centrosymmetric
  - Periodicity
    x: $\boldsymbol{P}$-periodic $\Rightarrow y$: $\boldsymbol{P}$-periodic
- Downsampling: $y = (\downarrow \boldsymbol{M})x$
  - Symmetry
    x: quadrantally centrosymmetric about $\boldsymbol{c}_x \in \mathbb{Z}^2$ and $\boldsymbol{M} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$
    $\Rightarrow y$: rotated quadrantally centrosymmetric about $\boldsymbol{M}^{-1}\boldsymbol{c}_x$
  - Periodicity
    x: $\boldsymbol{P}$-periodic and $\boldsymbol{M}^{-1}\boldsymbol{P}$: an integer matrix $\Rightarrow y$: $(\boldsymbol{M}^{-1}\boldsymbol{P})$-periodic.

Type-2 symmetric extension algorithm

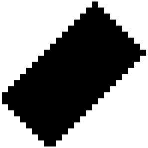


- Input sequence $\tilde{x}$: $\{0, 1, \ldots, L_0 - 1\} \times \{0, 1, \ldots, L_1 - 1\}$
- x: type-2 symmetric extension (quadrantally centrosymmetric about $\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix}^T$ and $\boldsymbol{P}$ -periodic with $\boldsymbol{P} = \begin{bmatrix} 2L_0 & 0 \\ 0 & 2L_1 - 2 \end{bmatrix}$)
- H_0: even-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_0 = [d_{0,0} \; d_{0,1}]^T$, $d_{0,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $d_{0,1} \in \mathbb{Z}$
- H_1: odd-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_1 = \begin{bmatrix} d_{1,0} & d_{1,1} \end{bmatrix}^T$, $d_{1,0} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ and $d_{1,1} \in \mathbb{Z}$;
- $\boldsymbol{d}_0 - \boldsymbol{d}_1 \in \mathrm{LAT}(\boldsymbol{M})$
- First channel
  - u_0: $\boldsymbol{P}$-periodic and quadrantally centrosymmetric about

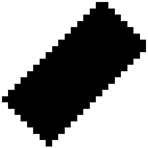
A simple example

- $H_0(z_0, z_1) = \frac{1}{2}(1 + z_0)$, $H_1(z_0, z_1) = 1 - z_0$, and $G_0(z_0, z_1) = 1 + z_0$, $G_1(z_0, z_1) = \frac{1}{2}(-1 + z_0)$
- Group delays of the analysis filters are both $\begin{bmatrix}-0.5 & 0\end{bmatrix}^T$
- Input $\tilde{x}$ is defined on $\{0, 1, \ldots, 31\} \times \{0, 1, \ldots, 15\}$ (512 samples)
- Subband sequences y_0 and y_1

264 samples in $\tilde{y}_0$



248 samples in $\tilde{y}_1$



Type-3 symmetric extension algorithm

- Similar to the type-2 algorithm
- x: type-3 symmetric extension of $\tilde{x}$
- Analysis filters satisfy
  - H_0: even-even quadrantal centrosymmetry with group delay $\boldsymbol{d}_0 = \begin{bmatrix} d_{0,0} & d_{0,1} \end{bmatrix}^T$, $d_{0,0} \in \mathbb{Z}$ and $d_{0,1} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
  - H_1: even-odd quadrantal centrosymmetry with group delay $\boldsymbol{d}_1 = \begin{bmatrix} d_{1,0} & d_{1,1} \end{bmatrix}^T$, $d_{1,0} \in \mathbb{Z}$ and $d_{1,1} \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$
  - $\boldsymbol{d}_0 - \boldsymbol{d}_1 \in \mathrm{LAT}(\boldsymbol{M})$
- Subbands y_0 and y_1
  - Periodic and rotated quatrantally centrosymmetric
  - Independent samples located in finite regions
  - $N_0 + N_1 = L_0 L_1$

Conclusion

- Different types of 2-D symmetries
- Four ways to extend a 2-D finite-extent sequence
- Preservations of symmetry and periodicity under convolution and downsampling
- Two types of symmetric extension algorithms for quincunx filter banks