ECE 515 Fall 2025 Assignment 1 Solutions

1. RVs X and Y (x, y)

$$p(0,0) = \frac{1}{2} \quad p(0,1) = \frac{1}{4} \quad p(1,0) = 0 \quad p(1,1) = \frac{1}{4}$$

$$p(X=0) = \frac{3}{4} \quad p(X=1) = \frac{1}{4}$$

$$p(y=0) = \frac{1}{2} \quad p(y=1) = \frac{1}{2}$$

a) $H(X) = -\frac{3}{4}\log\frac{3}{4} - \frac{1}{4}\log\frac{1}{4} = .811$ bit

b) $H(Y) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$ bit

c) $$H(X|Y) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(y_j)}$$

$$= -\frac{1}{2}\log\frac{1/2}{1/2} - \frac{1}{4}\log\frac{\frac{1}{4}}{1/2} - \frac{1}{4}\log\frac{1/4}{1/2}$$

$$= 0 - \frac{1}{2}\log\frac{1}{4} = 0.5 \text{ bit}$$

d) $$H(Y|X) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j) \frac{p(x_i, y_j)}{p(x_i)}$$

$$= -\frac{1}{2}\log\frac{1/2}{3/4} - \frac{1}{4}\log\frac{1/4}{3/4} - \frac{1}{4}\log\frac{1/4}{1/4}$$

$$H(Y|X) = -\frac{1}{2}\log\frac{2}{3} - \frac{1}{4}\log 1/3 - \frac{1}{4}\log 1$$

$$= .292 + .396 = .689 \text{ bit}$$

e) $H(XY) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

$$= .811 + .689 = 1.5 \text{ bits}$$

f) $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$

$$= .811 - .5 = .311 \text{ bit}$$

2 $\quad 2^n+1$ symbols x_i

a) $p(x_1) = 0$

lower bound - let $p(x_i) = 1$ for some $i \neq 1$

then $H(X) = -1 \log 1 = 0$ bit

upper bound - let $p(x_i) = \frac{1}{2^n}$ for $i \neq 1$

the $H(X) = -\sum_{i=2}^{2^n+1} \frac{1}{2^n} \log \frac{1}{2^n} = \log 2^n = n$ bits

b) $p(x_1) = \frac{1}{2}$

lower bound - let $p(x_i) = \frac{1}{2}$ for some $i \neq 1$

then $H(X) = -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = \log 2 = 1$ bit

upper bound - let $p(x_i) = \frac{1}{2^{n+1}}$ for $i \neq 1$

then

$$H(X) = -\frac{1}{2} \log \frac{1}{2} - \sum_{i=2}^{2^n+1} \frac{1}{2^{n+1}} \log \frac{1}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{1}{2} \log 2^{n+1} = \frac{1}{2} + \frac{n+1}{2} = \frac{n}{2} + 1 \text{ bits}$$

3 jar with 5 black balls and 10 white balls

a) Experiment X is randomly drawing a ball

$p_X(\text{black}) = \frac{1}{3}$ $p_X(\text{white}) = 2/3$

$H(X) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = 0.918$ bit

b) Experiment Y, the probabilities if the outcome of X is black are

$p(\text{black}|\text{black}) = \frac{4}{14}$ $p(\text{white}|\text{black}) = \frac{10}{14}$

$H(Y|\text{black}) = -\frac{2}{7}\log\frac{2}{7} - \frac{5}{7}\log\frac{5}{7} = 0.863$ bit

c) Experiment Y, the probabilities if the outcome of X is white are

$p(\text{black}|\text{white}) = \frac{5}{14}$ $p(\text{white}|\text{white}) = \frac{9}{14}$

$H(Y|\text{white}) = -\frac{5}{14}\log\frac{5}{14} - \frac{9}{14}\log\frac{9}{14} = 0.940$ bit

d) $H(Y|X) = p_X(\text{black})H(Y|\text{black}) + p_X(\text{white})H(Y|\text{white})$

$= \frac{1}{3}(.863) + \frac{2}{3}(.940) = 0.914$ bit

4.1(a) $H(Y) = H(X) = 8$ bits

(b) $H(Y|X) = 0$ bits since X provides complete information about Y

(c) $H(X|Y) = 0$

(d) $I(X;Y) = H(X) - H(X|Y) = 8$ bits

(e) If $Y(X)$ is not invertible, different values of X may correspond to the same values of Y

therefore $H(Y) < H(X)$

Since knowledge of Y no longer determines X completely

$8 \geq H(X|Y) > 0$

5 Stanley Cup Final teams A and B

X is the outcome RV

Y is the number of games RV

POSSIBLE OUTCOMES

AAAA	ABAAA	ABBAAA	BBBAAAA
BBBB	BABBB	BAABBB	AAABBBB
	⋮	⋮	⋮

$Y = 4$ 2 outcomes $\quad p(Y=4) = \left(\frac{1}{2}\right)^4 \cdot 2 = .125$

$Y = 5$ 8 outcomes $\quad p(Y=5) = \left(\frac{1}{2}\right)^5 \cdot 8 = .25$

$Y = 6$ 20 outcomes $\quad p(Y=6) = \left(\frac{1}{2}\right)^6 \cdot 20 = .3125$

$Y = 7$ 40 outcomes $\quad p(Y=7) = \left(\frac{1}{2}\right)^7 \cdot 40 = .3125$

(b) $H(Y) = -.125 \log .125 - .25 \log .25 - .625 \log .3125$

$= .375 + .5 + 1.049 = 1.924$ bits

$p(X=4) = \left(\frac{1}{2}\right)^4 = .0625$

$p(X=5) = \left(\frac{1}{2}\right)^5 = .03125$

$p(X=6) = \left(\frac{1}{2}\right)^6 = .015625$

$p(X=7) = \left(\frac{1}{2}\right)^7 = .0078125$

(a) $H(X) = -.125 \log .0625 - .25 \log .03125$

$- .3125 \log .015625 - .3125 \log .0078125$

$= .5 + 1.25 + 1.875 + 2.1875 = 5.8125$ bits

(c) $H(Y|X) = 0$ since knowing the outcome reveals how many games have been played

(d) $H(XY) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

$\therefore H(X|Y) = H(X) - H(Y)$

$= 3.889$ bits

6. if a person is blond, there are two possible eye colors

blue with probability $\frac{3}{4}$
and not blue with probability $\frac{1}{4}$

(a) $H(\text{eye color} \mid \text{blond}) = -\frac{1}{4}\log\frac{1}{4} - \frac{3}{4}\log\frac{3}{4} = .811$ bit

(b) this requires $p(\text{blond} \mid \text{blue})$ and $p(\text{not blond} \mid \text{blue})$

$$p(\text{blond} \mid \text{blue}) = \frac{p(\text{blond})\,p(\text{blue} \mid \text{blond})}{p(\text{blue})} = \frac{\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{2}} = \frac{3}{8}$$

because a person is either blond or not blond

$$p(\text{not blond} \mid \text{blue}) = 1 - \frac{3}{8} = \frac{5}{8}$$

$$H(\text{hair color} \mid \text{blue}) = -\frac{3}{8}\log\frac{3}{8} - \frac{5}{8}\log\frac{5}{8} = .954 \text{ bit}$$

(c) this requires the four probabilities

$p(\text{blond, blue}) = \frac{3}{16}$ $\quad p(\text{blond, not blue}) = \frac{1}{16}$
$p(\text{not blond, blue}) = \frac{5}{16}$ $\quad p(\text{not blond, not blue}) = \frac{7}{16}$

$$H(\text{hair color, eye color}) = -\frac{3}{16}\log\frac{3}{16} - \frac{1}{16}\log\frac{1}{16} - \frac{5}{16}\log\frac{5}{16} - \frac{7}{16}\log\frac{7}{16}$$
$$= 1.749 \text{ bits}$$

7. (a) $H(\text{colour}) = -\frac{1}{19}\log\frac{1}{19} - \frac{9}{19}\log\frac{9}{19} - \frac{9}{19}\log\frac{9}{19}$

$= 1.24$ bits

(b) $H(\text{colour, number}) = H(\text{number}) = \log 38$

$= 5.25$ bits

(c) $H(\text{number} \mid \text{colour}) = H(\text{colour, number}) - H(\text{colour})$

$= 5.25 - 1.24 = 4.01$ bits