ECE 515 FALL 2025 MIDTERM TEST SOLUTIONS

1. $p(0,0) = 1/3 \quad p(0,1) = 1/3 \quad p(1,0) = 0 \quad p(1,1) = 1/3$

$p(x=0) = p(0,0) + p(0,1) = 2/3$

$p(x=1) = p(1,0) + p(1,1) = 1/3$

$p(y=0) = p(0,0) + p(1,0) = 1/3$

$p(y=1) = p(0,1) + p(1,1) = 2/3$

a) $H(X) = -\frac{2}{3}\log\frac{2}{3} - \frac{1}{3}\log\frac{1}{3} = .528 + .390 = .918$ bit

b) $H(Y) = -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3} = .918$ bit

c) $H(X|Y) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j)\log\frac{p(x_i, y_j)}{p(y_j)}$

$$= -1/3\log\frac{1/3}{1/3} - 1/3\log\frac{1/3}{2/3} - 0 - 1/3\log\frac{1/3}{2/3}$$

$$= -2/3\log 1/2 = 2/3 \text{ bit}$$

d $H(XY) = H(Y) + H(X|Y)$

$$= .918 + .667 = 1.585 \text{ bits}$$

e) $I(X;Y) = H(X) - H(X|Y) = .918 - .667 = .251$ bit

f) $H(Y|X) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i)}$

$= -\frac{1}{3}\log\frac{1/3}{2/3} - \frac{1}{3}\log\frac{1/3}{1/3} - 0 - \frac{1}{3}\log\frac{1/3}{2/3}$

$= -\frac{2}{3}\log\frac{1}{2} = \frac{2}{3}$ bit

2) source X with $N = 8$ symbols

the smallest entropy occurs when X is deterministic

$p(x_i) = 1$ for some i

$H(X) = 0$ bit

the largest entropy occurs when the symbols are equiprobable

$p(x_i) = \frac{1}{8}$ for all i

$H(X) = \log_2 8 = 3$ bits

3. $H(X) = H(X|Y) + I(X;Y)$

$H(Y) = H(Y|X) + I(X;Y)$

$H(XY) = H(X|Y) + H(Y|X) + I(X;Y)$

$\therefore \quad H(X) + H(Y) - H(XY) = I(X;Y) \geq 0$

Equality holds when $I(X;Y) = 0$

this occurs when X and Y are statistically independent

4. the relative entropy is minimized when

$q(X) = p(X)$

therefore $\quad q(x_1) = 1/2 \quad q(x_2) = 1/4$

$q(x_3) = q(x_4) = 1/8$