ECE 515 FALL 2025 ASSIGNMENT 3 SOLUTIONS

1. $p(x_1)=.5$ $p(x_2)=.2$ $p(x_3)=.2$ $p(x_4)=.1$

$q(x_1)=a$ $q(x_2)=2a$ $q(x_3)=b$ $q(x_4)=2b$

$$H(p,q)=-\sum_{i=1}^{4} p(x_i)\log q(x_i)$$

$$=-.5\log a-.2\log 2a-.2\log b-.1\log 2b$$

$$\sum_{i=1}^{4} q(x_i)=3a+3b=1$$

$$\therefore b=1/3-a$$

$$H(p,q)=-.5\log(a)-.2\log(2a)-.2\log(1/3-a)$$
$$-.1\log[2(1/3-a)]$$

$$=-.5\log a-.2\log 2-.2\log a-.2\log(1/3-a)$$
$$-.1\log 2-.1\log(1/3-a)$$

$$=-.7\log a-.3\log(1/3-a)-0.3$$

to minimize the cross entropy, set

$$\frac{d\,H(p,q)}{da} = 0$$

$$\frac{d\,H(p,q)}{da} = -.7\left(\frac{1}{a\ln 2}\right) - .3\left(\frac{1}{(1/3-a)\ln 2}\right)(-1)$$

$$-\frac{.7}{a} + \frac{.3}{(1/3-a)} = 0$$

$$-.7(1/3-a) + .3a = 0$$

$$-\frac{.7}{3} = -a \Rightarrow \quad a = .7/3 = .233$$

$$2a = .467 \quad b = 1/3 - a = .1 \qquad 2b = .2$$

$$H(p,q) = .5\log .233 - .2\log .467 - .2\log .1 + .1\log .2$$

$$= 1.050 + .220 + .664 + .232 = 2.166 \text{ bits}$$

$$H(p(x)) = -.5\log .5 - .2\log .2 - .2\log .2 - .1\log .1$$

$$= .5 + .929 + .332 = 1.761 \text{ bits}$$

$$D[p(x)\|q(x)] = \sum_{i=1}^{4} p(x_i)\log\left[\frac{p(x_i)}{q(x_i)}\right]$$

$$= .5\log 2.143 + .2\log .429 + .2\log 2 + .1\log 0.5$$

$$= .550 - .244 + .2 - .1$$

$$= .405 \text{ bit}$$

$$H(p(x)) + D[p(x)\|q(x)]$$

$$= 1.761 + .405 = 2.166 \text{ bits}$$

$$= H(p,q)$$

2. Prove that $H(p,q) \geq 0$

$$H(p,q) = -\sum_{i=1}^{N} p(x_i) \log q(x_i)$$

we have that $0 \leq p(x_i) \leq 1$, $0 \leq q(x_i) \leq 1$

$\therefore$ $\log q(x_i) \leq 0$ and $-\log q(x_i) \geq 0$

so $-p(x_i) \log q(x_i) \geq 0$

and therefore $-\sum_{i=1}^{N} p(x_i) \log q(x_i) \geq 0$

since every term in $-H(p,q)$ is ≥ 0,

to achieve equality every term

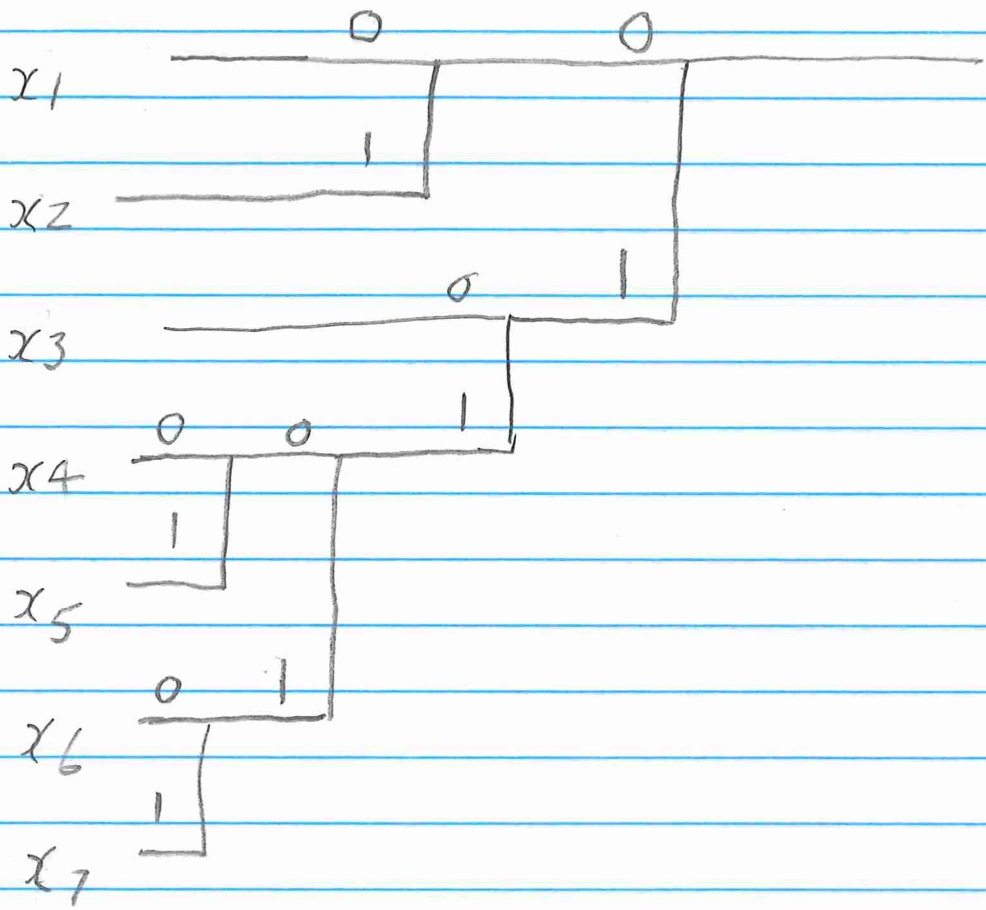
must be 0

this occurs when

$p(x_i) = q(x_i) = 1$ for some i

3. $p_1 > p_2 > p_3 > p_4 > p_5 > p_6 \quad p_7$

$$p_3 > p_4 + p_5 + p_6 + p_7$$



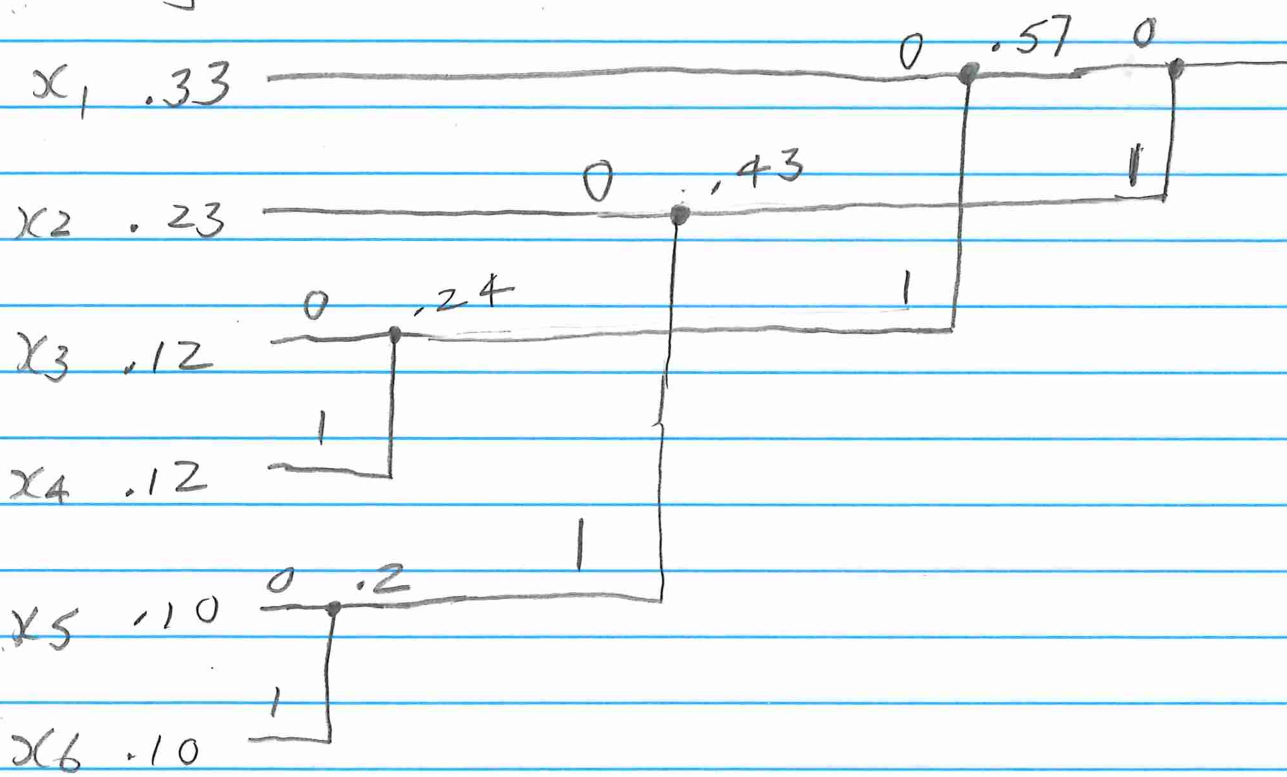
x_1	00
x_2	01
x_3	10
x_4	1100
x_5	1101
x_6	1110
x_7	1111

4. Course Notes Problem 2.3

a) $p(x_1) = .33$ $p(x_2) = .23$ $p(x_3) = .12$

$p(x_4) = .12$ $p(x_5) = .10$ $p(x_6) = .10$

binary code



x_1 00
x_2 10
x_3 010
x_4 011
x_5 110
x_6 111

ternary code

x_1 .33

x_2 .23

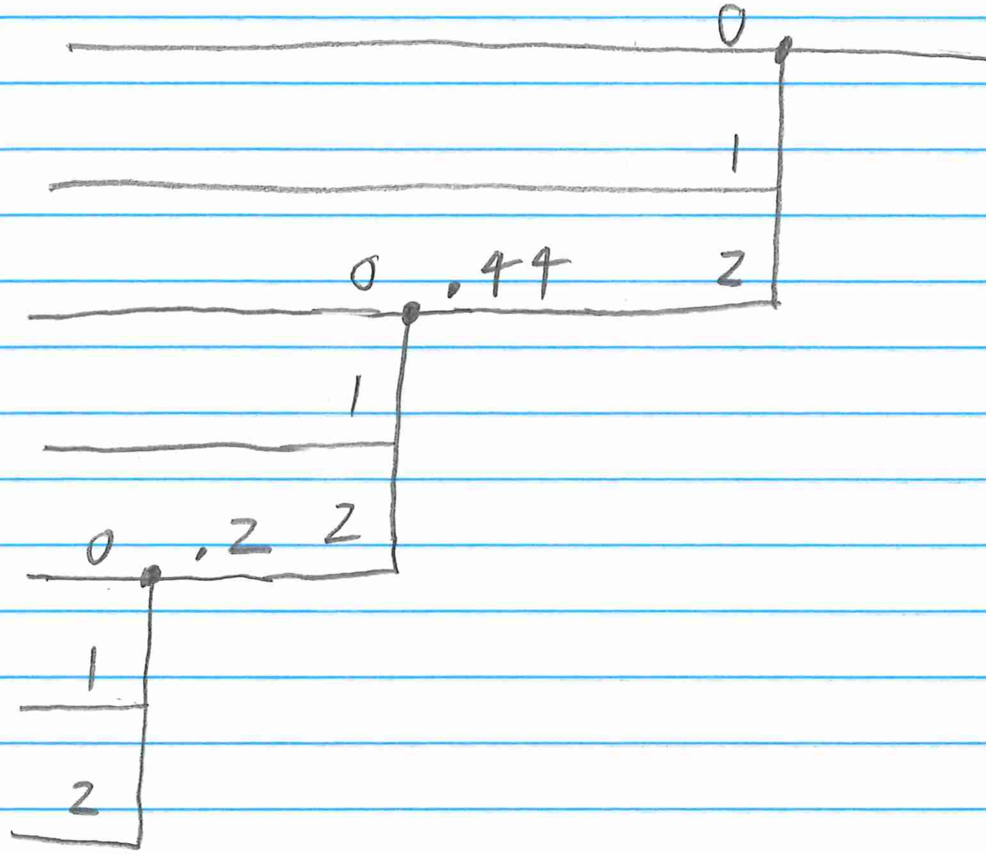
x_3 .12

x_4 .12

x_5 .10

x_6 .10

x_7 0



add a dummy symbol

x_1 0
x_2 1
x_3 20
x_4 21
x_5 220
x_6 221

$$L(C)_{binary} = 2(.56) + 3(.44) = 2.44 \text{ bits/symbol}$$

$$L(C)_{ternary} = 1(.56) + 2(.24) + 3(.20) = 1.64 \text{ trits/symbol}$$

$$H(X) = -.33\log .33 - .23\log .23 - .24\log .12 + .2\log .1$$

$$= .528 + .488 + .734 + .664$$

$$= 2.414 \text{ bits}$$

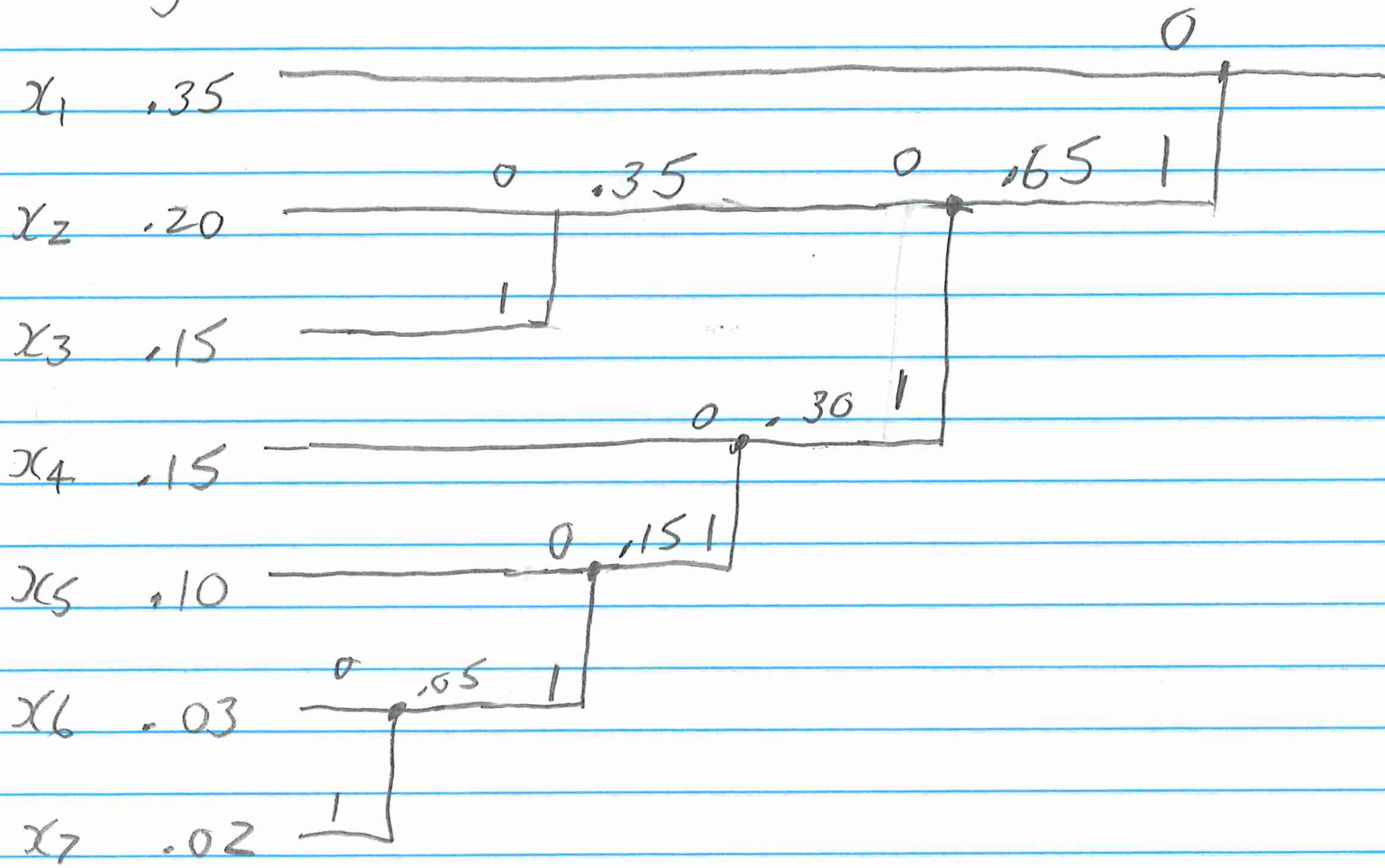
$$\eta_{binary} = \frac{H(X)}{L(C)} = 98.9\%$$

$$\eta_{ternary} = \frac{H(X)}{L(C)\log_2 3} = 92.9\%$$

b) $p(x_1) = .35$ $p(x_2) = .20$ $p(x_3) = .15$ $p(x_4) = .15$

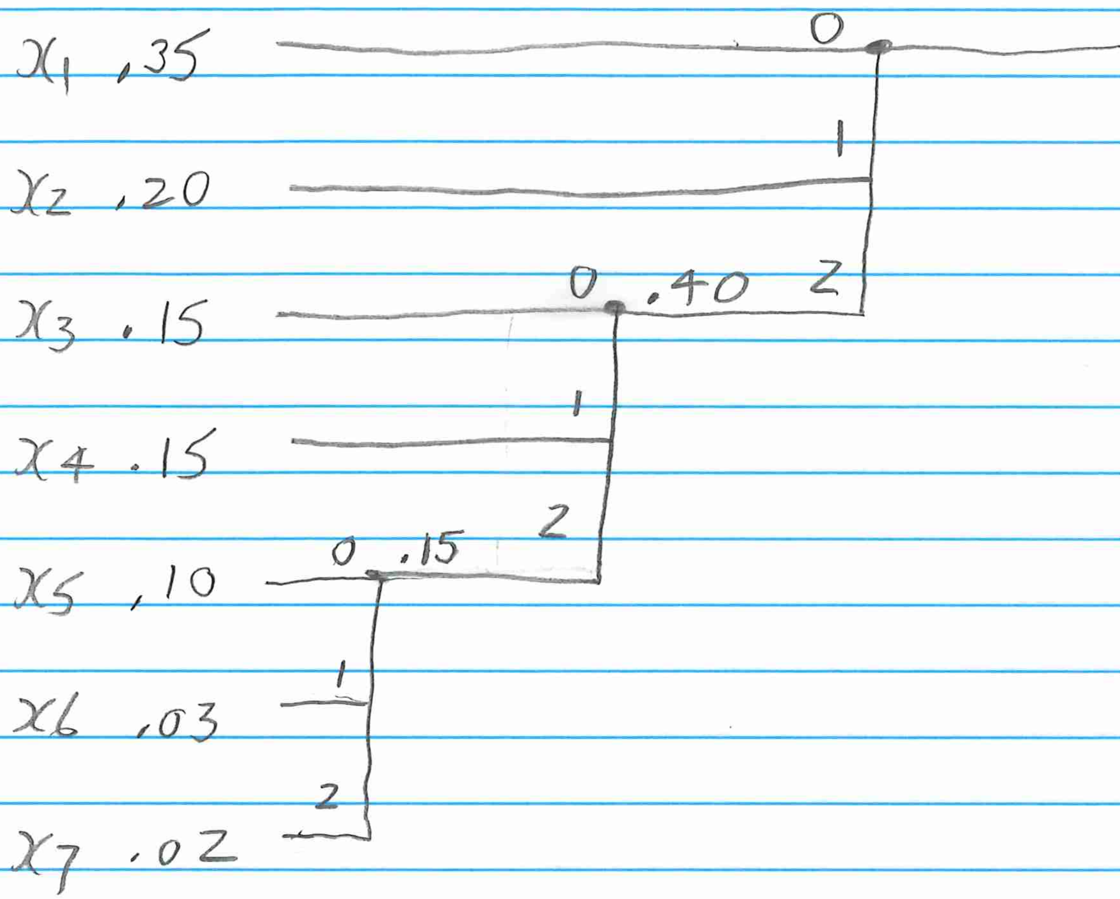
$p(x_5) = .10$ $p(x_6) = .03$ $p(x_7) = .02$

binary code 1



x_1	0
x_2	1 0 0
x_3	1 0 1
x_4	1 1 0
x_5	1 1 1 0
x_6	1 1 1 1 0
x_7	1 1 1 1 1

ternary code



x_1 0
x_2 1
x_3 20
x_4 21
x_5 220
x_6 221
x_7 222

$$L(C)_{binary} = .35 + 3(.5) + 4(.10) + 5(.05)$$
$$= 2.5 \text{ bits/symbol}$$

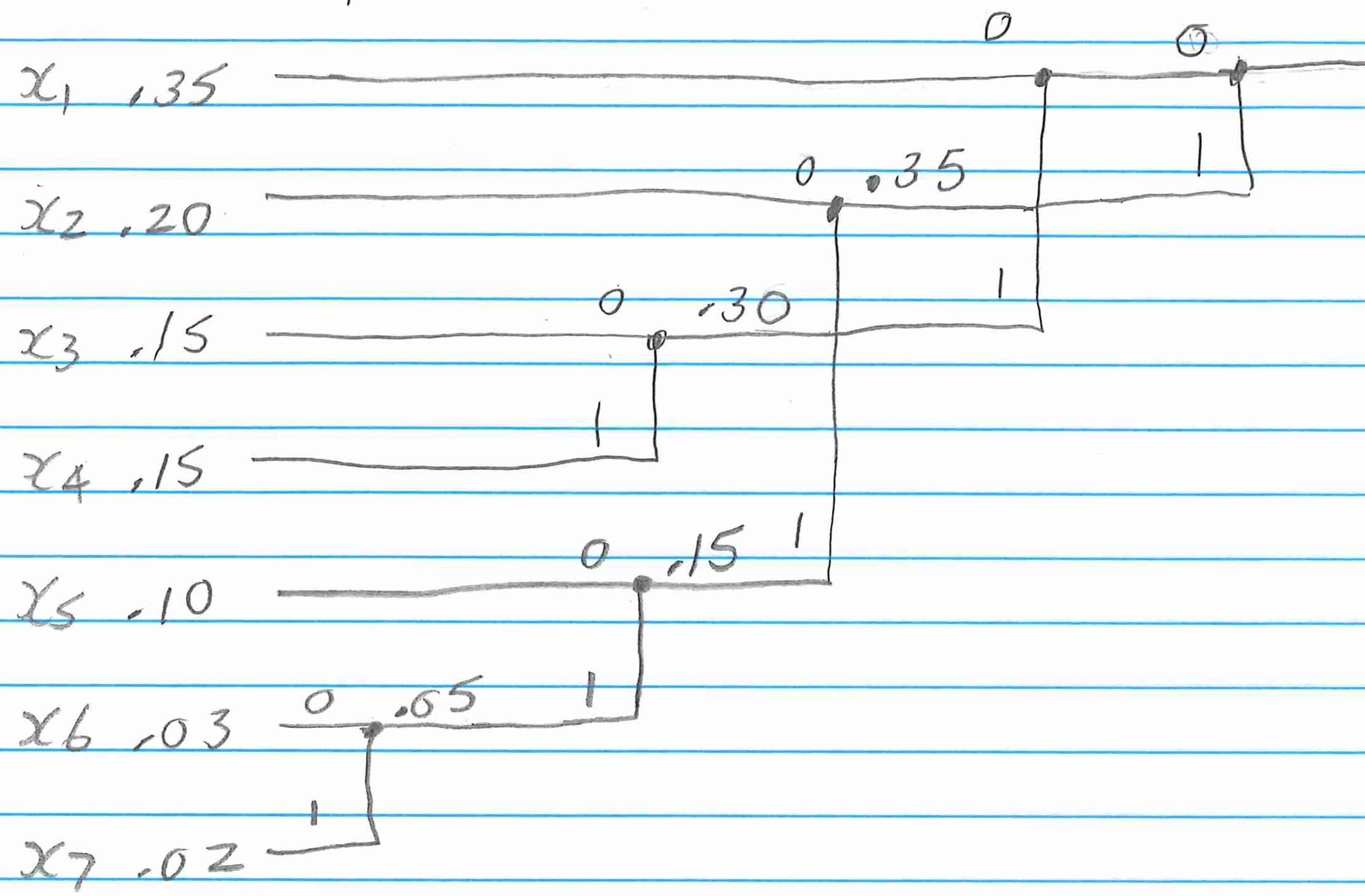
$$L(C)_{ternary} = .55 + 2(.30) + 3(.15)$$
$$= 1.6 \text{ trits/symbol}$$

$$H(X) = .35 \log .35 + .2 \log .2 + .3 \log .15 + .1 \log .1$$
$$+ .03 \log .03 + .02 \log .02$$
$$= .530 + .464 + .821 + .332 + .152 + .113$$
$$= 2.412 \text{ bits}$$

$$\eta_{binary} = \frac{H(X)}{L(C)} = 96.5\%$$

$$\eta_{ternary} = \frac{H(X)}{L(C) \log_2 3} = 95.1\%$$

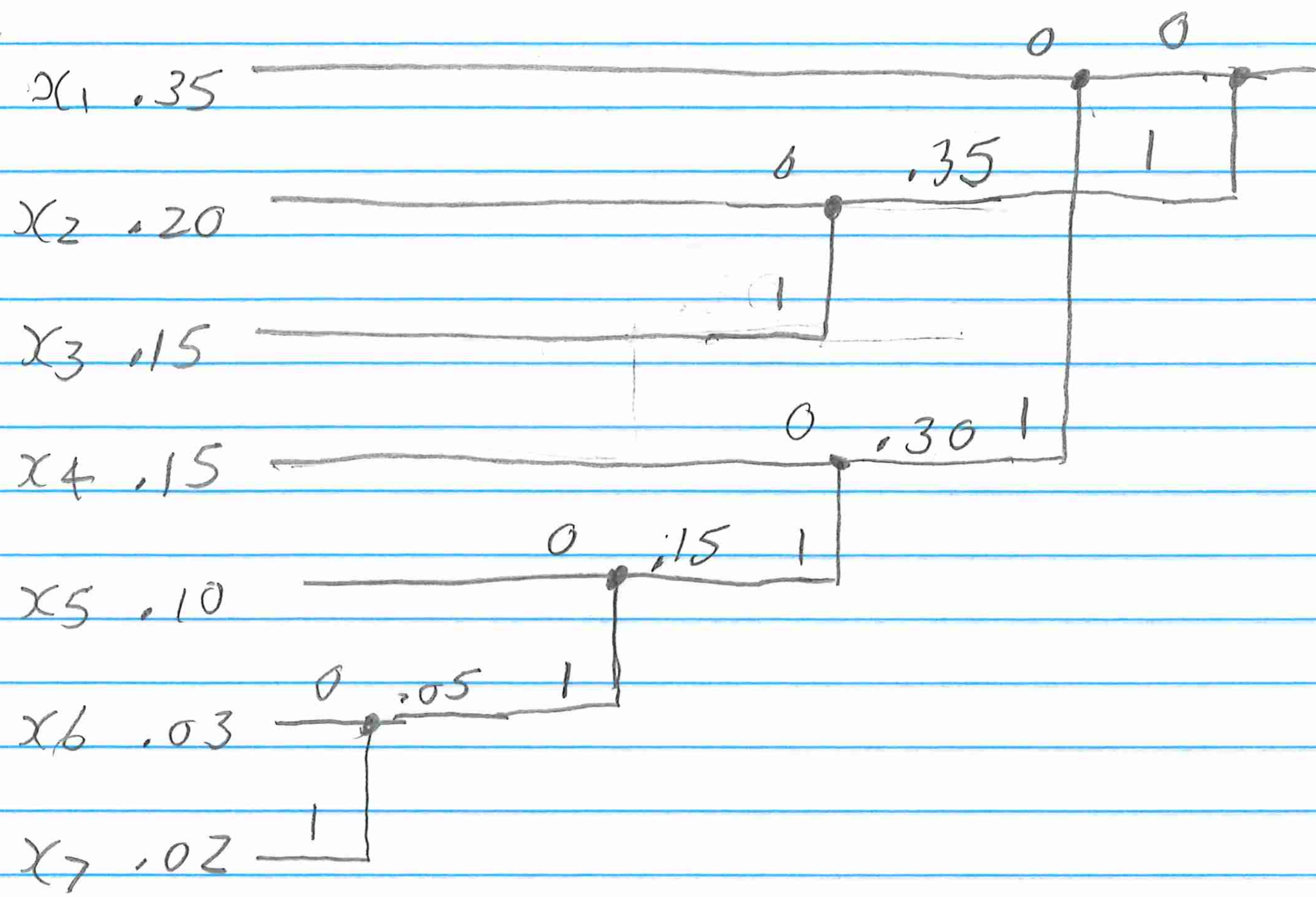
(c) other possibilities: code 2



x_1	00
x_2	10
x_3	010
x_4	011
x_5	110
x_6	1110
x_7	1111

$L(C) = 2(.55) + 3(.40) + 4(.05) = 2.5$ bits/symbol

code 3



x_1 00
x_2 10
x_3 11
x_4 010
x_5 0110
x_6 01110
x_7 01111

$$L(C) = 2(.7) + 3(.15) + 4(.1) + 5(.05)$$

$$= 2.5 \text{ bits/symbol}$$

code 1 has lengths 1,3,3,3,4,5,5

code 2 has lengths 2,2,3,3,3 4,4

code 3 has lengths 2,2,2,3,4,5,5

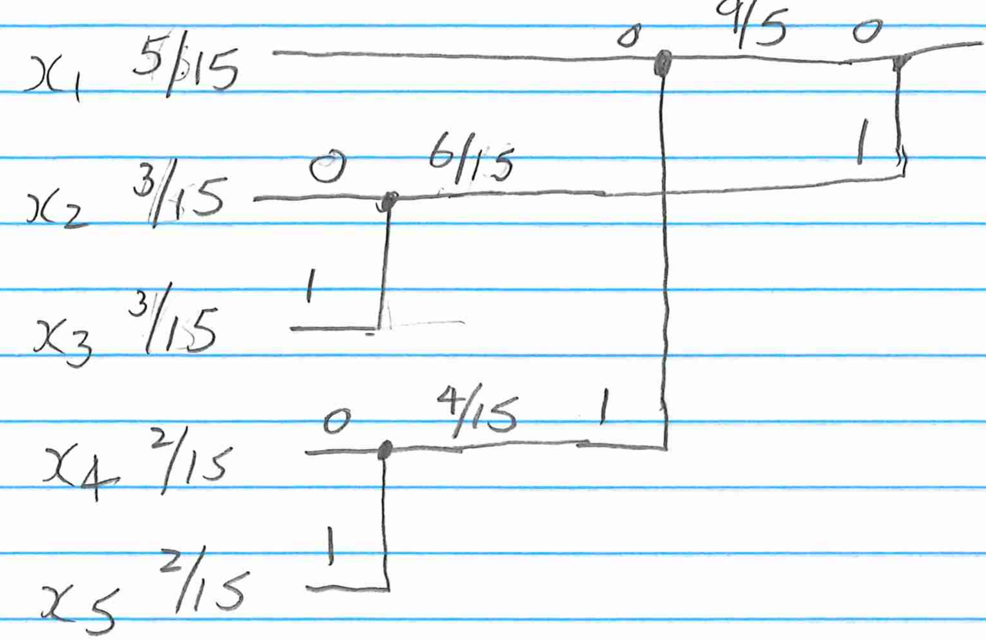
- code 2 has no codewords of length 1 or 5
  thus it has the smallest variance in length

- code 3 has no code words of length 1
  thus it has a smaller variance than code 1

- a smaller variance is preferable as the code will be more robust to errors in the probabilities

- thus the order of preference is:
  code 2, code 3, code 1

5(a) source Y probabilities $(1/3, 1/5, 1/5, 2/15, 2/15)$



x_1 00
x_2 10
x_3 11
x_4 010
x_5 011

(b) probabilities $(1/5, 1/5, 1/5, 1/5, 1/5)$

$H(X) = \log_2 5 = 2.322$ bits

the codeword lengths are l_1, l_2, l_3, l_4, l_5

the average codeword length is

$$L(C) = \sum_{i=1}^{5} \frac{l_i}{5} = \frac{n}{5} \text{ for some integer } n$$

and $L(C) \geq H(X) = 2.322$

$\therefore n > 11.6$

the smallest possible value is $n = 12$

the code in part (a) has lengths

2, 2, 2, 3, 3

so the sum of the lengths is 12

which meets the lower bound for n

6. Determine which of the following codes is uniquely decodable and which are prefix codes

(a) $\{1, 00, 01\}$ prefix code so UD

(b) $\{00, 1, 10\}$ UD but not prefix

(c) $\{1, 10, 01\}$ not UD so also not prefix

101 could be 1 01 or 10 1

(d) $\{1, 10, 1000\}$ UD but not prefix

(e) $\{00, 01, 10, 001\}$ NOT UD so also not prefix

001001 could be 001 001 or 00 10 01

7. $L(C) < H(X) + 1$

show that this result cannot be improved upon

consider a binary source with

$p(x_1) = 1 - \delta \qquad p(x_2) = \delta$

for the code to be uniquely decodable both codewords must have length at least 1 so

$$L(C) \geq 1$$

as $\delta \to 0 \quad H(X) \to 0$

so $L(C) \geq H(X) + 1 - \epsilon$ with $\epsilon \to 0$