ECE 515 Fall 2025 Assignment 5 Solutions

1. Lempel Ziv algorithm

01101110011010101111000111

position	S_n		Numerical Representation	Binary Codeword
1	S_1	0		
2	S_2	1		
3	S_3	01	1 2	0001 1
4	S_4	10	2 1	0010 0
5	S_5	11	2 2	0010 1
6	S_6	100	4 1	0100 0
7	S_7	110	5 1	0101 0
8	S_8	101	4 2	0100 1
9	S_9	011	3 2	0011 1
10	S_{10}	1100	7 1	0111 0
11	S_{11}	0111	9 2	1001 1

Note that a 4 bit root is needed

$$H(x) = -0.7\log 0.7 - 0.2\log 0.2 - 0.1\log 0.1$$

$$= .3602 + .4644 + .3322$$

$$= 1.157 \text{ bits}$$

for 8 symbols the expected length is

$$L = 9.25 \text{ bits}$$

the actual code word is longer because the low probability symbols occur more frequently than expected.

2. Lempel-Ziv decoding

		S_1 0
		S_2 1
00011	S_1 1	S_3 01
00010	S_1 0	S_4 00
00100	S_2 0	S_5 10
01010	S_5 0	S_6 100
01001	S_4 1	S_7 001
01000	S_4 0	S_8 000
00101	S_2 1	S_9 11
00110	S_3 0	S_{10} 010
00111	S_3 1	S_{11} 011
01100	S_6 0	S_{12} 1000
10000	S_8 0	S_{13} 0000
11001	S_{12} 1	S_{14} 10001

The decoded sequence is

010010100001000110100111000000010001

36 bits   12 1s   24 0s

Thus it can be concluded that

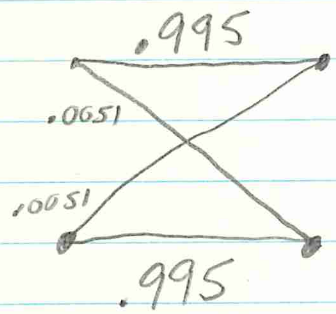
$p(0) > p(1)$ or $p(0) \approx 2/3$ $p(1) \approx 1/3$

3. DMC with channel transition probability matrix

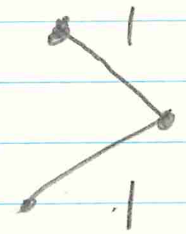
$$P = \begin{bmatrix} .985 & .005 \\ .010 & .010 \\ .005 & .985 \end{bmatrix}$$

a) this defines a weakly symmetric channel composed of the following strongly symmetric channels

$q_1 = .99$



$q_2 = .01$



the capacity is given by

$$C_1 = .995 \log .995 + .0051 \log .0051 + \log 2$$

$$= -.007268 - .03853 + 1 = .9542$$

$$C_2 = 0$$

$$C = .99 C_1 + .01 C_2 = .9447$$

b) if there are no erasures, the channel becomes a BSC with

$$P = \begin{bmatrix} .99 & .0.1 \\ .01 & .99 \end{bmatrix}$$

$$C = .99 \log .99 + .01 \log .01 + \log 2$$

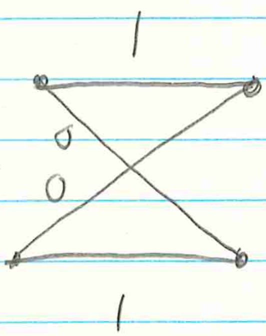
$$= -.01435 - .06644 + 1 = .9192$$

c)

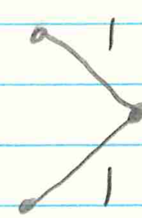
$$P = \begin{bmatrix} .985 & 0 \\ .015 & .015 \\ 0 & .985 \end{bmatrix}$$

this is a weakly symmetric channel (BEC)

$q_1 = .985$



$q_2 = .015$



$$C_1 = \log 1 + \log 2 = 1$$

$$C_2 = 0$$

$$C = .985(1) + .01(0) = .985$$

d) the BEC channel provides the highest capacity of .985 because there are no errors

the errors and erasures channel provides the next best capacity of .9447

the BSC provides the lowest capacity of .9192

CONCLUSION: the use of erasures can improve the performance

4. The transition probability matrix is

$$P = \begin{bmatrix} 1-p & 0 & 0 & p \\ p & 1-p & 0 & 0 \\ 0 & p & 1-p & 0 \\ 0 & 0 & p & 1-p \end{bmatrix}$$

a) the mutual information is

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= H(Y) - \sum_x H(Y|X=x)\,p(x)$$

$$H(Y|X=x) = -(1-p)\log(1-p) - p\log p = h(p)$$

$$I(X|Y) = H(Y) - h(p) \leq \log_2 4 - h(p) = 2 - h(p)$$

$\therefore$ $I(X|Y)$ is a maximum when

Y is equiprobable $p(y_j) = \frac{1}{4}$

As this is a strongly symmetric channel

this occurs when X is equiprobable $p(x_i) = \frac{1}{4}$

the capacity is then $C = 2 - h(p)$

b) if $p = \frac{1}{2}$, $C = 2 - h(p) = 1$ bit

if $p(X=0) = \frac{1}{2}$ and $p(X=2) = \frac{1}{2}$

there is no ambiguity at the output about the input

$\therefore C = H(X) = 1$ bit as required

5. $I(X;Y) = H(Y) - H(Y|X)$

a) $$H(Y|X) = \sum_{i=0}^{L-1} p(x_i) \sum_{j=0}^{L-1} p(y_j|x_i) \log \frac{1}{p(y_j|x_i)}$$

Now $p(y_j|x_i) = p(z_{(j-i) \bmod L})$

thus $$\sum_{j=0}^{L-1} p(y_j|x_i) \log \frac{1}{p(y_j|x_i)} = \sum_{k=0}^{L-1} p(z_k) \log \frac{1}{p(z_k)}$$

since X and Z are independent

$$H(Y|X) = \sum_{i=0}^{L-1} p(x_i) \sum_{k=0}^{L-1} p(z_k) \log \frac{1}{p(z_k)}$$

$$= \sum_{k=0}^{L-1} p(z_k) \log \frac{1}{p(z_k)} = H(Z)$$

$\therefore I(X;Y) = H(Y) - H(Z)$

b) the capacity is achieved by maximizing $H(Y)$

since Z is noise

because of the symmetry of the noise with respect to the inputs, capacity occurs with equiprobable inputs

Thus $C = \log L - H(Z)$

6. ternary code $N=11$ $M=729$

can correct 2 errors or less

for a ternary symmetric channel with probability of error p

$$P_e = \sum_{i=3}^{11} \binom{11}{i} 2^i \left(\frac{p}{2}\right)^i (1-p)^{11-i}$$

this is the probability of 3 or more errors

the probability of correct decoding is

$$P_c = 1 - P_e = \sum_{i=0}^{2} \binom{11}{i} 2^i \left(\frac{p}{2}\right)^i (1-p)^{11-i}$$

$$= (1-p)^{11} + 11\,p(1-p)^{10} + 55\,p^2(1-p)^9$$

(a) $p=.1$ $.0896$

(b) $p=.01$ 1.55×10^{-4}

(c) $p=.001$ 1.64×10^{-7}

(d) $p=.0001$ 1.65×10^{-10}

7. Binary code of length $N=7$

(a) there are 7 sequences a distance 1 from a given codeword and 1 sequence a distance 0 from the codeword

thus there are $7+1=8$ sequences a distance less than or equal to 1 from c_m

(b) the maximum number of sequences (vectors) of length 7 is

$$2^7=128$$

to be able to correct single errors, the 8 sequences in (a) associated with each codeword must be disjoint from the sequences associated with the other codewords

thus the maximum possible number of codewords is

$$M=\left\lfloor \frac{128}{8} \right\rfloor = 16$$

(d) $R = \frac{\log_2 M}{N} = \frac{\log_2 16}{7} = \frac{4}{7} = .571$

(c) the following set of 16 codewords differ in at least 3 places and thus all single errors can be corrected

0000000
1000101
0100111
0010110
0001011
1100010
1010011
1001110
0110001
0101100
0011101
1110100
1101001
1011000
0111010
1111111