

ECE 515 FALL 2026 ASSIGNMENT 1 SOLUTIONS

1. RV X has q symbols $\{x_1, x_2, x_3, x_4, \dots, x_q\}$

$$p(x_1) = \frac{1}{2} \quad p(x_2) = \frac{1}{8} \quad p(x_3) = \frac{1}{8}$$

$$p(x_4) = p(x_5) = a$$

$$p(x_6) = p(x_7) = p(x_8) = p(x_9) = b$$

$$\text{then } \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + 2a + 4b = 1$$

$$\Rightarrow a + 2b = \frac{1}{8}$$

$$H(X) = - \sum_{i=1}^q p(x_i) \log_2 p(x_i)$$

$$= -\frac{1}{2} \log_2 \frac{1}{2} - \frac{1}{4} \log_2 \frac{1}{8} - 2a \log_2 a - 4b \log_2 b$$

$$= \frac{5}{4} - 2a \log_2 a - 4b \log_2 b$$

upper bound

entropy is maximized when the six variable probability symbols have equal probabilities

$$a = b$$

from $a + 2b = \frac{1}{8}$ $3a = \frac{1}{8} \Rightarrow a = b = \frac{1}{24}$

the probabilities are then

$$\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{24}, \frac{1}{24}, \dots, \frac{1}{24}, \frac{1}{24}\right)$$

$$\begin{aligned} H_{\max} &= \frac{5}{4} - 6 \left(\frac{1}{24} \log_2 \frac{1}{24} \right) \\ &= \frac{5}{4} + \frac{1}{4} \log_2 24 \approx 2.40 \text{ bits} \end{aligned}$$

Lower bound - see next page

the minimum occurs at an extreme (a number of probabilities equal to 0)

$$b = 0 \Rightarrow a = \frac{1}{8}$$

the probabilities are then $\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0, 0, 0, 0\right)$

$$H(X) = \frac{5}{4} - \frac{1}{4} \log_2 \frac{1}{8} = \frac{5}{4} + \frac{3}{4} = 2 \text{ bits}$$

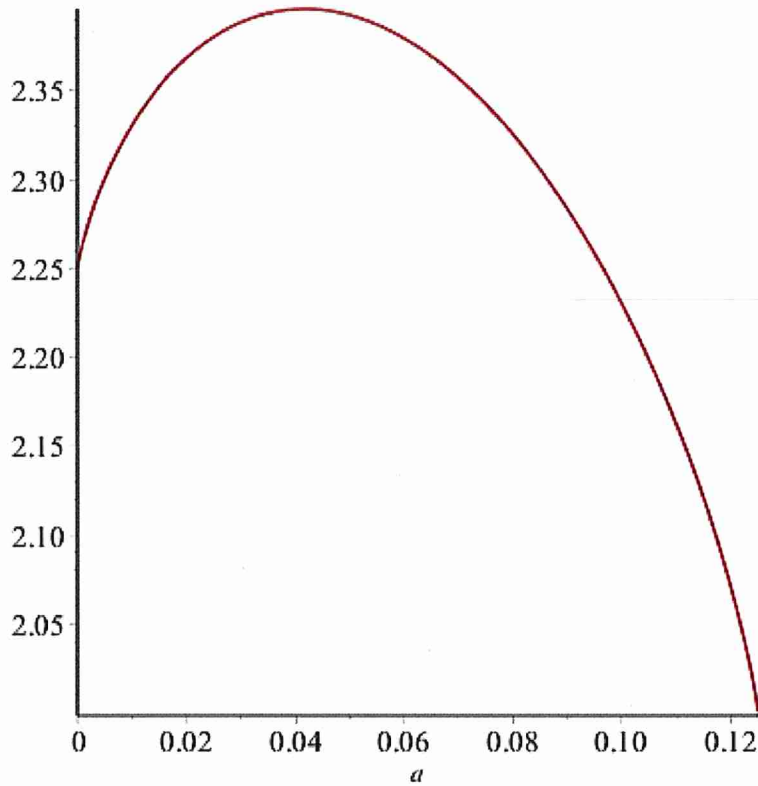
$$a = 0 \Rightarrow b = \frac{1}{16}$$

$$H(X) = \frac{5}{4} + \frac{1}{4} \log_2 16 = 2.25 \text{ bits}$$

∴ $H_{\min} = 2 \text{ bits}$

2A

$H(x)$



$$a + 2b = \frac{1}{8} \Rightarrow 2b = \frac{1}{8} - a \text{ or } b = \frac{1}{16} - \frac{a}{2}$$

substitute in $H(x)$

$$H(x) = \frac{5}{4} - 2a \log_2 a - 4\left(\frac{1}{16} - \frac{a}{2}\right) \log_2 \left(\frac{1}{16} - \frac{a}{2}\right)$$

$$0 \leq a \leq \frac{1}{8}$$

2. a) the most efficient sequence of questions is in order of probability

1. is it A?
2. is it B?
3. is it C?
- ⋮

the last question can either be
is it H or is it I?

b) each question is binary (yes/no) and reduces the remaining uncertainty by a factor of $\frac{1}{2}$ (1 bit)

c) the average number of questions is

$$\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 3 + \frac{1}{16} \cdot 4 + \frac{1}{32} \cdot 5 + \frac{1}{64} \cdot 6 + \frac{1}{128} \cdot 7 + \frac{2}{256} \cdot 8 = 1.992 \text{ questions}$$

2 there are $6^2 = 36$ possible outcomes when throwing each die, each with the same probability,

$$1/36$$

thus each throw provides

$$\log_2 36 = 5.17 \text{ bits of information}$$

since it is given that the sum of the faces is 7, of the 36 possible outcomes, 6 remain which can have occurred, so the remaining uncertainty is

$$\log_2 6 = 2.58 \text{ bits}$$

the amount of information given by the fact that the sum of the faces is 7 is then

$$\begin{aligned} \log_2 36 - \log_2 6 &= 5.17 - 2.58 \\ &= 2.59 \text{ bits} \end{aligned}$$

$$6. \quad p(0,0) = \frac{1}{2} \quad p(0,1) = \frac{1}{4} \quad p(1,0) = \frac{1}{8} \quad p(1,1) = \frac{1}{8}$$

$$p(X=0) = \frac{3}{4} \quad p(X=1) = \frac{1}{4}$$

$$p(Y=0) = \frac{5}{8} \quad p(Y=1) = \frac{3}{8}$$

$$H(X) = -\frac{3}{4} \log_2 \frac{3}{4} - \frac{1}{4} \log_2 \frac{1}{4} = .811 \text{ bits}$$

$$H(Y) = -\frac{5}{8} \log_2 \frac{5}{8} - \frac{3}{8} \log_2 \frac{3}{8} = \frac{5}{8} (.678) + \frac{3}{8} (1.415) \\ = .424 + .531 = .954 \text{ bits}$$

$$H(X|Y) = - \sum_i \sum_j p(x_i, y_j) \log_2 \frac{p(x_i, y_j)}{p(y_j)}$$

$$= -\frac{1}{2} \log_2 \frac{\frac{1}{2}}{\frac{5}{8}} - \frac{1}{4} \log_2 \frac{\frac{1}{4}}{\frac{3}{8}} - \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{5}{8}} - \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{3}{8}}$$

$$= -\frac{1}{2} \log_2 \frac{4}{5} - \frac{1}{4} \log_2 \frac{2}{3} - \frac{1}{8} \log_2 \frac{1}{5} - \frac{1}{8} \log_2 \frac{1}{3}$$

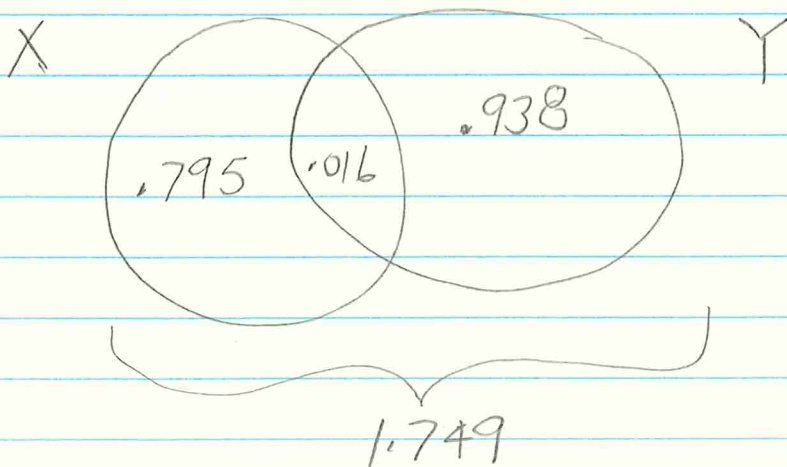
$$= .161 + .146 + .290 + .198$$

$$= .795 \text{ bits}$$

$$\begin{aligned}
H(Y|X) &= - \sum_i \sum_j p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(x_i)} \\
&= - \frac{1}{2} \log_2 \frac{\frac{1}{2}}{\frac{3}{4}} - \frac{1}{4} \log_2 \frac{\frac{1}{4}}{\frac{3}{4}} - \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{1}{4}} - \frac{1}{8} \log_2 \frac{\frac{1}{8}}{\frac{1}{4}} \\
&= -\frac{1}{2} \log_2 \frac{2}{3} - \frac{1}{4} \log_2 \frac{1}{3} - \frac{1}{4} \log_2 \frac{1}{2} \\
&= .292 + .396 + .250 \\
&= .938 \text{ bits}
\end{aligned}$$

$$\begin{aligned}
H(XY) &= H(X) + H(Y|X) = .811 + .938 = 1.749 \text{ bits} \\
&= H(Y) + H(X|Y) = .954 + .795 = 1.749 \text{ bits}
\end{aligned}$$

$$\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) = .811 - .795 = .016 \text{ bits} \\
&= H(Y) - H(Y|X) = .954 - .938 = .016 \text{ bits}
\end{aligned}$$



5. the four situations are:

being suitable or not
and rejected or not

Denote these as $s, \bar{s}, r, \bar{r}$

$$p(\bar{s}) = \frac{1}{4} \quad p(r|\bar{s}) = \frac{3}{4} \quad p(r) = \frac{1}{2}$$

$$\text{ooo } p(s) = 1 - p(\bar{s}) = \frac{3}{4}$$

$$p(\bar{r}|\bar{s}) = \frac{1}{4}$$

$$p(\bar{r}) = \frac{1}{2}$$

$$\text{Using Bayes' Theorem } p(\bar{s}|r) = \frac{p(r|\bar{s})p(\bar{s})}{p(r)} = \frac{3}{8}$$

$$p(s|r) = 1 - p(\bar{s}|r) = \frac{5}{8}$$

$$p(\bar{s}|\bar{r}) = \frac{p(\bar{r}|\bar{s})p(\bar{s})}{p(\bar{r})} = \frac{1}{8}$$

$$p(s|\bar{r}) = 1 - p(\bar{s}|\bar{r}) = \frac{7}{8}$$

Let X be the RV $s, \bar{s}$
and Y be the RV $r, \bar{r}$

$$(a) \quad H(Y | X = \bar{s})$$

$$= -p(r|\bar{s}) \log p(r|\bar{s}) - p(\bar{r}|\bar{s}) \log p(\bar{r}|\bar{s})$$

$$= -\frac{3}{4} \log \frac{3}{4} - \frac{1}{4} \log \frac{1}{4} = .811 \text{ bits}$$

(b) if the selection is made by tossing a coin every student has an equal chance of selection

$$\text{Thus } p(\bar{r}|\bar{s}) = p(r|\bar{s}) = p(\bar{r}|s) = p(r|s) = \frac{1}{2}$$

$$H(Y | X = \bar{s})$$

$$= -\frac{1}{2} \log \frac{1}{2} - \frac{1}{2} \log \frac{1}{2} = 1 \text{ bit}$$

(c) since being suitable or not does not play a role in case (b), there is no prior information about being selected.

Thus less information is received in case (a) compared to case (b)

.811 bit versus 1 bit

6. Let X denote if it rains or not

Y denote if rain is predicted or not

The desired quantity is $I(X; Y) = H(X) - H(X|Y)$

$$H(X) = -\frac{3}{16} \log \frac{3}{16} - \frac{13}{16} \log \frac{13}{16} = .453 + .243 = .696 \text{ bits}$$

for the weatherman

$$H(X|Y) = -\sum_{i=1}^2 \sum_{j=1}^2 p(x_i, y_j) \log \frac{p(x_i, y_j)}{p(y_j)}$$

$$= -\frac{1}{8} \log \frac{1/8}{5/16} - \frac{3}{16} \log \frac{3/16}{5/16} - \frac{1}{16} \log \frac{1/16}{11/16} - \frac{10}{16} \log \frac{10/16}{11/16}$$

$$= .165 + .138 + .216 + .086 = .605 \text{ bits}$$

$$I(X; Y) = .696 - .605 = .091 \text{ bits}$$

for the student

$$H(X|Y) = H(X) \text{ since } Y \text{ gives no information about } X$$

$$I(X; Y) = .696 - .696 = 0$$

Thus, predicting no rain all the time provides no information about the weather whereas the weatherman provides some information

1. Let $y_1 = \text{truth}$

$y_2 = \text{lie}$

$y_3 = \text{refuse}$

$$p = p(C) \quad 1-p = p(H)$$

$$p(y_1) = 0.5p + 0.3(1-p)$$

$$p(y_2) = 0.3p + 0.5(1-p)$$

$$p(y_3) = 0.2p + 0.2(1-p) = 0.2$$

$$H(Y) = -(0.3 + 0.2p) \log_2 (0.3 + 0.2p)$$

$$-(0.5 - 0.2p) \log_2 (0.5 - 0.2p)$$

$$-0.2 \log 0.2$$

$$H(Y|X) = p H(Y|X=C) + (1-p) H(Y|X=H)$$

$$= -p(0.5 \log_2 0.5 + 0.3 \log_2 0.3 + 0.2 \log_2 0.2)$$

$$-(1-p)(0.3 \log_2 0.3 + 0.5 \log_2 0.5 + 0.2 \log_2 0.2)$$

$$= -0.5 \log 0.5 - 0.3 \log 0.3 - 0.2 \log 0.2$$

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= -(0.3 + 0.2p) \log_2 (0.3 + 0.2p)$$

$$- (0.5 - 0.2p) \log_2 (0.5 - 0.2p)$$

$$+ 0.5 \log_2 0.5 + 0.3 \log_2 0.3$$

$$\frac{dI}{dp} = -0.2 \left[\log_2 (0.3 + 0.2p) - \log_2 (0.5 - 0.2p) \right]$$

$$- (0.3 + 0.2p) \cdot \frac{0.2}{(0.3 + 0.2p) \ln 2}$$

$$+ (0.5 - 0.2p) \cdot \frac{0.2}{(0.5 - 0.2p) \ln 2}$$

$$= -0.2 \log \left[\frac{(0.5 - 0.2p)}{(0.3 + 0.2p)} \right]$$

this is zero when

$$0.5 - 0.2p = 0.3 + 0.2p$$

$$\text{or } 0.4p = 0.2 \Rightarrow p = \frac{1}{2}$$

$$\begin{aligned} I_{\max} &= -0.4 \log_2 0.4 - 0.4 \log_2 0.4 \\ &\quad + 0.5 \log_2 0.5 + 0.3 \log_2 0.3 \\ &= 1.0575 - \frac{1}{2} - .5211 = .0364 \text{ bits} \end{aligned}$$