University of Victoria

Department of Electrical and Computer Engineering

ECE 515 Information Theory

Fall 2026

Midterm Test

INSTRUCTIONS
One sheet of $8.5" \times 11"$ paper allowed. There are 25 marks total.
State any assumptions that may be necessary. Please write clearly.

Question 1 [12 marks total]

Consider two binary random variables X and Y with joint probability distribution $\mathrm{p}(x, y)$ as follows: $\mathrm{p}(0,0) = 1/2$, $\mathrm{p}(0,1) = 1/4$, $\mathrm{p}(1,0) = 0$, and $\mathrm{p}(1,1) = 1/4$.

a) [2 marks] Find $H(X)$.

b) [2 marks] Find $H(Y)$.

c) [2 marks] Find $H(X|Y)$.

d) [2 marks] Find $H(Y|X)$.

e) [2 marks] Find $H(XY)$.

f) [2 marks] Find $I(X;Y)$.

Question 2 [4 marks]

Consider a Binary Symmetric Channel (BSC) with input X and output Y.

a) [2 marks] What input distribution maximizes $I(X;Y)$?

b) [2 marks] For crossover probability $p = 0.1$, compute $I(X;Y)$ for the distribution in (a).

Question 3 [4 marks]

Let X be a random variable taking on a finite number of values. For the following, determine the sign $>, \geq, =, \leq, <$ and justify your answer.

a) [2 marks] $H(X)$ versus $H(\cos(X))$

b) [2 marks] $H(X)$ versus $H(3^X)$

Question 4 [5 marks]

Let XYZ be a discrete joint ensemble of three random variables. Prove the validity of the inequality $I(XY;Z) \geq I(X;Z)$, and if true find conditions for equality. Venn diagrams are not sufficient.

ECE 515 FALL 2025 MIDTERM TEST SOLUTIONS

1. $p(0,0) = \frac{1}{2}$ $p(0,1) = \frac{1}{4}$ $p(1,0) = 0$ $p(1,1) = \frac{1}{4}$

$p(X=0) = \frac{3}{4}$ $p(X=1) = \frac{1}{4}$

$p(Y=0) = \frac{1}{2}$ $p(Y=1) = \frac{1}{2}$

a) $H(X) = -\frac{3}{4}\log_2\frac{3}{4} - \frac{1}{4}\log_2\frac{1}{4} = .811$ bit

b) $H(Y) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} = 1$ bit

c) $$H(X|Y) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j)\log_2\frac{p(x_i, y_j)}{p(y_j)}$$

$$= -\frac{1}{2}\log_2\frac{1/2}{1/2} - \frac{1}{4}\log_2\frac{1/4}{1/2} - \frac{1}{4}\log_2\frac{1/4}{1/2}$$

$$= 0 - \frac{1}{2}\log_2\frac{1}{2} = \frac{1}{2} \text{ bit}$$

d) $$H(Y|X) = \sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i, y_j)\log_2\frac{p(x_i, y_j)}{p(x_i)}$$

$$= -\frac{1}{2}\log_2\frac{1/2}{3/4} - \frac{1}{4}\log_2\frac{1/4}{3/4} - \frac{1}{4}\log_2\frac{1/4}{1/4}$$

$$= -\frac{1}{2}\log_2\frac{2}{3} - \frac{1}{4}\log_2\frac{1}{3} - 0$$

$H(Y|X) = .2925 + .3962 = .689$ bit

e) $H(XY) = H(X) + H(Y|X) = H(Y) + H(X|Y)$

$= .811 + .689 = 1 + 0.5 = 1.5$ bits

f) $I(X;Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$

$= .811 - 0.5 = 1 - .689 = .311$ bit

2 BSC input X output Y

a) $I(X;Y)$ is maximized with an equiprobable input distribution

$$p(x_1) = p(x_2) = \frac{1}{2}$$

b) $p = 0.1$

for an equiprobable input distribution

$$I(X;Y) = 1 - h(p) \text{ bits}$$

$$= 1 - 0.1 \log 0.1 - 0.9 \log 0.9$$

$$= 1 - .3322 - .1368$$

$$= 1 - .469 = .531 \text{ bit}$$

3a) $H(X) \geq H(\cos(X))$

since $X \to \cos(X)$ is not necessarily a one-to-one mapping

equality occurs if cosine is one-to-one on the range of X

b) $H(X) = H(3^X)$

since $X \to 3^X$ is a one-to-one mapping, the probabilities of the values do not change

4) Prove $I(XY;Z) \geq I(X;Z)$

$I(XY;Z) = I(X;Z) + I(Y;Z|X)$

since $I(Y;Z|X) \geq 0$

(conditioning cannot decrease the mutual information)

therefore

$I(XY;Z) \geq I(X;Z)$

Equality holds when $I(Y;Z|X) = 0$

i.e., iff conditioned on X

Y and Z are independent