B-8-26.

$$|G(j\omega)| = \frac{\sqrt{a^2\omega^2+1}}{\omega^2}, \qquad \angle G(j\omega) = \tan^{-1} a\omega - 180^\circ$$

The phase margin of 45° at $\omega = \omega_1$ requires that

$$\frac{\sqrt{a^2\omega_1^2+1}}{\omega_1^2} = 1$$

$$\tan^{-1} a\omega_1 - 180^\circ = 45^\circ - 180^\circ$$

Thus, we have

$$a^2\omega_1^2 + 1 = \omega_1^4, \qquad a\omega_1 = 1$$

Solving for a, we obtain

$$a = \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}} = 0.841$$

B-8-29.

$$G(s) = \frac{K}{s(s^2+s+4)} = \frac{0.25K}{s(0.25s^2+0.25s+1)}$$

The quadratic term in the denominator has the undamped natural frequency of 2 rad/sec and the damping ratio of 0.25. Define the frequency corresponding to the angle of -130° to be ω_1.

$$\underline{/G(j\omega_1)} = -\underline{/j\omega_1} - \underline{/1 - 0.25\omega_1^2 + j\,0.25\omega_1}$$

$$= -90° - \tan^{-1}\frac{0.25\omega_1}{1-0.25\omega_1^2} = -130°$$

Solving this last equation for ω_1, we find $\omega_1 = 1.491$. Thus, the phase angle becomes equal to $-130°$ at $\omega = 1.491$ rad/sec. At this frequency, the magnitude must be unity, or $|G(j\,\omega_1)| = 1$. The required gain K can be determined from

$$|G(j1.491)| = \left|\frac{0.25K}{(j1.491)(-0.555 + j\,0.3725 + 1)}\right| = 0.2890K$$

Setting $|G(j1.491)| = 0.2890K = 1$, we find

$$K = 3.46$$

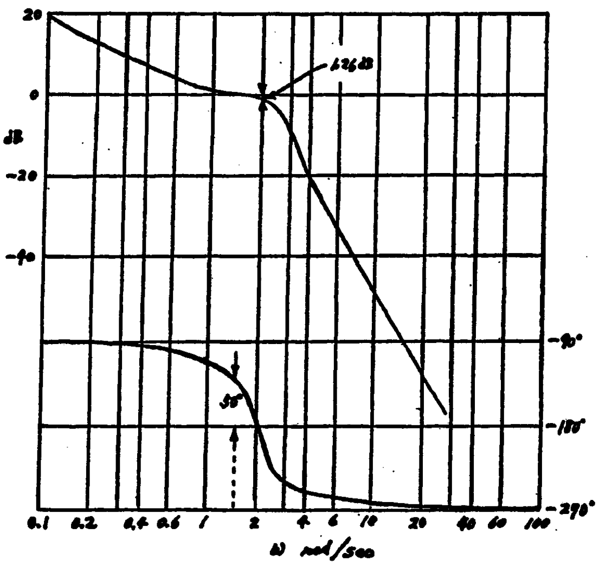
Note that the phase crossover frequency is at $\omega = 2$ rad/sec, since

$$\underline{/G(j2)} = -\underline{/j2} - \underline{/-0.25\times 2^2 + 0.25\times j2 + 1} = -90° - 90° = -180°$$

The magnitude $|G(j2)|$ with K = 3.46 becomes

$$|G(j2)| = \left|\frac{0.865}{(j2)(-1+0.5j+1)}\right| = 0.865 = -1.26\ dB$$

Thus, the gain margin is 1.26 dB. The Bode diagram of $G(j\omega)$ with K = 3.46 is shown below.



B-8-31. Note that

$$G(s) = \frac{K}{s(s^2 + s + 0.5)} = \frac{2K}{s(2s^2 + 2s + 1)}$$

We shall first plot a Bode diagram of $G(j\omega)$ when $K = 0.5$. That is, we plot a Bode diagram for

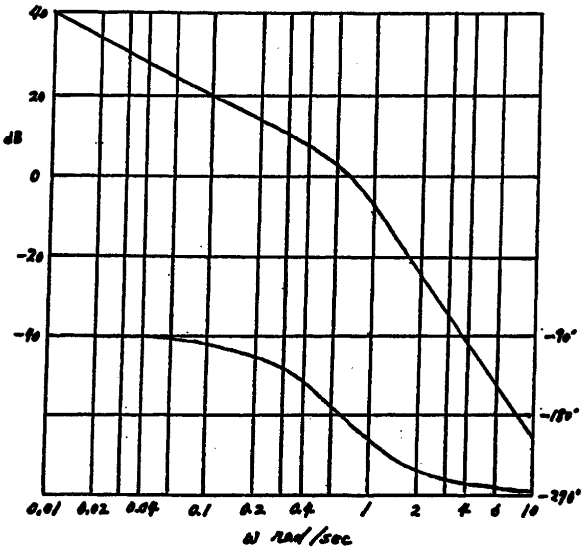
$$G(j\omega) = \frac{1}{j\omega[2(j\omega)^2 + 2j\omega + 1]}$$

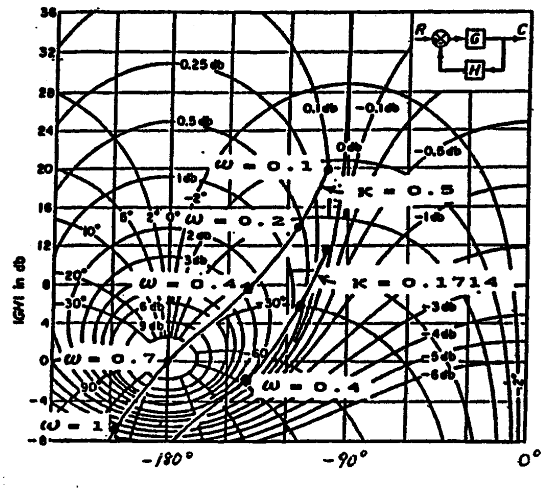
It is shown below. By reading the magnitude and phase angle values at each frequency point condidered, the log-magnitude versus phase curve can be plotted as shown below the Bode diagram. By moving the curve vertically, we can shift the curve to be tangent to the M = 2 dB locus. The vertical shift needed is 9.3 dB. That is, if we lower the curve by 9.3 dB, then it is tangent to the M = 2 dB locus. Therefore, we set

$$2K = -9.3 \ dB$$

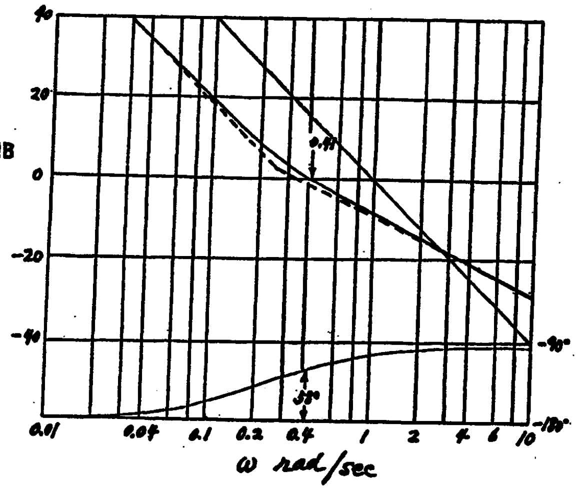
Solving this equation for K determines the desired value of K as

$$K = 0.1714$$





B-9-4. Choose the gain crossover frequency to be approximately 0.4 rad/sec and the phase margin to be approximately 60°. Draw the high frequency asymptote having the slope of -20 dB/dec to cross the 0 dB line at about $\omega = 0.35$ rad/sec. Choose the corner frequency to be 0.25 rad/sec. Then the low-frequency asymptote can be drawn on Bode diagram. See the Bode diagram shown below.



The actual magnitude curve crosses the 0 dB line at about $\omega = 0.41$ rad/sec and the phase margin is approximately 58°.

Since we have chosen the corner frequency to be 0.25 rad/sec, we get

$$T_d = 4$$

From the Bode diagram, K_d must be chosen to be -21.4 dB, or

$$K_d = -21.4 \text{ dB} = 0.0851$$

Thus

$$K_d(1 + T_d s) = 0.0851\,(1 + 4s)$$

Then, the open-loop transfer function becomes

$$G(s) = \frac{0.0851\,(1+4s)}{s^2}$$

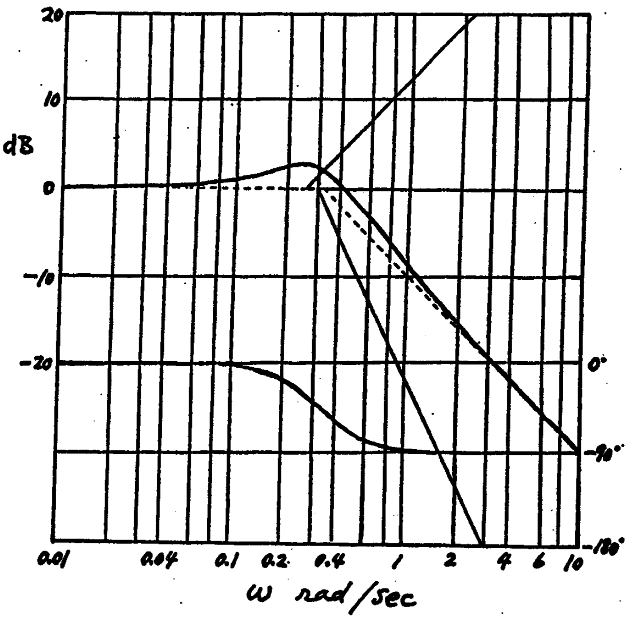
The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{0.0851\,(1+4s)}{s^2 + 0.0851\,(1+4s)} = \frac{4s+1}{11.751\,s^2 + 4s + 1}$$

A Bode diagram of

$$\frac{C(j\omega)}{R(j\omega)} = \frac{4j\omega + 1}{11.751\,(j\omega)^2 + 4j\omega + 1}$$

is shown on the next page. From this diagram we see that the bandwidth is approximately 0.5 rad/sec.



B-9-5. Let us use the following lead compensator:

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Since K_v is specified as 4.0 $\sec^{-1}$, we have

$$K_v = \lim_{s \to 0} s K_c \alpha \frac{Ts+1}{\alpha Ts+1} \frac{K}{s(0.1s+1)(s+1)} = K_c \alpha K = 4$$

Let us set K = 1 and define $K_c \alpha = \hat{K}$. Then

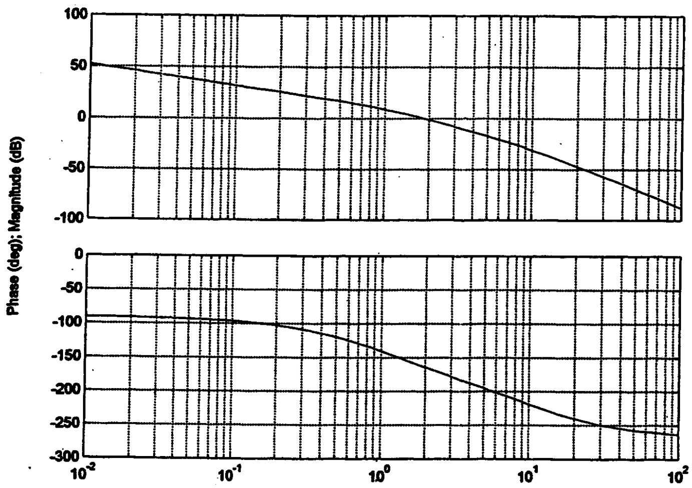
$$\hat{K} = 4$$

Next, plot a Bode diagram of

$$\frac{4}{s(0.1s+1)(s+1)} = \frac{4}{0.1s^3+1.1s^2+s}$$

The following MATLAB program produces the Bode diagram shown on the next page.

```
% ***** Bode diagram *****

num = [0  0  0  4];
den = [0.1  1.1  1  0];
bode(num,den)
title('Bode Diagram of G(s) = 4/[s(0.1s+1)(s+1)]')
```



From this plot, the phase and gain margins are 17° and 8.7 dB, respectively.

Since the specifications call for a phase margin of 45°, let us choose

$$\phi_m = 45° - 17° + 12° = 40°$$

(This means that 12° has been added to compensate for the shift in the gain crossover frequency.) The maximum phase lead is 40°. Since

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha} \qquad (\phi_m = 40°)$$

α is determined as 0.2174. Let us choose, instead of 0.2174, α to be 0.21, or

$$\alpha = 0.21$$

Next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. Note that the maximum phase-lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha}\, T)$. The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}\, T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left| \frac{1 + j\omega T}{1 + j\omega\alpha T} \right|_{\omega = \frac{1}{\sqrt{\alpha} T}} = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.21}} = 2.1822 = 6.7778 \text{ dB}$$

We need to find the frequency point where, when the lead compensator is added, the total magnitude becomes 0 dB. The magnitude $G(j\omega)$ is -6.7778 dB corresponds to ω= 2.81 rad/sec. We select this frequency to be the new gain crossover frequency ω_c. Then we obtain

$$\frac{1}{T} = \sqrt{\alpha}\,\omega_c = \sqrt{0.21} \times 2.81 = 1.2877$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{2.81}{\sqrt{0.21}} = 6.1319$$

Hence

$$G_c(s) = K_c \frac{s+1.2877}{s+6.1319}$$

and

$$K_c = \frac{\hat{K}}{\alpha} = \frac{4}{0.21}$$

Thus

$$G_c(s) = \frac{4}{0.21} \frac{s+1.2877}{s+6.1319} = 4 \frac{0.7766s+1}{0.163085s+1}$$

The open-loop transfer function becomes as

$$G_c(s)\,G(s) = 4 \frac{0.7766s+1}{0.163085s+1} \frac{1}{s(0.1s+1)(s+1)}$$

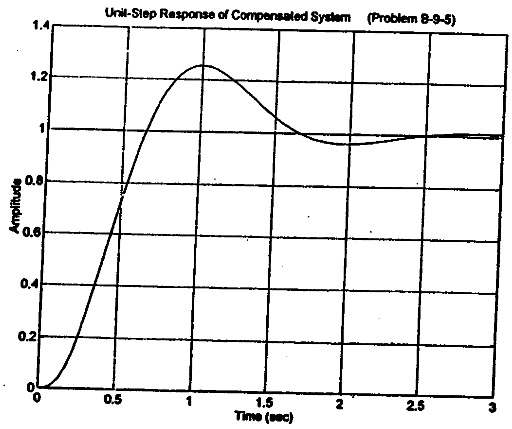
$$= \frac{3.1064s+4}{0.01631s^4+0.2794s^3+1.2631s^2+s}$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{3.1064s+4}{0.01631s^4+0.2794s^3+1.2631s^2+4.1064s+4}$$

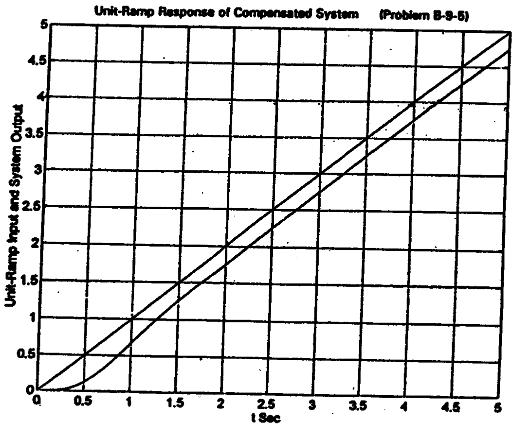
The following MATLAB program produces the unit-step response curve as shown on the next page.

```
% ***** Unit-step response *****

numc = [0  0  0  3.1064  4];
denc = [0.01631  0.2794  1.2631  4.1064  4];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System    (Problem B-9-5)')
```



Similarly, the following MATLAB program produces the unit-ramp response curve as shown below.

```
% ***** Unit-ramp response *****

numc = [0  0  0  0  3.1064  4];
denc = [0.01631  0.2794  1.2631  4.1064  4  0];
t = 0:0.01:5;
c = step(numc,denc,t);
plot(t,c,t,t)
grid
title('Unit-Ramp Response of Compensated System     (Problem B-9-5)')
xlabel('t Sec')
ylabel('Unit-Ramp Input and System Output')
```



B-9-6. To satisfy the requirements, try a lead compensator $G_c(s)$ of the form

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha T s+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

Define

$$G_1(s) = K\,G(s) = \frac{K}{s(s+1)}$$

where $K = K_c \alpha$. Since the static velocity error constant K_v is given as 50 $\sec^{-1}$, we have

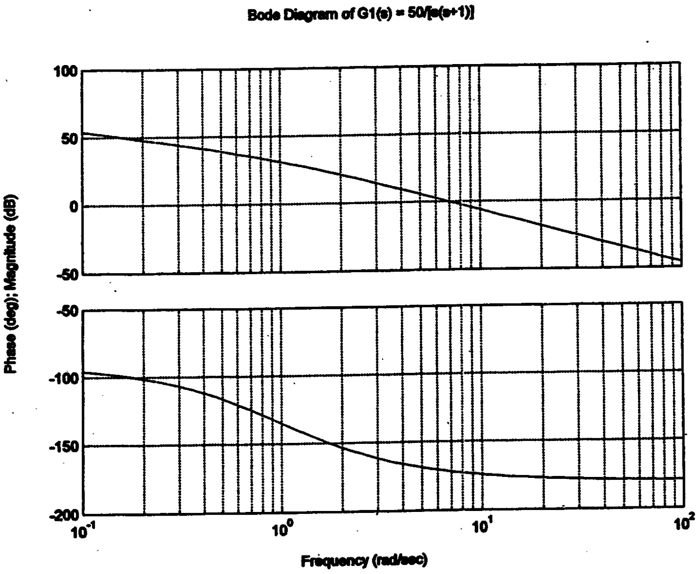
$$K_v = \lim_{s\to 0} s\,G_c(s)G(s) = \lim_{s\to 0} s \frac{Ts+1}{\alpha T s+1} \frac{K}{s(s+1)} = K = 50$$

We shall now plot a Bode diagram of

$$G_1(s) = \frac{50}{s(s+1)}$$

The following MATLAB program produces the Bode diagram shown below.

```
% ***** Bode diagram *****

num = [0  0  50];
den = [1  1  0];
w = logspace(-1,2,100);
bode(num,den,w);
title('Bode Diagram of G1(s) = 50/[s(s+1)]')
```



From this plot, the phase margin is found to be 7.8°. The gain margin is $+\infty$ dB. Since the specifications call for a phase margin of 50°, the additional phase lead angle necessary to satisfy the phase margin requirement is 42.2°. We may assume the maximum phase lead required to be 48°. This means that 5.8° has been added to compensate for the shift in the gain crossover frequency. Since

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha}$$

$\phi_m = 48^\circ$ corresponds to $\alpha = 0.14735$. (Note that $\alpha = 0.15$ corresponds to $\phi_m = 47.657^\circ$.) Whether we choose $\phi_m = 48^\circ$ or $\phi_m = 47.657^\circ$ does not make much difference in the final solution. Hence, we choose $\alpha = 0.15$.

The next step is to determine the corner frequencies $\omega = 1/T$ and $\omega = 1/(\alpha T)$ of the lead compensator. Note that the maximum phase-lead angle ϕ_m occurs at the geometric mean of the two corner frequencies, or $\omega = 1/(\sqrt{\alpha}T)$. The amount of the modification in the magnitude curve at $\omega = 1/(\sqrt{\alpha}T)$ due to the inclusion of the term $(Ts + 1)/(\alpha Ts + 1)$ is

$$\left| \frac{1+j\omega T}{1+j\omega\alpha T} \right|_{\omega = \frac{1}{\sqrt{\alpha}T}} = \left| \frac{1+j\frac{1}{\sqrt{\alpha}}}{1+j\alpha\frac{1}{\sqrt{\alpha}}} \right| = \frac{1}{\sqrt{\alpha}}$$

Note that

$$\frac{1}{\sqrt{\alpha}} = \frac{1}{\sqrt{0.15}} = 2.5820 = 8.239 \text{ dB}$$

We need to find the frequency point where, when the lead compensator is added, the total magnitude becomes 0 dB. The frequency at which the magnitude of $G_1(j\omega)$ is equal to -8.239 dB occurs between $\omega = 10$ and 100 rad/sec. From the Bode diagram we find the frequency point where $|G_1(j\omega)| = -8.239$ dB occurs at $\omega = 11.4$ rad/sec. Noting that this frequency corresponds to $1/(\sqrt{\alpha}T)$, or

$$\omega_c = \frac{1}{\sqrt{\alpha}\,T}$$

we obtain

$$\frac{1}{T} = \omega_c\sqrt{\alpha} = 11.4\sqrt{0.15} = 4.4152$$

$$\frac{1}{\alpha T} = \frac{\omega_c}{\sqrt{\alpha}} = \frac{11.4}{\sqrt{0.15}} = 29.4347$$

The lead compensator thus determined is

$$G_c(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c \frac{s+4.4152}{s+29.4347}$$

where K_c is determined as

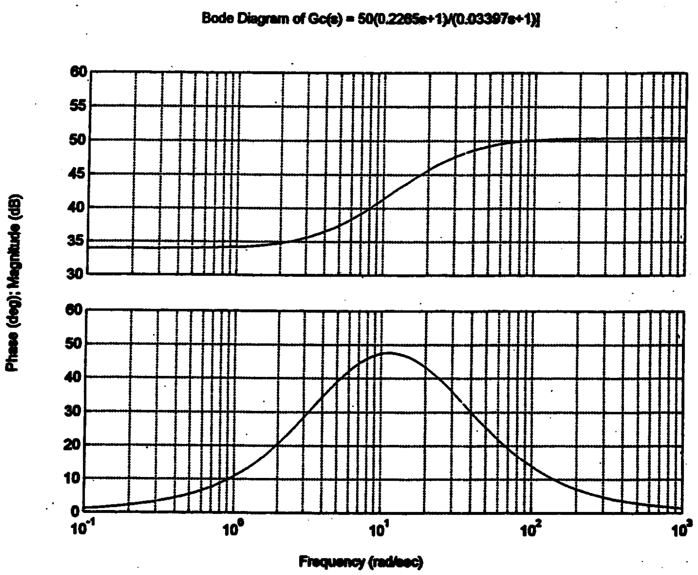
$$K_c = \frac{K}{\alpha} = \frac{50}{0.15} = \frac{1000}{3}$$

Thus,

$$G_c(s) = \frac{1000}{3}\,\frac{s+4.4152}{s+29.4347} = 50\,\frac{0.22655s+1}{0.03397s+1}$$

The following MATLAB program produces the Bode diagram of the lead compensator just designed, as shown below.

```
% ***** Bode diagram *****

num1 = [11.325  50];
den1 = [0.03397  1];
w = logspace(-1,3,100);
bode(num1,den1,w);
title('Bode Diagram of Gc(s) = 50(0.2265s+1)/(0.03397s+1)]')
```



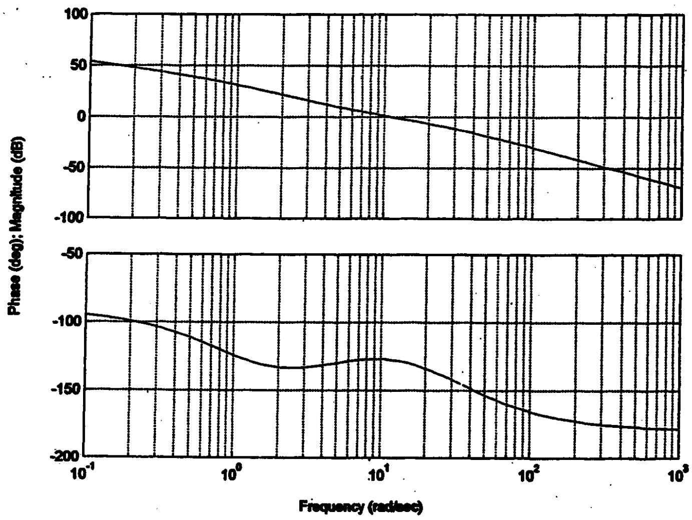
The open-loop transfer function of the designed system is

$$G_c(s)\,G(s) = \frac{1000}{3}\left(\frac{s+4.4152}{s+29.4347}\right)\frac{1}{s(s+1)}$$

The following MATLAB program produces the Bode diagram of $G_C(s)G(s)$, which is shown on the next page.

```
% ***** Bode diagram *****

num = [0  0  1000  4415.2];
den = [3  91.3041  88.3041  0];
w = logspace(-1,3,100);
bode(num,den,w);
title('Bode Diagram of Gc(s)G(s) = 1000(s+4.4152)/[3(s+29.4347)s(s+1)]')
```

Bode Diagram of Gc(s)G(s) = 1000(s+4.4152)/[3(s+29.4347)s(s+1)]



From this diagram, it is clearly seen that the phase margin is approximately 52°, the gain margin is $+\infty$ dB, and K_v = 50 sec^{-1}; all specifications are met. Thus, the designed system is satisfactory.

Next, we shall obtain the unit-step and unit-ramp responses of the original uncompensated system and the compensated system. The original uncompensated system has the following closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{1}{s^2 + s + 1}$$

The closed-loop transfer function of the compensated system is

$$\frac{C(s)}{R(s)} = \frac{1000\,(s + 4.4152)}{3(s+29.4347)s(s+1) + 1000\,(s+4.4152)}$$

$$= \frac{1000\,s + 4415.2}{3s^3 + 91.3041\,s^2 + 1088.3041\,s + 4415.2}$$

The closed-loop poles of the compensated system are as follows:

s = -11.1772 + j7.5636

s = -11.1772 - j7.5636

s = -8.0804

The MATLAB program given at the top of next page produces the unit-step responses of the uncompensated and compensated systems. The resulting response curves are shown on the next page.

B-9-9. Let us assume that the compensator $G_c(s)$ has the following form:

$$G_c(s) = K_c \frac{(T_1 s+1)(T_2 s+1)}{\left(\frac{T_1}{\beta} s+1\right)(\beta T_2 s+1)} = K_c \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{\beta}{T_1}\right)\left(s+\frac{1}{\beta T_2}\right)}$$

Since K_v is specified as 20 $\sec^{-1}$, we have

$$K_v = \lim_{s\to 0} s\, G_c(s) \frac{1}{s(s+1)(s+5)} = K_c \frac{1}{5} = 20$$

Hence

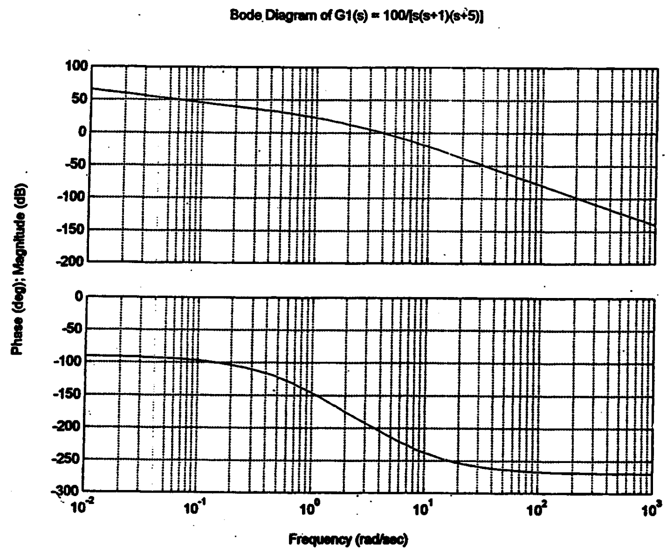
$$K_c = 100$$

Define

$$G_1(s) = 100\, G(s) = \frac{100}{s(s+1)(s+5)}$$

The following MATLAB program produces the Bode diagram of $G_1(s)$ as shown on the next page.

```
% ***** Bode diagram *****

num = [0  0  0  100];
den = [1  6  5  0];
w = logspace(-2,3,100);
bode(num,den,w);
title('Bode Diagram of G1(s) = 100/[s(s+1)(s+5)]')
```



From this diagram we find the phase crossover frequency to be $\omega = 2.25$ rad/sec. Let us choose the gain crossover frequency of the designed system to be 2.25 rad/sec so that the phase lead angle required at $\omega = 2.25$ rad/sec is 60°.

Once we choose the gain crossover frequency to be 2.25 rad/sec, we can determine the corner frequencies of the phase lag portion of the lag-lead compensator. Let us choose the corner frequency $1/T_2$ to be one decade below the new gain crossover frequency, or $1/T_2 = 0.225$. For the lead portion of the compensator, we first determine the value of β that provides $\phi_m = 65°$. (5° added to 60°.) Since

$$\sin \phi_m = \frac{1-\frac{1}{\beta}}{1+\frac{1}{\beta}} = \frac{\beta-1}{\beta+1}$$

we find $\beta = 20$ corresponds to 64.7912°. Since we need 65° phase margin, we may choose $\beta = 20$. Thus

$$\beta = 20$$

Then, the corner frequency $1/(\beta T_2)$ of the phase lag portion becomes as follows:

$$\frac{1}{\beta T_2} = \frac{1}{20 \times \frac{1}{0.225}} = \frac{0.225}{20} = 0.01125$$

Hence, the phase lag portion of the compensator becomes as

$$\frac{s+0.225}{s+0.01125} = 20\,\frac{4.4444s+1}{88.8889s+1}$$

For the phase lead portion, we first note that

$$G_1(j2.25) = 10.35 \text{ dB}$$

If the lag-lead compensator contributes -10.35 dB at ω = 2.25 rad/sec, then the new gain crossover frequency will be as desired. The intersections of the line with slope +20 dB/dec [passing through the point (2.25, -10.35 dB)] and the 0 dB line and -26.0206 dB line determine the corner frequencies. Such intersections are found as ω = 0.3704 and ω = 7.4077 rad/sec, respectively. Thus, the phase lead portion becomes

$$\frac{s+0.3704}{s+7.4077} = \frac{1}{20}\left(\frac{2.6998\,s+1}{0.1350\,s+1}\right)$$

Hence the compensator can be written as

$$G_c(s) = 100\left(\frac{4.4444\,s+1}{88.8889\,s+1}\right)\left(\frac{2.6998\,s+1}{0.1350\,s+1}\right)$$

$$= 100\left(\frac{s+0.225}{s+0.01125}\right)\left(\frac{s+0.3704}{s+7.4077}\right)$$

Then the open-loop transfer function $G_c(s)G(s)$ becomes as follows:

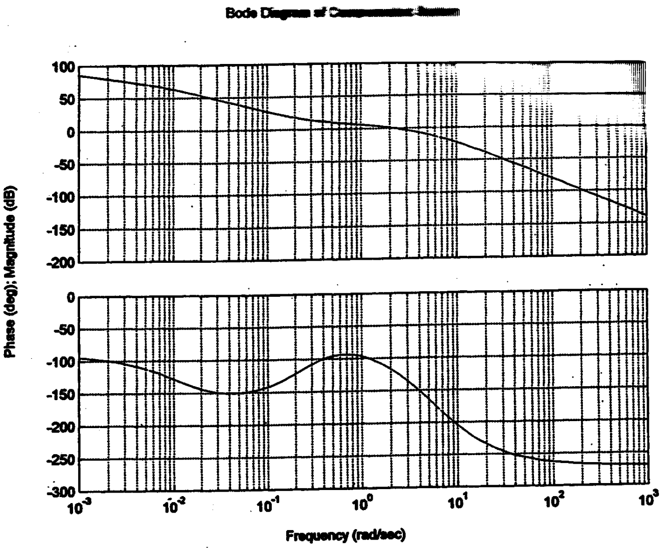
$$G_c(s)\,G(s) = 100\left(\frac{4.4444\,s+1}{88.8889\,s+1}\right)\left(\frac{2.6998\,s+1}{0.1350\,s+1}\right)\frac{1}{s(s+1)(s+5)}$$

$$= \frac{1199.90\,s^2 + 714.42\,s + 100}{12s^5 + 161.0239\,s^4 + 595.1434\,s^3 + 451.1195\,s^2 + 5s}$$

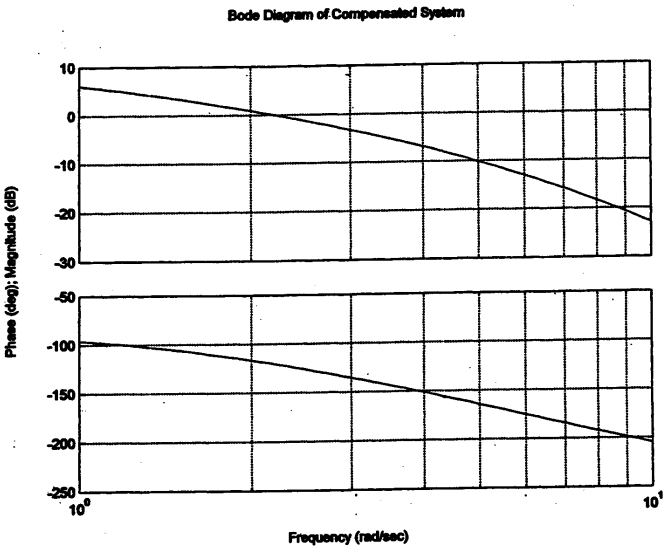
The following MATLAB program produces the Bode diagram of the open-loop transfer function.

```
% ***** Bode diagram *****

num = [0  0  0  1199.90  714.42  100];
den = [12  161.0239  595.1434  451.1195  5  0];
w = logspace(-3,3,100);
bode(num,den,w);
title('Bode Diagram of Compensated System')
```

The resulting Bode diagram is shown on the next page.



To read the phase margin and gain margin precisely, we need to expand the diagram between $\omega = 1$ and $\omega = 10$ rad/sec. This can be done easily by modifying the preceding MATLAB program. [Simply change the command w = logspace (-3,3, 100) to w = logspace(0,1,100).] The resulting Bode diagram is shown below.



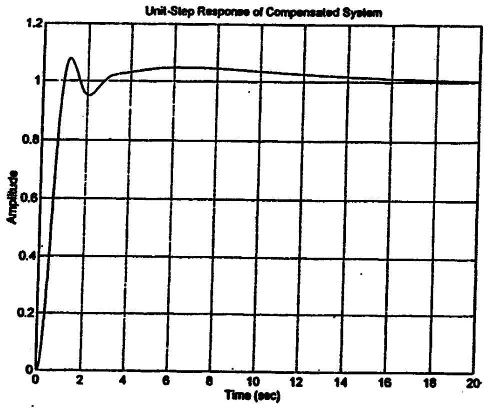
From this diagram we find that the phase margin is approximately 60° and gain margin is 14.35 dB. The static velocity error constant is 20 sec^{-1}.

The closed-loop transfer function of the designed system is

$$\frac{C(s)}{R(s)} = \frac{1199.90 s^2 + 714.42 s + 100}{12 s^5 + 161 s^4 + 595.1 s^3 + 1651 s^2 + 719.4 s + 100}$$

The following MATLAB program produces the unit-step response. The resulting unit-step response curve is shown below.

```
% ***** Unit-step response *****

numc = [0  0  0  1199.90  714.42  100];
denc = [12  161  595.1  1651  719.4  100];
step(numc,denc)
grid
title('Unit-Step Response of Compensated System')
```



The closed-loop poles can be obtained by entering the following MATLAB program into the computer.

```
roots(denc)

ans =

 -9.7022
 -1.6110 + 3.0494i
 -1.6110 - 3.0494i
 -0.2463 + 0.1076i
 -0.2463 - 0.1076i
```

Notice that there are two zeros (s = -0.225 and s = -0.4939) near the closed-loop poles at s = -0.2463 ± j0.1076. Such a pole-zero combination generates a long tail with small amplitude in the unit-step response.

The following MATLAB program will produce the unit-ramp response as shown below.

```
% ***** Unit-ramp response *****

numc = [0  0  0  0  1199.90  714.42  100];
denc = [12  161  595.1  1651  719.4  100  0];
t = 0:0.05:20;
c = step(numc,denc,t);
plot(t,c,'-',t,t,'.')
grid
title('Unit-Ramp Response of Compensated System')
xlabel('t Sec')
ylabel('Output and Ramp Input')
text(11,7,'Output'); text(1,7,'Ramp Input')
```