ELEC 360 Control Theory and Systems I

Part A: Introduction
Laplace Transforms
Mathematical Modeling of Dynamic Systems

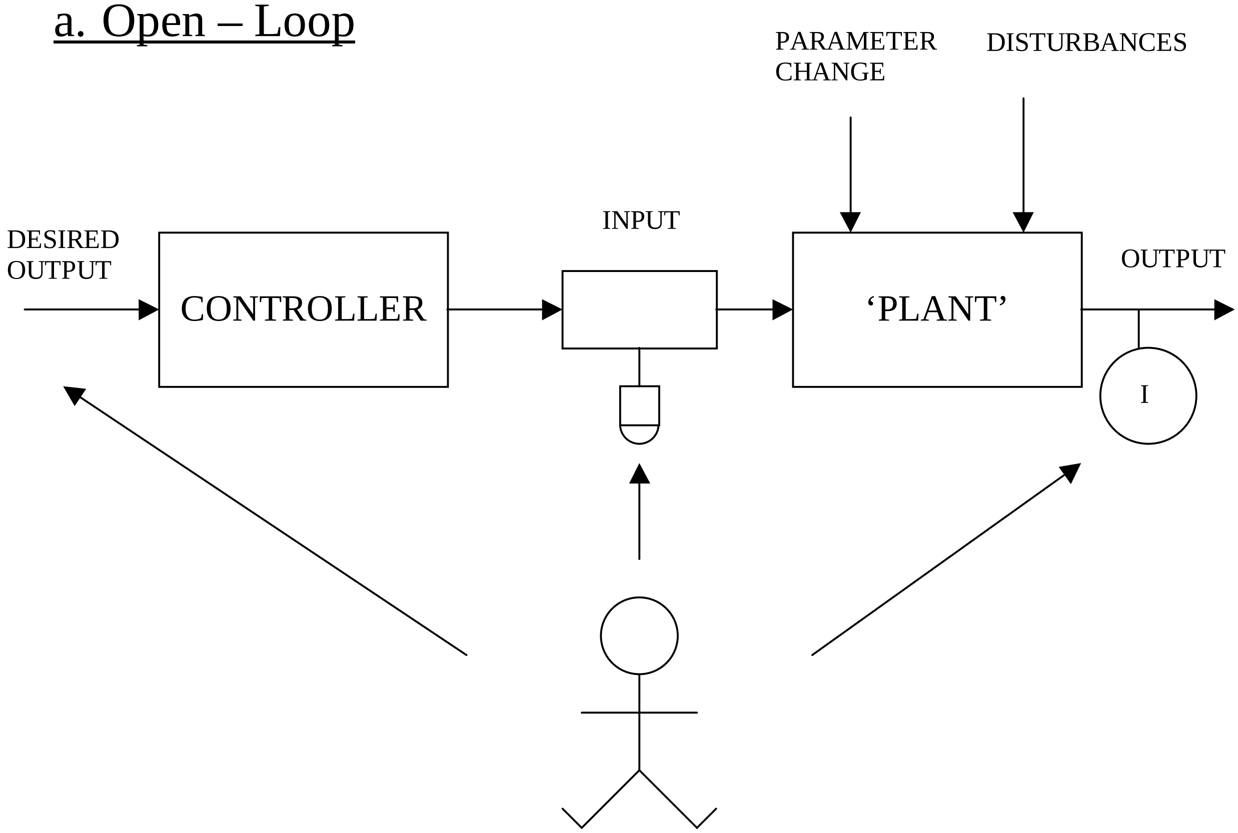
Part B: Transient Response Analysis
Steady State Response Analysis

Part C: Root Locus Analysis
Frequency Response Analysis

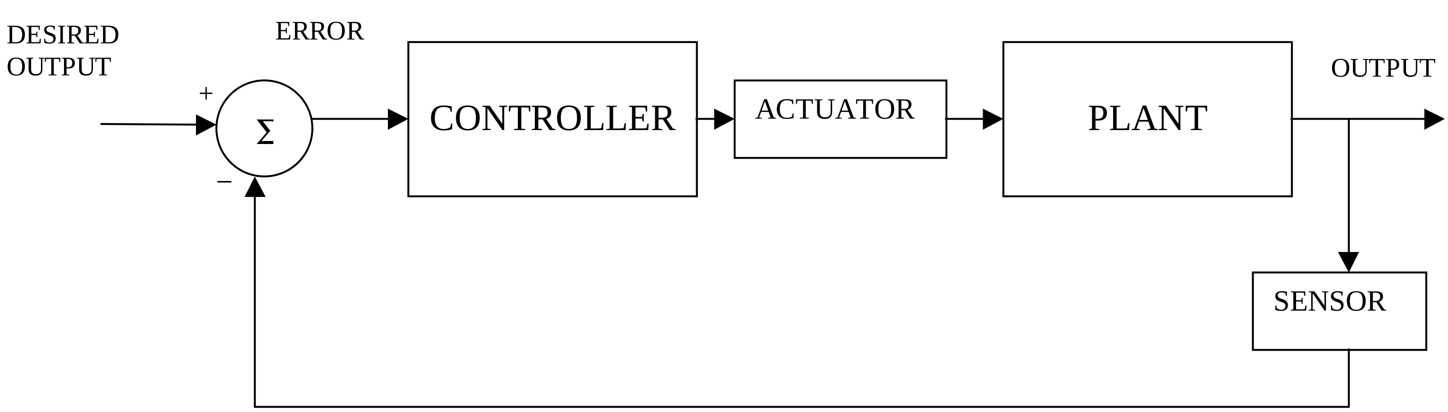
Part D: Control System Design

Open-loop and Closed-loop Control

a. Open – Loop



b. Closed – Loop



Some History

Date	*Technology*	*Problem*	*People*	*Method*
300 – 0 BC		Water clocks Oil lamps	Ktesibios Philon	
		Pneumatica	Heron of Alexandria	
16th – 17th century		Temperature and pressure regulators	Cornelis Drebbel D. Papin	
		Speed regulators	J. Watt I. Polzunov	
19 century	Steam engine	Stability	Maxwell	DE
		Stability	Ruth Hurwitz	
1920	Ship Steering	Stab/design	Minorsky	DE
1927-32	Feedback amps	Stab/design	Bode Nyquist	LT
1930's	Power drives	Stab/design	Brown	LT
1940's	Gun & radar syst.	Stab/design	Many	LT

1950's	Aircraft control	Stab/time resp.	Evans	Root locus
	General theory	'Optimal' control	Wiener Pontryakin	C of V
1960s	Aerospace	Multivariable State space Optimal Control	Kalman Bellman Russian work	SS
1970s	Industrial control	Disturbance rejection Computational methods & many others	Many	SS
1980s	Industrial control	Worst-case design Plant changes Robust control	Many	Operator theory

<u>Comments:</u>

- This is main stream – many other branches (computer control, chemical process control, etc.).

- Recent work has confirmed that the *techniques developed for SISO systems by Bode, Nyquist, Evans & others* are (when intelligently used) capable of producing excellent designs.

The Controller Design Process

Modeling

1. Model the system to be controlled.

2. Simplify the model, if necessary, so that it's tractable.

3. Analyze the model & simulate if necessary; decide what sensors and actuators are needed and where they should be placed.

4. (Usually) identify – by experiment – the values of model parameters.

5. Verify (by simulation and comparison with plant behavior) that the model adequately represents the plant – if not, repeat from 1.

6. Decide on performance specifications.

Design

7. Decide on the type of controller to be used.

8. Design a controller to meet the specs; if impossible or overly complex, repeat from 6.

Verification

9. Simulate the controlled system (computer model – step 1 – or pilot plant); if unsatisfactory, repeat from appropriate point.

Implementation

10. Choose hardware & software; implement and test the controller.

11. Tune the controller on-line if needed; train operators and maintainers to get best use out of the system.

MATHEMATICAL BACKGROUND

Complex Number $s = \sigma + j\omega$

Complex Function $G(s) = G_R + jG_I$

$$\frac{dG(s)}{ds} = \lim_{\Delta s \to 0} \frac{G(s + \Delta s) - G(s)}{\Delta s}$$

- G(s) is *analytic* in a region if G(s) and all its derivatives exist in this region.

- *Rational function*:

$$G(s) = \frac{A(s)}{B(s)} = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)}$$

$-z_i$: zeros, $-p_j$: poles, $m \leq n$

Rational functions are analytic in the s-plane except at isolated points called *singularities*.

Poles are singularities of G(s).

Laplace Transforms

Consider f(t), such that f(t) = 0 for t < 0

$$£\left[f(t)\right] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$£^{-1}\left[f(t)\right] = f(t) = \frac{1}{2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

£[*f(t)*] exists if:

1. f(t) is sectionally continuous in every finite interval in the range t > 0

2. f(t) is of exponential order as t approaches infinity, i.e., there exists a real and positive constant σ such that

$$e^{-\sigma t}\left|f(t)\right| \to 0 \qquad \text{for } t \to \infty \text{ and } \sigma > \sigma_c$$

where σ_c is the *abscissa of convergence.*

Remark 1: $f(t) = e^{t^2}$ for $0 \le t \le \infty$

→ does not have a Laplace transform

$f(t) = e^{t^2}$ for $0 \le t \le T < \infty$ and
$f(t) = 0$ for $t > T$

→ does have a Laplace transform

Remark 2:

$$£_{+}\left[f(t)\right] = \int_{0+}^{\infty} f(t)e^{-st}\,dt$$

$$£_{-}\left[f(t)\right] = \int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$£_{-}\left[f(t)\right] = £_{+}\left[f(t)\right] + \int_{0-}^{0+} f(t)e^{-st}\,dt$$

$£_{+}$ and $£_{-}$ are equal iff

$$\int_{0-}^{0+} e^{-st} f(t)\,dt = 0$$

Remark 3:

$$£_{+}\left[\delta(t)\right] = \int_{0+}^{\infty} \delta(t)e^{-st}\,dt = 0$$

$$£_{-}\left[\delta(t)\right] = \int_{0-}^{\infty} \delta(t)e^{-st}\,dt = 1$$

Laplace Transform Theorems

f(t) is a Laplace transformable function and

F(s) its Laplace trasform

- $\pounds\left[f(t-\alpha)\cdot u(t-\alpha)\right] = e^{-\alpha s}F(s)$ *u(t)* : step
- $\pounds\left[e^{-\alpha t} f(t)\right] = F(s+\alpha)$
- $\pounds\left[f\left(\frac{t}{\alpha}\right)\right] = \alpha F(\alpha s)$
- $\pounds\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$

$$\int_0^\infty f(t) e^{-st} dt = f(t)\frac{e^{-st}}{s}\Big|_0^\infty - \int_0^\infty \left(\frac{d}{dt} f(t)\right)\frac{e^{-st}}{-s} dt$$

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\pounds\left[\frac{d}{dt} f(t)\right]$$

- $\pounds\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots sf^{n-2}(0) - f^{n-1}(0)$
- $\pounds\left[\int f(t) dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t) dt\right]_{t=0}}{s}$

Final value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt} f(t)$ are Laplace transformable
- $\lim_{t\to\infty} f(t)$ exists
- F(s) analytic in Re(s) ≥ 0 except for a single pole at s = 0

Based on the Laplace Transform theorems:

$$\lim_{s \to 0} \int_0^\infty \left[\frac{d}{dt} f(t) \right] e^{-st}\, dt = \lim_{s \to 0} [sF(s) - f(0)]$$

since $\lim_{s \to 0} e^{-st} = 1$

→ $\int_0^\infty \frac{d}{dt} f(t) dt = f(\infty) - f(0) = \lim_{s\to 0} [sF(s) - f(0)]$

→ $f(\infty) = \lim_{s\to 0} sF(s)$

Initial value Theorem

$$f(0+) = \lim_{s \to \infty} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt} f(t)$ are both Laplace transformable
- $\lim_{s \to \infty} sF(s)$ exists

From the Laplace Transform Theorems:

$$\lim_{s\to\infty} £_{+}\left[\frac{d}{dt} f(t)\right] = \lim_{s\to\infty} \int_{0+}^{\infty}\left[\frac{d}{dt} f(t)\right] e^{-st} dt = 0$$

$$= \lim_{s\to\infty} sF(s) - f(0+) = 0$$

Convolution

$$f_1(t) \circ - - \bullet F_1(s)$$

$$f_2(t) \circ - - \bullet F_2(s)$$

$$f_3(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau = £^{-1}[F_1($$

Example:

Consider

$$f_1(t) = \begin{Bmatrix} 1 & for \quad 0 \le t \le T \\ 0 & else \end{Bmatrix}$$

$$f_2(t) = \begin{Bmatrix} 1 & for \quad 0 \le t \le \frac{T}{2} \\ 0 & else \end{Bmatrix}$$

Find $f_3(t)$

$$f_3(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau = £^{-1}[F_1(s) \cdot F_2(s)]$$

$$f_3(t)=\int_0^t d\tau = t \qquad \text{for} \qquad 0 \le t \le \frac{T}{2}$$

$$f_3(t)=\int_0^{\frac{T}{2}} d\tau = \frac{T}{2} \qquad \text{for} \qquad \frac{T}{2} \le t \le T$$

$$f_3(t)=\int_{t-T}^{\frac{T}{2}} d\tau = \frac{3T}{2} - t \qquad \text{for} \qquad T \le t \le \frac{3T}{2}$$

$$f_3(t)=0 \qquad \text{else}$$

Using $\quad f_3(t)=\pounds^{-1}\left[F_1(s)\cdot F_2(s)\right]$

$$F_1(s)=\frac{1-e^{-sT}}{s}$$

$$F_2(s)=\frac{1-e^{\frac{-sT}{2}}}{s}$$

$$F_3(s)=\frac{\left(1-e^{-sT}\right)\left(1-e^{\frac{-sT}{2}}\right)}{s^2}$$

$$f_3(t)=t-\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)-(t-T)u(t-T)+\left(t-\frac{3T}{2}\right)u\left(t-\frac{3T}{2}\right)$$

for $t \ge 0$ *u(t): step*

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

Consider

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)}$$

1. All roots are *distinct and real:*

$$F(s) = \frac{a_1}{s+p_1} + \ldots + \frac{a_n}{s+p_n}$$

$$a_k = \left[\frac{B(s)}{A(s)}(s+p_k)\right]_{s=-p_k}$$

$$£^{-1}\left[\frac{a_k}{s+p_k}\right] = a_k e^{-p_k t}$$

2. *Complex conjugate* poles

$$F(s)=\frac{B(s)}{A(s)}=\frac{a_1 s+a_2}{(s+p_1)(s+p_2)}+\frac{a_3}{s+p_3}+\ldots$$

$$p_1 = p_2 *$$

$$(a_1 s + a_2)_{s=-p_1} = \left[\frac{B(s)}{A(s)}(s+p_1)(s+p_2)\right]_{s=-p_1}$$

equating real and imaginary parts → α_1, α_2

Obtain *f(t)* using α_1, α_2 and the following table entries:

$$£[e^{-at}\cos\omega t]=\frac{s+a}{(s+a)^2+\omega^2}$$

$$£[e^{-at}\sin\omega t]=\frac{\omega}{(s+a)^2+\omega^2}$$

2. *Multiple real* poles

$$F(s)=\frac{B(s)}{A(s)}=\frac{B(s)}{(s+p_1)^r(s+p_{r+1})\ldots}$$

$$=\frac{b_r}{(s+p_1)^r}+\frac{b_{r-1}}{(s+p_1)^{r-1}}+\ldots+\frac{b_1}{s+p_1}+\frac{a_{r+1}}{s+p_{r+1}}\ldots$$

where

$$b_r = \left[\frac{B(s)}{A(s)} (s + p_1)^r \right]_{s=-p_1}$$

$$b_{r-1} = \left\{ \frac{d}{ds} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

$$b_{r-j} = \frac{1}{j!} \left\{ \frac{d^j}{ds^j} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

$$b_1 = \frac{1}{(r-1)!} \left\{ \frac{d^{r-1}}{ds^{r-1}} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

and

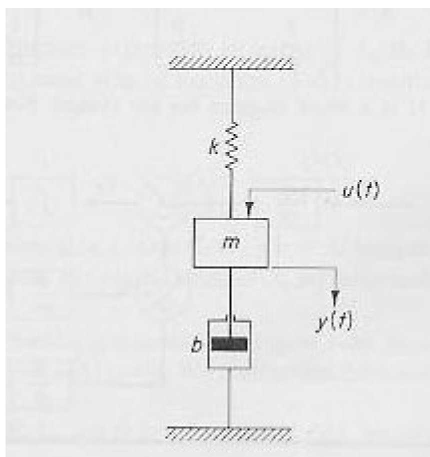
$$£^{-1}\left[\frac{1}{(s+p_1)^n} \right] = \frac{t^{n-1}}{(n-1)!} e^{-p_1 t}$$

Solution of linear differential equations using the Laplace transform

1. Take the Laplace transform of each term and convert the differential equation into an algebraic equation. Obtain the Laplace transform of the dependent variable.

2. The solution is obtained after inverse Laplace transform of the dependent variable.

Example: Simple mechanical system



$$m\ddot{y} + ky = u(t) \qquad\qquad m, k > 0$$

$$m\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + kY(s) = U(s)$$

$$Y(s) = \frac{U(s)}{ms^2 + k} + \frac{msy(0) + m\dot{y}(0)}{ms^2 + k}$$

for U(s) = 1/s

$$y(t) = \left(\frac{1}{k} - \frac{1}{k}\cos\sqrt{\frac{k}{m}}t\right) + \left(y(0)\cos\sqrt{\frac{k}{m}}t + \dot{y}(0)\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t\right)$$

Transfer function

$$a_0 \overset{n}{y} + \ldots + a_n y = b_0 \overset{m}{u} + \ldots b_m u \qquad m \leq n$$

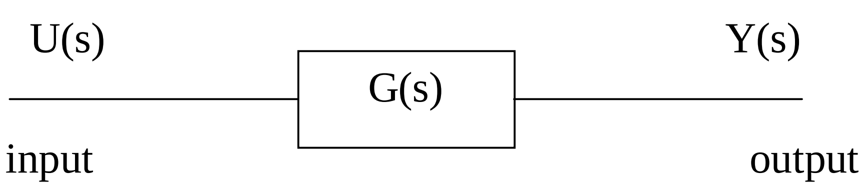
Take the Laplace transforms
(assuming zero initial conditions)

$$\frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} = \frac{b_0 s^m + \ldots + b_m}{a_0 s^n + \ldots + a_n} = G(s)$$

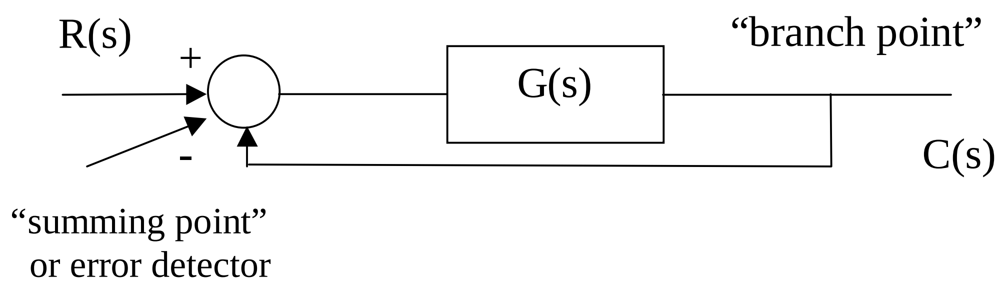
G(s) is the *transfer function* from U(s) to Y(s)

Block diagrams

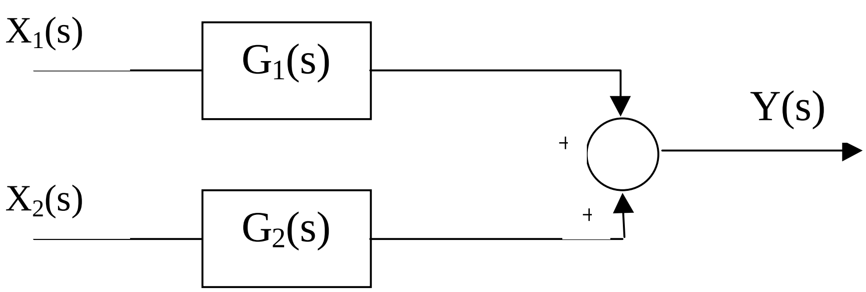
Open-loop



Closed-loop

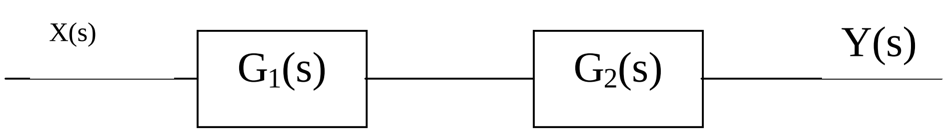


Parallel



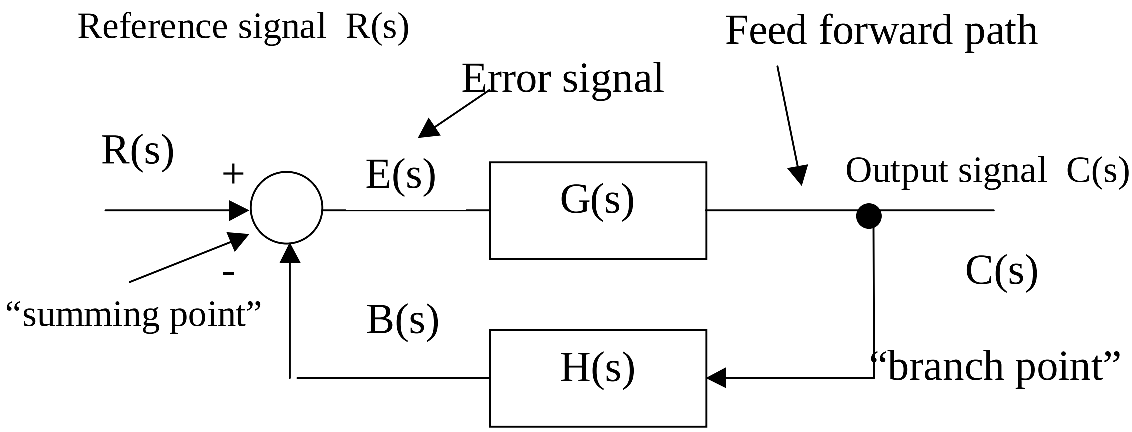
$$Y(s) = G_1(s)X_1(s) + G_2(s)X_2(s)$$

Series (Cascade)



$$Y(s) = G_1(s)G_2(s)X(s)$$

Closed-loop



Feed forward transfer function: $\frac{C(s)}{E(s)} = G(s)$

Feed back transfer function: $\frac{B(s)}{C(s)} = H(s)$

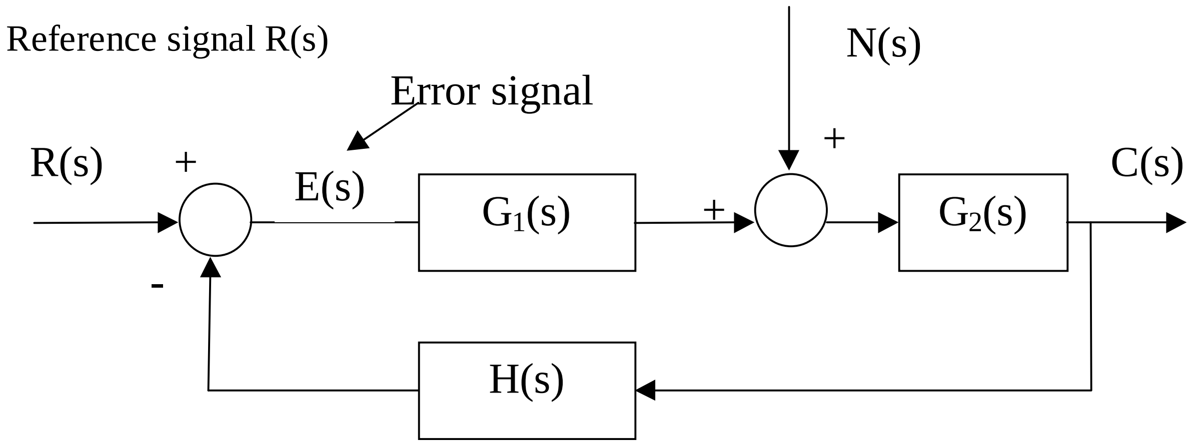
Open-loop transfer function: $\frac{B(s)}{E(s)} = G(s)H(s)$

Closed-loop transfer function: $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$C(s) = G(s)E(s) = G(s)[R(s) - B(s)] = G(s)R(s) - G(s)H(s)C(s)$$

$$\rightarrow \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Closed-loop subjected to disturbance



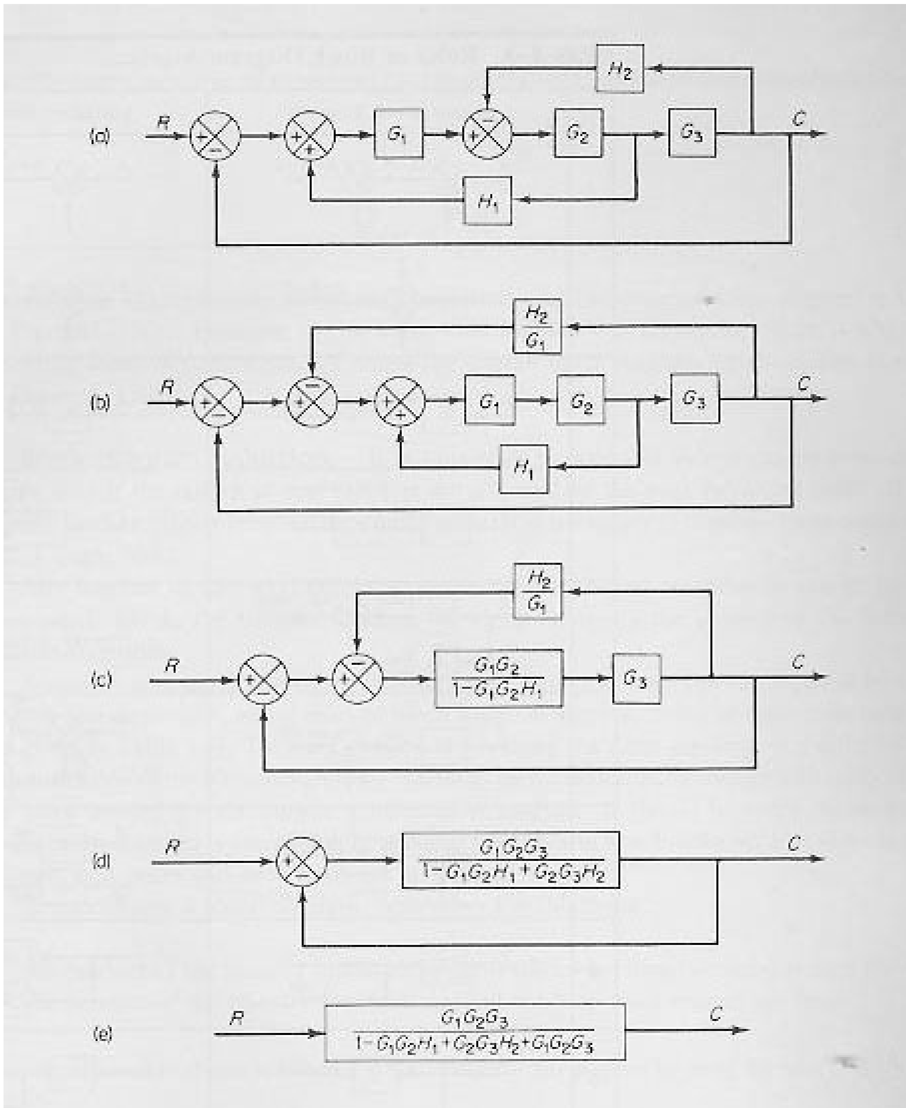
$$\frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$ assuming R(s) = 0

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$ assuming N(s) = 0

$$C(s) = C_N(s) + C_R(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + N(s)\right]$$

$$C(s) = \left[\frac{G_1(s)\cdot G_2(s)}{1+G_1(s)G_2(s)H(s)}, \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]\begin{bmatrix} R(s) \\ N(s) \end{bmatrix}$$

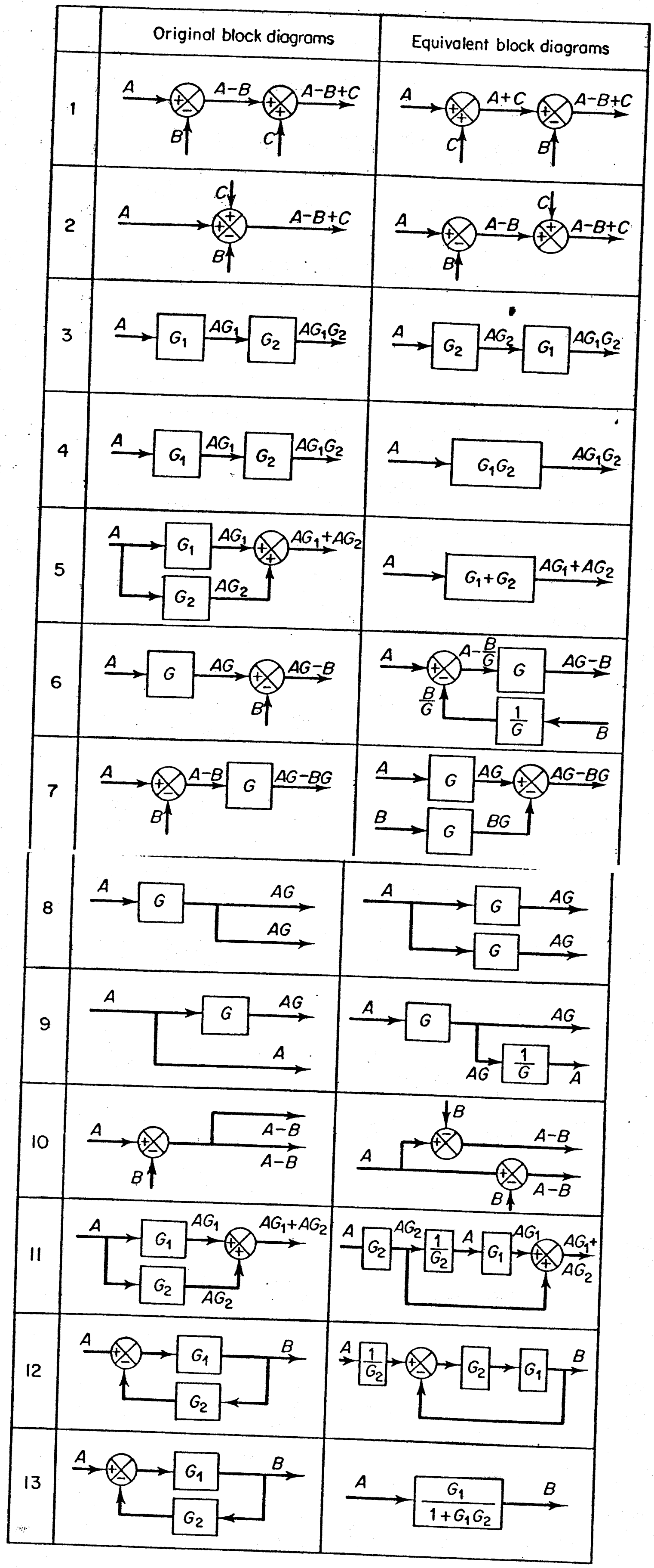
Block Diagram Reduction



(a) Multiple-loop system;

(b) -(e) successive reduction of the block diagram shown in (a)

Rules of Block Diagram Algebra

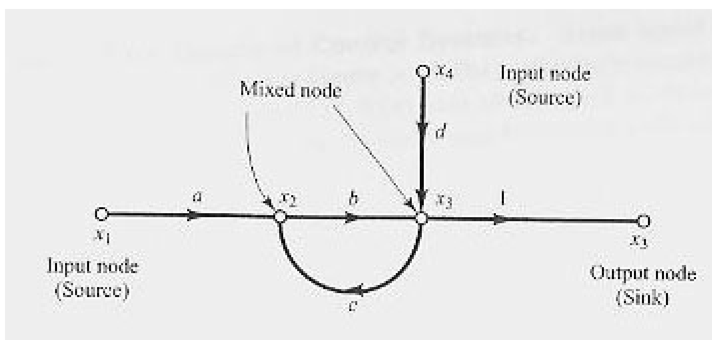


Signal flow Graphs (SFG)

A *Signal Flow Graph (SFG)* is a diagram which represents a set of simultaneous linear equations.

It consists of a network in which nodes are connected by directed branches.

Example:

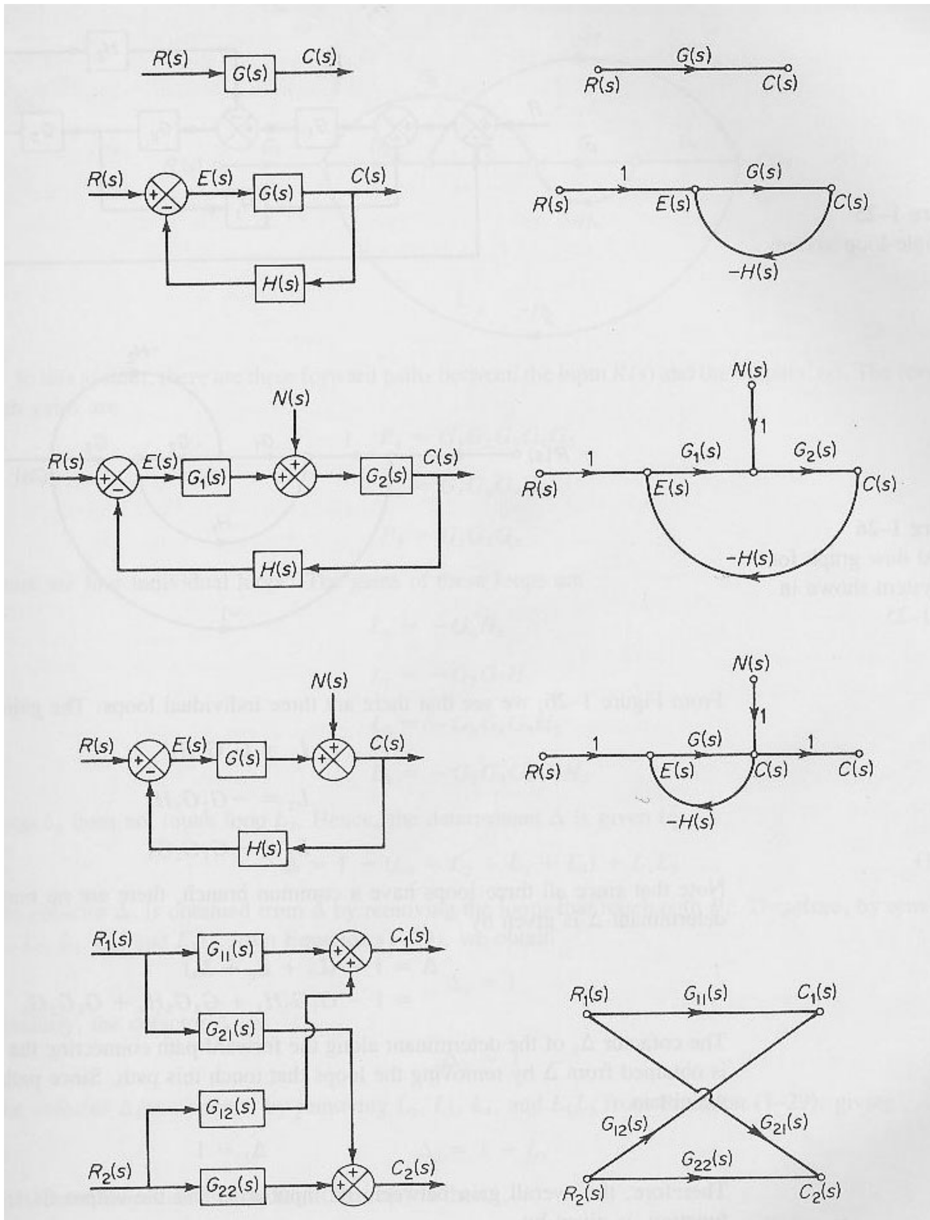


$$x_3 = abx_1 + dx_4 + bcx_3$$

Note that summations are taken over all possible paths from input to output

Definitions:

Node:	Point representing a variable or a signal.
Branch:	Gain between two nodes
Source:	A node without outgoing branches (input node)
Sink:	A node with incoming branches only (output nodes)
Mixed node:	Both incoming + outgoing branches
Path:	Connected branches in the direction of the branch arrows. If a node is crossed more than once, it is *closed*.
Forward path:	From input node to output node without crossing nodes more than once.
Loop:	Closed path.
Non touching loops:	Loops that do not possess common nodes.



Block diagrams and corresponding signal flow graphs

Properties of Signal Flow Graphs

1. A branch indicates the functional dependence of one signal on another. A signal passes through only in the direction specified by the arrow of the branch.

2. A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

3. A mixed node, which has both incoming and outgoing branches, may be treated as an output node (sink) by adding an outgoing branch of unity transmittance. However, we can not change a mixed node to a source by this method.

4. For a given system, the signal flow graph is not unique. Many different signal flow graphs can be drawn for a given system by writing the system equations differently.

Mason's Formula **for Signal Flow Graphs**

$$P = \frac{1}{\Delta} \sum_k P_k \Delta_k$$

P: Overall transmittance between input and output node.

Δ*: determinant of path* =

1 – (sum of all different loop gains)
+ (sum of gain products of all combinations of *2 non-touching loops*)
- (sum of gain products of all combinations of *3 non-touching loops*)
+…..

$$\Delta = 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \ldots$$

P_k: Gain of k^{th} forward path

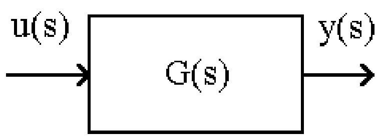
Δ_k: Cofactor of the k^{th} forward path.
Δ_k is determined as the determinant Δ but with the loops touching the k^{th} forward path removed.

STATE - SPACE DESCRIPTION

Input - Output

↕

Transfer Function

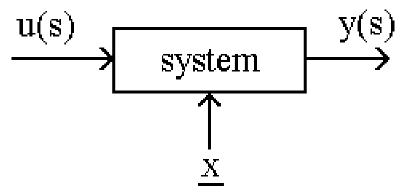


differential equation of order n

State - Space Description

↕

State vector $\underline{x} = [x_1, \ldots, x_n]^T$



n differential equations of first order

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{cx} + du$$

State: A set of variables $\underline{x}(t) = [x_1(t), \ldots, x_n(t)]^T$ such that the knowledge of these variables at $t = t_0$ together with the input u(t) for $t \geq t_0$ determines the behaviour of the system for any time $t \geq t_0$.

Minimal state: A set of variables $\underline{x}(t)$ with n minimal.

SISO (Single Input Single Output) system description:

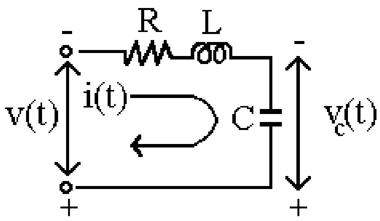
$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{cx} + du$$

$\underline{x}$: state vector

y: output
u: input

A: system matrix
$\underline{b}$: input vector
$\underline{c}$: output vector
d: direct input/output transmission

Example:



$$\underline{x} = [i(t), \quad v_c(t)]^T$$
$$y = v_C(t)$$
$$u = v(t)$$

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

$$C\frac{dv_c}{dt} = i$$

$$L\frac{di}{dt} + Ri + v_c = v$$

State-space description

$$\dot{x} = \begin{bmatrix} di/dt \\ dv_C/dt \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} i \\ v_c \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\underline{b}} v$$

$$y = v_c = \underbrace{[0 \quad 1]}_{\underline{c}} \begin{bmatrix} i \\ v_C \end{bmatrix} + \underbrace{[0]}_{d} v$$

Input-Output:

$$G(s) = \frac{V_C(s)}{V(s)}$$ from the above equations.

$$G(s) = d + \underline{c}\,(sI - A)^{-1}\,\underline{b}$$

Remark: One could also choose for state variables

$$z_1 = v_C(t) + Ri(t)$$
$$z_2 = v_C(t)$$

Choice of state variables is not unique.

Minimal number of state variables = order of system = order of differential equation is unique.

Nonuniqueness of State-Space Realizations:

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$
$$\Downarrow$$
$$\underline{x} = T\underline{z}$$

$\underline{z}$: new state

T : similarity transformation
det $T \neq 0$

$$\dot{\underline{z}} = T^{-1}AT\underline{z} + T^{-1}\underline{b}u$$
$$y = \underline{c}T\underline{z} + du$$

$$\dot{\underline{z}} = A_1\underline{z} + b_1u$$
$$y = \underline{c}_1\underline{z} + du$$

$$A_1 = T^{-1}AT$$
$$\underline{b}_1 = T^{-1}\underline{b}$$
$$\underline{c}_1 = \underline{c}T$$

Example (from previous page):

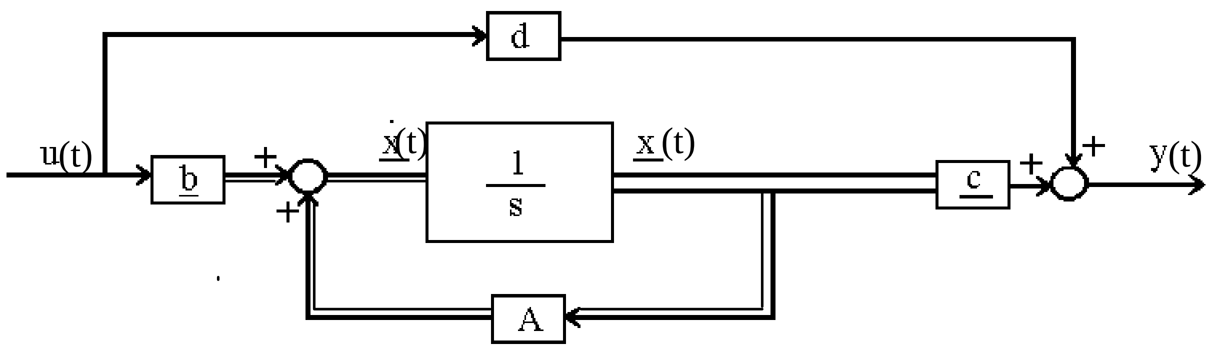
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} R & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^{-1}\underline{x}$$

$$T = \frac{1}{R}\begin{bmatrix} 1 & -1 \\ 0 & R \end{bmatrix}$$

CONTINUOUS STATE-SPACE EQUATIONS

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \qquad \underline{x} \in R^n$$

$$y = \underline{c}\underline{x} + du$$



STATE-SPACE REPRESENTATIONS OF CONTINUOUS SYSTEMS

Observable Canonical Form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{b_0 + b_1 s^{-1} + \ldots + b_n s^{-n}}{1 + a_1 s^{-1} + \ldots + a_n s^{-n}}$$

$$Y(s) = b_0 U(s) - s^{-1}[a_1 Y(s) - b_1 U(s)] - s^{-2}[a_2 Y(s) - b_2 U(s)] - \ldots - s^{-n}[a_n Y(s) - b_n U(s)]$$

$$Y(s) = b_0 U(s) + s^{-1}\{b_1 U(s) - a_1 Y(s) + s^{-1}(b_2 U(s) - a_2 Y(s) + s^{-1}(b_3 \ldots))\}$$

$$Y(s) = b_0 U(s) + X_n(s)$$

$$X_n(s) = s^{-1}[b_1U(s) - a_1Y(s) + X_{n-1}(s)]$$

$$X_{n-1}(s) = s^{-1}[b_2U(s) - a_2Y(s) + X_{n-2}(s)]$$

$$\vdots$$

$$X_2(s) = s^{-1}[b_{n-1}U(s) - a_{n-1}Y(s) + X_1(s)]$$

$$X_1(s) = s^{-1}[b_nU(s) - a_nY(s)]$$

replace Y(s) in the above equations with $Y(s) = b_0U(s) + X_n(s)$ and transform them in the time domain

$$\dot{x}_n = x_{n-1} - a_1x_n + (b_1 - a_1b_0)\,u$$

$$\dot{x}_{n-1} = x_{n-2} - a_2x_n + (b_2 - a_2b_0)\,u$$

$$\dot{x}_1 = -a_nx_n + (b_n - a_nb_0)\,u$$

and

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \ldots & \ldots & 0 & -a_n \\ 1 & & & & -a_{n-1} \\ 0 & & & & \vdots \\ 0 & & & & \vdots \\ 0 & \ldots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_nb_0 \\ \vdots \\ \vdots \\ \vdots \\ b_1 - a_1b_0 \end{bmatrix} u$$

$$y = [0, \quad \ldots \quad \ldots, \quad 0, \quad 1] \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + b_0u$$

<u>Special Case (Observability Canonical Form)</u>

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = u$$

Define the following state variables:

$$x_1 = y, \quad x_2 = \dot{y}, \quad \ldots, \quad x_n = y^{(n-1)}$$

Then

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \ldots, \quad \dot{x}_{n-1} = x_n$$

$$\dot{x}_n = y^{(n)} = -a_n x_1 - \ldots - a_1 x_n + u$$

and

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{c}\underline{x} + du$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & & : \\ : & & & . & : \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \ldots & \ldots & -a_1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ : \\ : \\ 1 \end{bmatrix}$$

$\underline{c} = [1 \quad 0 \quad .. \quad 0]$ and $d = 0$

Observability Canonical Form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots + b_{n-1}\dot{u} + b_n u$$

Define states:

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \quad \text{-->} \quad \dot{x}_1 = x_2 + \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \ldots - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \ldots - a_1 x_n + \beta_n u$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\vdots$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \ldots - a_{n-1} \beta_1 - a_n \beta_0$$

and

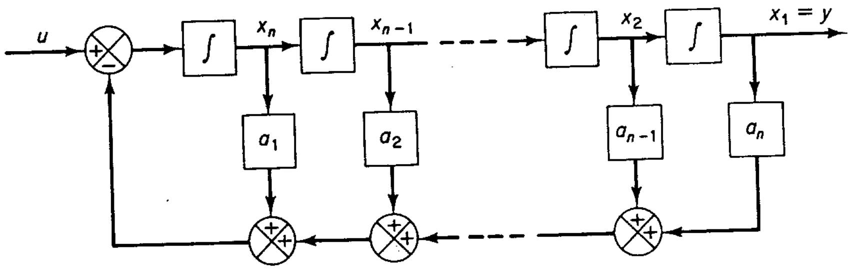
$$\underline{\dot{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{cx} + du$$

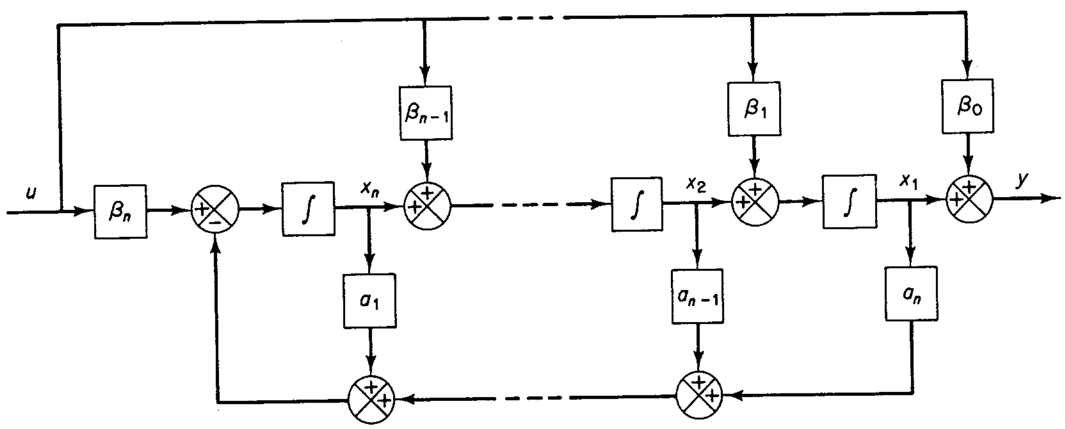
A and $\underline{c}$ as in the previous case,

$$\underline{b} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \qquad d = [\beta_0]$$

Special Case:



Observability Canonical Form



Comments:

- Choice of state variables is not unique

- Infinite many possibilities for a state-space description of a dynamic system described by a differential equation.

- Forms with special structure (like the above) are called <u>canonical forms.</u>

Input-Output Description from State-Space Description

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{c}\underline{x} + du$$

$$sX(s)=AX(s)+\underline{b}U(s)$$

$$(sI-A)X(s)=\underline{b}U(s)$$

$$X(s) = (sI - A)^{-1}\underline{b}U(s)$$

$$\frac{Y(s)}{U(s)} = \underline{c}(sI - A)^{-1}\underline{b} + d$$

$$Y(s) = [\underline{c}(sI - A)^{-1}\underline{b} + d]U(s)$$

Matrix inversion, special case n=2

use $$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

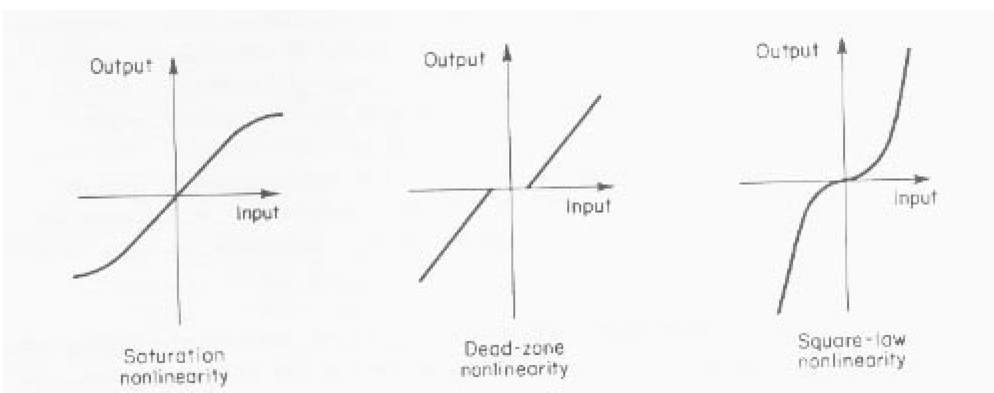
Mathematical Models

Model: The mathematical description of the dynamic characteristics of a system.

- Simplicity versus accuracy
- Time-variant versus time-invariant
- Linear versus nonlinear
  - Linear systems are described by linear differential equations.
  - Nonlinear, such as

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + (x-1)\frac{dx}{dt} = A\sin\ x$$

are more complicate.



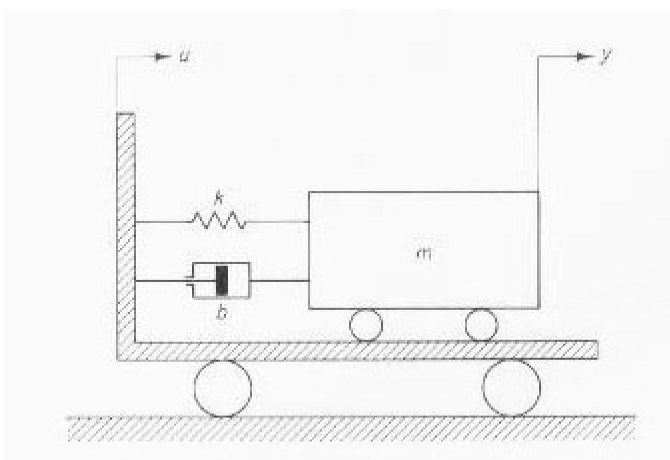
Examples of mathematical modes of nonlinear dynamic systems

Real Systems are nonlinear and time variant. However,
linear time-invariant approximations
around an operating point
are usually good approximations.

Mathematical Models of Simple Systems

Translational Mechanical Systems

Newton's Law: mass x acceleration = $\sum$ Forces



Spring-mass-dashpot system mounted on a cart

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y-u)$$

$$m\frac{d^2y}{dt}+b\frac{dy}{dt}+ky=b\frac{du}{dt}+ku$$

which leads to the following *transfer function*:

$$\frac{Y(s)}{U(s)}=G(s)=\frac{bs+k}{ms^2+bs+k}$$

or to the following *state-space model*

$$\ddot{y}+\frac{b}{m}\dot{y}+\frac{k}{m}y=\frac{b}{m}\dot{u}+\frac{k}{m}u$$

$$\alpha_1 \quad \alpha_2 \quad b_1 \quad b_2$$

Using

$$\beta_0=b_0=0$$

$$\beta_1=b_1-a_1\beta_0=\frac{b}{m}$$

$$\beta_2=b_2-a_1\beta_1-a_2\beta_0=\frac{k}{m}-\left(\frac{b}{m}\right)^2$$

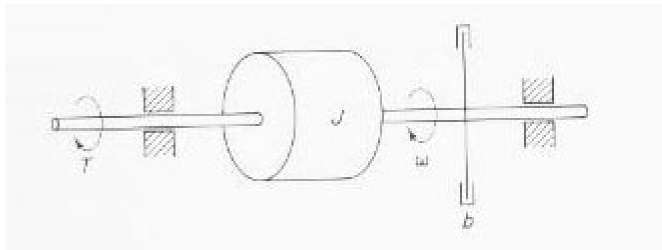
$$x_1=y-\beta_0 u=y$$

$$x_2=\dot{x}_1-\beta_1 u=\dot{x}_1-\frac{b}{m}u$$

we obtain

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}\frac{b}{m}\\ \frac{k}{m}-\left(\frac{b}{m}\right)^2\end{bmatrix}\cdot u \qquad y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Mechanical Rotational Systems



Mechanical rotational system

$$Ja = \sum T$$

Torque

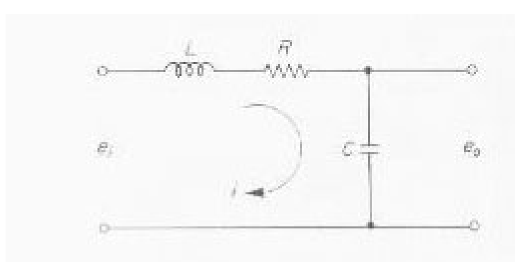
Angular acceleration

Inertia

$$J\dot{\omega} + f\omega = T \qquad \frac{\Omega(s)}{T(s)} = \frac{1}{Js + f}$$

Electrical Systems

RLC-Circuit



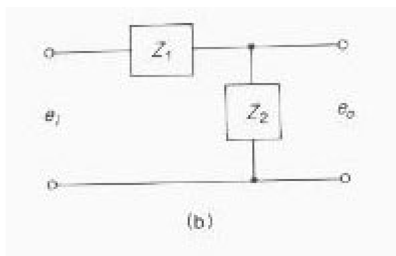
Simple RLC-circuit

The *transfer function* can be obtained from

$$\ddot{e}_0 + \frac{R}{L}\dot{e}_0 + \frac{1}{LC}e_0 = \frac{1}{LC}e_i$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

or from



Electrical circuit

$$Z_1 = Ls + R \qquad Z_2 = \frac{1}{LC}$$

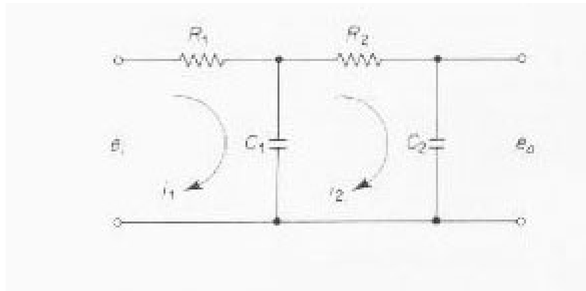
$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_i} = \frac{1}{LCs^2 + RCs + 1}$$

A *state-space model* is obtained using

$$x_1 = e_0 \quad x_2 = \dot{e}_0 \quad u = e_i \quad y = e_0 = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{LC} \end{bmatrix} \cdot u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Loading effect in series connection



RC Circiut

The transfer function can be obtained from

$$\frac{1}{C_1}\int(i_1 - i_2)dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int(i_2 - i_1)dt + R_2 i_2 \cdot \frac{-1}{C_2}\int i_2 dt = -e_0$$

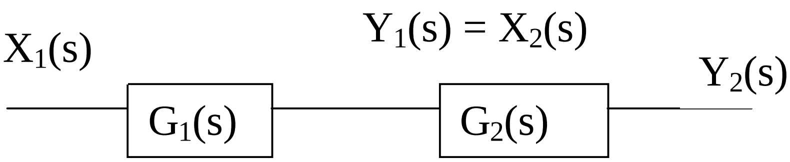
$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) - \frac{1}{C_2 s} I_2(s) = -E_0(s)$$

and

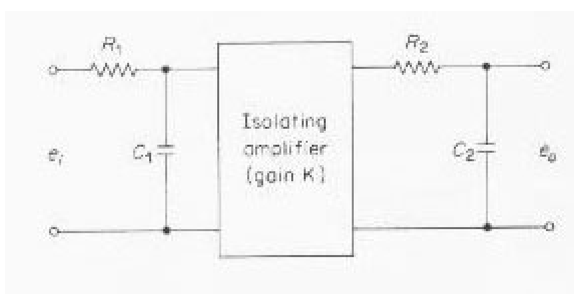
$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_1)s + 1}$$

Transfer functions of cascaded elements



$$\frac{Y_2(s)}{X_1(s)} = G_1(s)G_2(s)$$

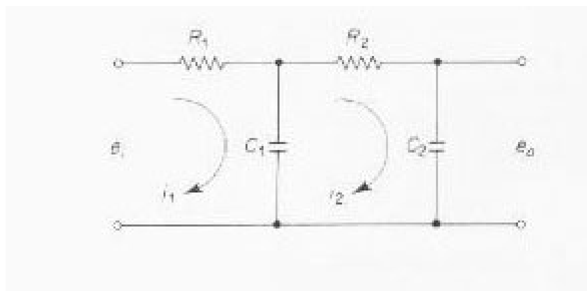
Example:



Cascade of two *non-loading* RC Citcuits

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{(R_1C_1s+1)}K\frac{1}{(R_2C_2s+1)}$$

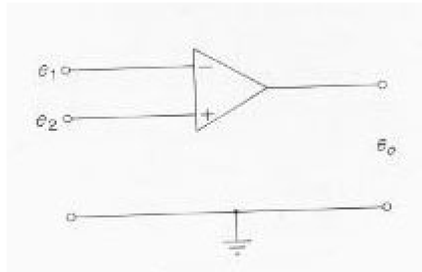
Note the difference in the transfer functions between the last and the previous circuit:



Cascade of two *loading* RC-Circuits

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_1)s + 1}$$

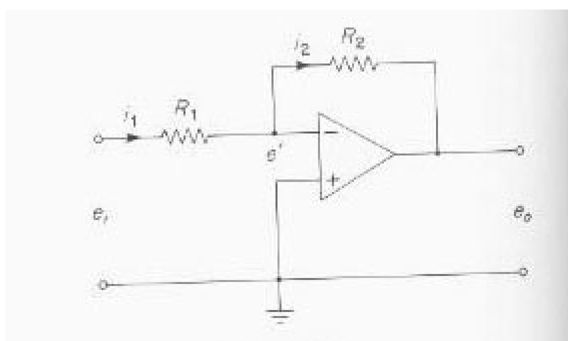
Operational Amplifiers



Operational amplifier

$$e_0 = K(e_2 - e_1) \qquad (K \text{ is} \sim 10^4)$$

Inverting amplifiers



Inverting amplifier

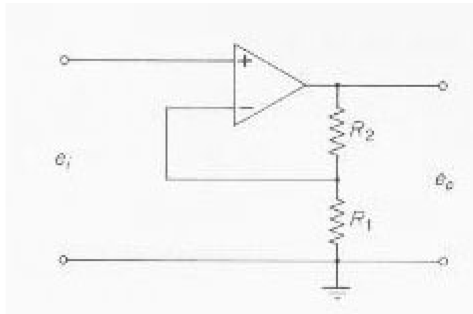
$$i_1 = \frac{e_i - e'}{R_1} \qquad\qquad i_2 = \frac{e' - e_0}{R_2}$$

Since $i_1 = i_2$, $e_0 = -K(0 - e')$ and $K >> 1$,

it follows that $e' = 0$ and

$$e_0 = -\frac{R_2}{R_1} e_i$$

Non-inverting amplifier



Non-inverting operational amplifier

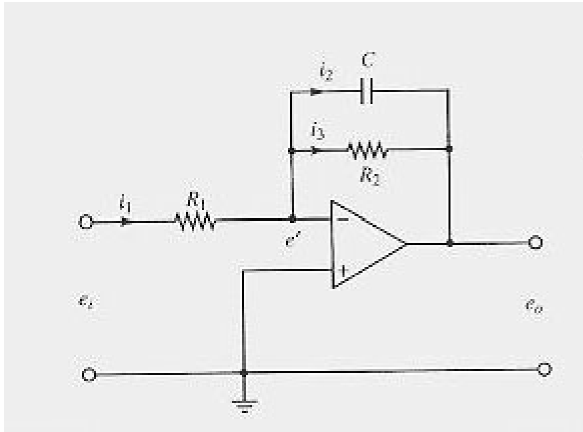
$$e_0 = Ke_i - \frac{KR_1}{R_1 + R_2} e_0$$

$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K} \right) e_0$$

Since K>>1 (K is ~ 10^4), and $\frac{R_1}{R_1 + R_2} >> \frac{1}{K}$

$$e_0 = \left(1 + \frac{R_2}{R_1} \right) e_i$$

First order circuit



First order lag circuit using operational amplifier

Since the current flowing into the amplifier is negligible

$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = C\frac{d(e' - e_0)}{dt} \qquad i_3 = \frac{e' - e_0}{R_2}$$

$i_1 = i_2 + i_3$

Hence,

$$\frac{e_i - e'}{R_1} = C\frac{d\left(e' - e_0\right)}{dt} + \frac{e' - e_0}{R_2}$$

Since $e' = 0$, we have

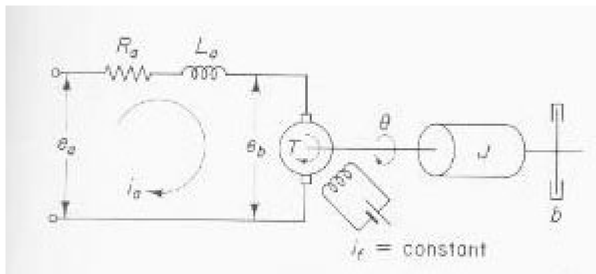
$$\frac{e_i}{R_1} = -C\frac{de_0}{dt} - \frac{e_0}{R_2} \qquad \text{and} \qquad \frac{E_i(s)}{R_1} = -\frac{R_2Cs + 1}{R_2}E_0(s)$$

leading to

$$\frac{E_0(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs + 1}$$

Electromechanical systems

Armature controlled DC Motors



Schematic diagram of an armature-controlled DC motor

Ψ:	Air gap flux (assumed constant)
T:	Torque
e_b:	Voltage induced in the armature due to rotation
i_a, i_f;	Armature and field currents

$\Psi = K_f i_f$ and $T = \Psi K_1 i_a = K_f K_1 i_f i_a = K_t i_a$

$$e_b = K_b \frac{d\theta(t)}{dt} = K_b \omega(t)$$

For the armature circuit, we have:

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a = V_s$$

For the mechanical part we have:

$$J\frac{d^2\theta}{dt} + f\frac{d\theta}{dt} = T = K_t i_a$$

The Laplace Transforms of the above three equations lead to

$$K_b s\Theta(s) = E_b(s)$$

$$(L_a s + R_a)I_a(s) + E_b(s) = E_a(s)$$

$$(Js^2 + fs)\Theta(s) = T(s) = K_t I_a(s)$$

and after eliminating $E_b(s)$ and $I_a(s)$, we have:

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_t}{s\left[L_a Js^2 + (L_a f + R_a J)s + R_a f + K_t K_b\right]}$$

For most small motors $L_a = 0$ → ignore it.

Using $\Omega(s) = s\Theta(s)$ and $E_a(s) = V_s(s)$

$$\frac{\Omega(s)}{V_s(s)} = \frac{K_t}{R_a Js + R_a f + K_t K_b} = \frac{K_m}{1 + s\tau_m}$$

where

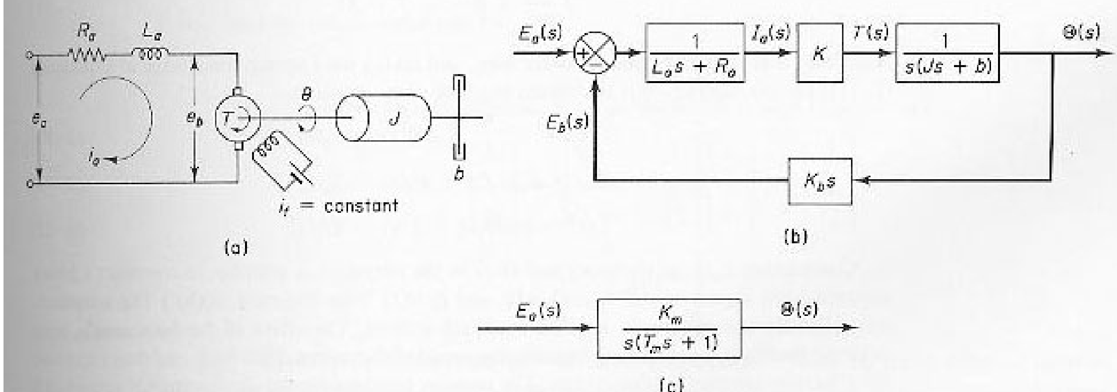
$$K_m = \frac{K}{R_a f + K_t K_b}$$

$$\tau_m = \frac{JR_a}{R_a f + K_t K_b}$$

Armature control of dc servomotors. Consider the armature-controlled dc servomotors shown in Figure 2–20(a), where the field current is held constant. In this system,

R_a = armature resistance, ohm
L_a = armature inductance, henry
i_a = armature current, ampere
i_f = field current, ampere
e_a = applied armature voltage, volt
e_b = back emf, volt
θ = angular displacement of the motor shaft, radian
T = torque developed by the motor, N-m
J = equivalent moment of inertia of the motor and load referred to the motor shaft, kg-m^2
b = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft, N-m/rad/sec

The torque T developed by the motor is proportional to the product of the armature current



(a) Schematic diagram of armature-controlled DC motor
(b) Block diagram obtained from Eqs.
(c) Simplified block diagram.

State-space description of armature controlled DC Motors

Form

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

we have

$$\ddot{\theta} + \frac{1}{\tau_m}\dot{\theta} = \frac{K_m}{\tau_m} e_a$$

Using the state variables

$$x_1 = \theta \quad \text{and} \quad x_2 = \dot{\theta}$$

and

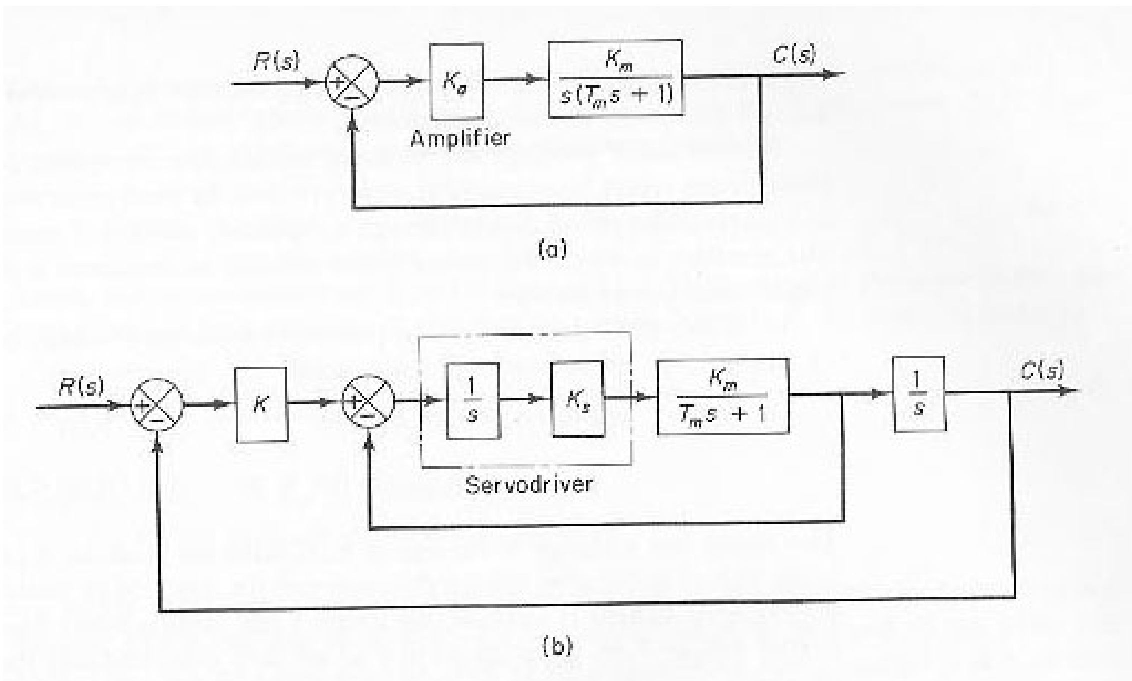
$$u = e_a \quad \text{and} \quad y = \theta = x_1$$

we obtain the following state-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{\tau_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_m}{\tau_m} \end{bmatrix} e_a$$

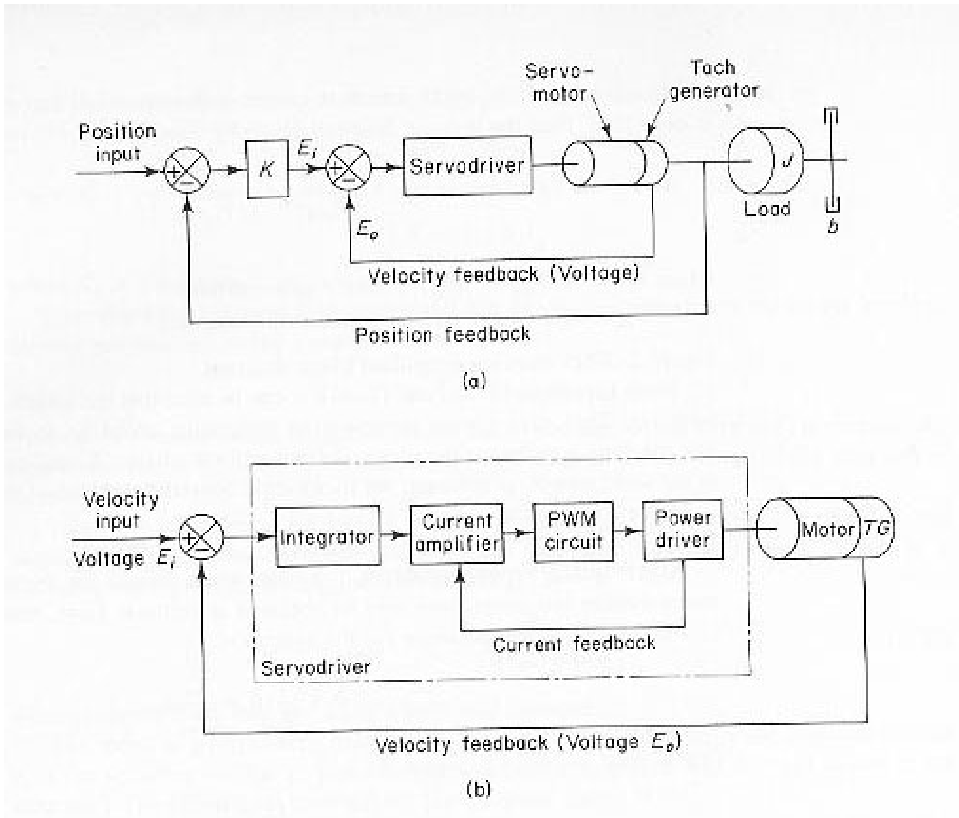
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Positional DC servo system



a) Simple, low-cost positional servo system.
b) High-speed, high-precision positional servo system.

Electronic motion control of DC servomotors



a) High-speed, high precision postional servo system with speed control using a servodriver, servomotor combination.
b) Functional diagram of a servodriver.

Linear approximation of non-linear systems

Consider the system described by:

y = f(x) where y: output
x: input
$(\bar{x}, \bar{y})$: operating point

The Taylor series expansion of y around $(x - \bar{x})$ gives

$$y = f(\bar{x}) + \left[\frac{df\,(y)}{dx}\right]_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2!}\left[\frac{d^2 f\,(y)}{dx^2}\right]_{x=\bar{x}} (x - \bar{x})^2 + \ldots$$

For small $(x - \bar{x})$ higher order derivatives are 0 and

$$y = \bar{y} + k(x - \bar{x})$$

is the linear approximation of the non-linear system

where

$$\bar{y} = f(\bar{x})$$

$$k = \left.\frac{df\,(y)}{dx}\right|_{x=\bar{x}}$$

Consider the system

$y = f(x_1,x_2)$ where y: output
x_1,x_2: inputs
$((\bar{x}_1,\bar{x}_2)\bar{y})$: operating point

The Taylor series expansion gives

$$y = f(\bar{x}_1,\bar{x}_2) + \left[\frac{\partial f(x_1,x_2)}{\partial x_1}\right]_{\substack{x_1=\bar{x}_1 \\ x_2=\bar{x}_2}} (x_1-\bar{x}_1) + \left[\frac{\partial f(x_1,x_2)}{\partial x_2}\right]_{\substack{x_1=\bar{x}_1 \\ x_2=\bar{x}_2}} (x_2-\bar{x}_2) + \ldots..$$

Since higher order terms can be considered 0

$$y = \bar{y} + k_1(x_1-\bar{x}_1) + k_2(x_2-\bar{x}_2)$$

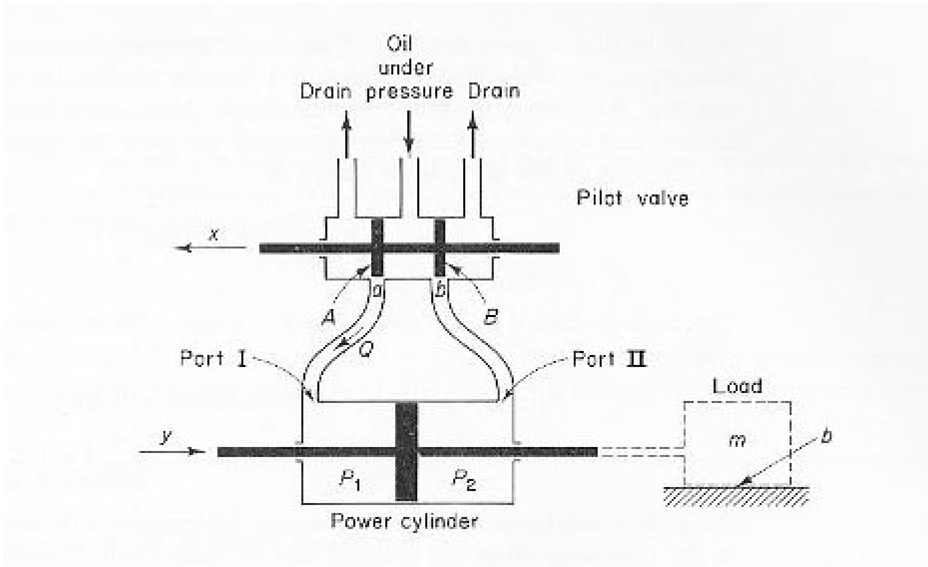
where

$$\bar{y} = f(\bar{x}_1,\bar{x}_2)$$

$$k_1 = \left.\frac{\partial f(x_1,x_2)}{\partial x_1}\right|_{\substack{x_1=\bar{x}_1 \\ x_2=\bar{x}_2}}$$

$$k_2 = \left.\frac{\partial y(x_1,x_2)}{\partial x_2}\right|_{\substack{x_1=\bar{x}_1 \\ x_2=\bar{x}_2}}$$

Example:



Schematic diagram of a hydraulic servomotor

Q :	Rate of flow to the power cylinder
$\Delta p = p_2 - p_1$:	Pressure difference in the two power cylinders
x :	Displacement of pilot valve

The relationship among the above variables is given by the nonlinear equation:

$$Q = f(x, \Delta p)$$

The linearized equation at $(\overline{Q}, \overline{x}, \Delta\overline{p}) = (0,0,0)$ is given by

$$Q - \overline{Q} = q = k_1(x - \overline{x}) - k_2(\Delta p - \Delta\overline{p})$$

where

$$k_1 = \left.\frac{\partial Q(x,\Delta p)}{\partial x}\right|_{\substack{x=\bar{x} \\ \Delta p=\Delta\bar{p}}} \qquad k_2 = \left.\frac{\partial Q(x,\Delta p)}{\partial \Delta p}\right|_{\substack{x=\bar{x} \\ \Delta p=\Delta\bar{p}}}$$

This gives

$$Q = k_1 x - k_2 \Delta p \qquad k_1, k_2 > 0$$

Using $Q\,dt = A\,p\,dy$ where

A:	Piston area
p:	Oil density
dy:	Displacement of mass

follows

$$\Delta p = \frac{1}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

The force developed by the power piston is given by:

$$A\Delta p = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

This force is applied to the mass m and including friction gives:

$$m\ddot{y} + f\,\dot{y} = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

where

f: friction coefficient

This gives:

$$m\ddot{y} + \left(f + \frac{A^2 p}{k_2} \right) \dot{y} = \frac{Ak_1}{k_2} x$$

Using

$$x \circ - - \bullet X(s)$$
$$y \circ - - \bullet Y(s)$$

where *X(s)* is the input
Y(s) is the output

$$\frac{Y(s)}{X(s)} = \frac{\frac{Ak_1}{k_2}}{s^2 m + \left(\frac{k_2 f + A^2 p}{k_2} \right) s} = \frac{k}{s(Ts+1)}$$

with

$$k = \frac{Ak_1}{k_2 f + A^2 p} \qquad T = \frac{mk_2}{k_2 f + A^2 p}$$