ROOT LOCUS

Consider the system

\[ \frac{C(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K \cdot G(s) \cdot H(s)} \]

Root locus presents the \textit{poles of the closed-loop system} when the gain \( K \) changes from 0 to \( \infty \)

\[ 1 + K \cdot G(s) \cdot H(s) = 0 \Rightarrow \begin{cases} |K \cdot G(s) \cdot H(s)| = 1 \quad \text{Magnitude Condition} \\ \angle G(s) \cdot H(s) = \pm 180^\circ \cdot (2k+1) \quad \text{Angle Condition} \\ k = 0, 1, 2, \ldots \]
Example:

\[ K \cdot G(s) \cdot H(s) = \frac{K}{s \cdot (s + 1)} \]

\[ 1 + K \cdot G(s) \cdot H(s) = 0 \implies s^2 + s + K = 0 \]

\[ s_{1,2} = -\frac{1}{2} \pm \frac{1}{2} \cdot \sqrt{1 - 4 \cdot K} \]

Angle condition:

\[ \angle \left( \frac{K}{s \cdot (s+1)} \right) = -\angle s - \angle (s+1) = -(180 - \theta_1) - \theta_2 = \pm 180 \]
Magnitude and Angle Conditions

\[ K \cdot G(s) \cdot H(s) = \frac{K \cdot (s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)} \]

\[ |K \cdot G(s) \cdot H(s)| = \frac{K \cdot B_1}{A_1 \cdot A_2 \cdot A_3 \cdot A_4} = 1 \]

\[ \angle G(s) \cdot H(s) = \varphi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 = \pm 180 \cdot (2 \cdot k + 1) \]

for \( k = 0, 1, 2, \ldots \)
Construction Rules for Root Locus

Open-loop transfer function:

\[ K H(s) \cdot G(s) = K \frac{B(s)}{A(s)} \]

m: order of open-loop numerator polynomial
n: order of open-loop denominator polynomial

**Rule 1: Number of branches**

The number of branches is equal to the number of poles of the open-loop transfer function.

**Rule 2: Real-axis root locus**

If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

**Rule 3: Root locus end-points**

The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=\(\infty\)) are at the open loop zeros and n-m branches terminate at infinity.
**Rule 4:** Slope of asymptotes of root locus as $s$ approaches infinity

\[
\gamma = \pm 180^\circ \cdot \left(2k + 1\right) \frac{n - m}{k = 0, 1, 2, ...}
\]

**Rule 5:** Abscissa of the intersection between asymptotes of root locus and real-axis

\[
\sigma_a = \frac{\sum_{i=1}^{n} (-p_i) - \sum_{i=1}^{m} (-z_i)}{n - m}
\]

(- $p_i$) = poles of open-loop transfer function
(- $z_i$) = zeros of open-loop transfer function
Rule 6: Break-away and break-in points

From the characteristic equation

\[ f(s) = A(s) + K \cdot B(s) = 0 \]

the break-away and -in points can be found from:

\[ \frac{dK}{ds} = - \frac{A'(s) \cdot B(s) - A(s) \cdot B'(s)}{B^2(s)} = 0 \]

Rule 7: Angle of departure from complex poles or zeros

Subtract from 180° the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

Rule 8: Imaginary-axis crossing points

Find these points by solving the characteristic equation for \( s = j\omega \) or by using the Routh’s table.
Rule 9: Conservation of the sum of the system roots

If the order of numerator is lower than the order of denominator by two or more, then the sum of the roots of the characteristic equation is constant.

Therefore, if some of the roots more towards the left as $K$ is increased, the other roots must more toward the right as $K$ in increased.
Discussion of Root Locus Construction Rules

Consider:

\[
K \cdot H(s) \cdot G(s) = K \cdot B(s) = K \cdot \frac{\sum_{i=0}^{m} b_i \cdot s^{m-i}}{\sum_{i=0}^{n} \alpha_i \cdot s^{n-i}}
\]

m: number of zeros of open-loop \( KH(s)G(s) \)

n: number of poles of open-loop \( KH(s)G(s) \)

Characteristic Equation: \( f(s) = A(s) + K \cdot B(s) = 0 \)
**Rule 1: Number of branches**

The characteristic equation has \( n \) zeros \( \Rightarrow \) the root locus has \( n \) branches

**Rule 2: Real-axis root locus**

Consider two points \( s_1 \) and \( s_2 \):

\[
\begin{align*}
\text{s}_1 \quad & \begin{cases}
\varphi_1 = 180, & \varphi_2 = 0 = \varphi_3, & \varphi_4 + \varphi_5 = 180.2 \\
\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 3.180
\end{cases}
\\
\text{s}_2 \quad & \begin{cases}
\varphi_1 = 180, & \varphi_2 = 180, & \varphi_3 = 0, & \varphi_4 + \varphi_5 = 360 \\
\varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 4.180
\end{cases}
\end{align*}
\]

Therefore, \( s_1 \) is on the root locus; \( s_2 \) is not.
Rule 3: Root locus end-points

Magnitude condition:

\[
\left| \frac{B(s)}{A(s)} \right| = \frac{1}{K} = \frac{\prod_{i=1}^{m} (s + z_i)}{\prod_{i=1}^{n} (s + p_i)}
\]

K=0 open loop poles
K=∞ m open loop zeros
m-n branches approach infinity

Rule 4: Slope of asymptotes of root locus as s approaches infinity

\[
K \cdot \frac{B(s)}{A(s)} = -1
\]

\[
\lim_{s \to \infty} \frac{K \cdot B(s)}{A(s)} = \lim_{s \to \infty} \frac{K}{s^{n-m}} = -1
\]

\[
s^{n-m} = -K \text{ for } s \to \infty
\]

Using the angle condition:

\[
\angle s^{n-m} = \angle -K = \pm 180^\circ \cdot (2 \cdot k + 1), \ k = 1, 2, 3, \ldots
\]

or
(n - m) \cdot \angle s = \pm 180^\circ \cdot (2 \cdot k + 1)

leading to

\[ \angle s = \gamma = \frac{\pm 180^\circ \cdot (2 \cdot k + 1)}{n - m} \]

**Rule 5: Abscissa of the intersection between asymptotes of root-locus and real axis**

\[
\frac{A(s)}{B(s)} = \frac{s^n + s^{n-1} \cdot \sum_{i=1}^{n} p_i + \ldots + \prod_{i=1}^{m} p_i}{s^m + s^{m-1} \cdot \sum_{i=1}^{m} z_i + \ldots + \prod_{i=1}^{n} z_i} = -K
\]

Dividing numerator by denominator yields:

\[
s^{n-m} - \left( \sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i \right) \cdot s^{n-m-1} + \ldots = -K
\]

For large values of s this can be approximated by:

\[
\left( \frac{\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i}{n - m} \right)^{n-m} = -K
\]

The equation for the asymptote (for \( s \to \infty \)) was found in Rule 4 as
\[ s^{n-m} = -K \]

\[
- \sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} -p_i - \sum_{i=1}^{m} z_i 
\]

this implies
\[
\sigma_{a} = -\frac{1}{n-m} \sum_{i=1}^{n} -p_i - \sum_{i=1}^{m} z_i 
\]

**Rule 6: Break-away and break-in points**

At break-away (and break-in) points the characteristic equation:

\[ f(s) = A(s) + K \cdot B(s) = 0 \]

has multiple roots such that:

\[
\frac{df(s)}{ds} = 0 \quad \Rightarrow \quad A'(s) + K \cdot B'(s) = 0 \quad \left(A'(s) = \frac{dA(s)}{ds}\right) 
\]

\[
\Rightarrow \quad \text{for} \quad K = -\frac{A'(s)}{B'(s)} , \quad f(s) \text{ has multiple roots} 
\]

Substituting the above equation into \( f(s) \) gives:

\[ A(s) \cdot B'(s) - A'(s) \cdot B(s) = 0 \]
Another approach is using:

\[ K = - \frac{A(s)}{B(s)} \quad \text{from} \quad f(s) = 0 \]

This gives:

\[ \frac{dK}{ds} = - \frac{A'(s) \cdot B(s) - A(s) \cdot B'(s)}{B^2(s)} \]

and break-away, break-in points are obtained from:

\[ \frac{dK}{ds} = 0 \]

**Extended Rule 6:**

Consider

\[ f(s) = A(s) + K \cdot B(s) = 0 \]

and

\[ K = - \frac{A(s)}{B(s)} \]

If the first \((y-1)\) derivatives of \(A(s)/B(s)\) vanish at a given point on the root locus, then there will be \(y\) branches approaching and \(y\) branches leaving this point.

The angle between two adjacent approaching branches is given by:

\[ \theta_y = \pm \frac{360^\circ}{y} \]
The angle between a leaving branch and an adjacent approaching branch is:

\[
\theta_y = \pm \frac{180^\circ}{y}
\]

**Rule 7: Angle of departure from complex pole or zero**

\[
\theta_2 = 90^\circ \\
\theta_3 = 180^\circ - (\theta_1 + \theta_2 - \varphi_1)
\]

**Rule 8: Imaginary-axis crossing points**

Example: \( f(s) = s^3 + b \cdot s^2 + c \cdot s + K \cdot d = 0 \)

\[
\begin{array}{c|ccc|c}
 s^3 & 1 & c & Kd \\
 s^2 & b & & \\
 s^1 & (bc-Kd)/b & Kd & \\
 s^0 & Kd & & \\
\end{array}
\]

For crossing points on the Imaginary axis:

\[
b \cdot c - K \cdot d = 0 \quad \Rightarrow \quad K = \frac{bc}{d}
\]

Further, \( b \cdot s^2 + K \cdot d = 0 \) leading to

\[
s_{1,2} = \pm j \cdot \sqrt{\frac{K \cdot d}{b}} = \pm j \omega
\]
The same result is obtained by solving \( f(j\omega) = 0 \).

**Rule 9: Conservation of the sum of the system roots**

From

\[
A(s) + K \cdot B(s) = \prod_{i=1}^{n} (s + r_i)
\]

we have

\[
\prod_{i=1}^{n} (s + p_i) + K \cdot \prod_{i=1}^{m} (s + r_i) = \prod_{i=1}^{n} (s + r_i)
\]

with

\[
A(s) = \prod_{i=1}^{n} (s + p_i) \quad \text{and} \quad B(s) = \prod_{i=1}^{m} (s + z_i)
\]

By equating coefficients of \( s^{n-1} \) for \( n \geq m + 2 \), we obtain the following:

\[
\sum_{i=1}^{n} -p_i = \sum_{i=1}^{n} -r_i
\]

i.e. the sum of closed-loop poles is independent of \( K \)!
Table 6-1  Open-Loop Pole–Zero Configurations and the Corresponding Root Loci
**Effect of Derivative Control and Velocity Feedback**

Consider the following three systems:

**Positional servo.**
Closed-loop poles:
\[ s = -0.1 \pm j \cdot 0.995 \]

**Positional servo with derivative control.**
Closed-loop poles:
\[ s = -0.5 \pm j \cdot 0.866 \]

**Positional servo with velocity feedback.**
Closed-loop poles:
\[ s = -0.5 \pm j \cdot 0.866 \]

Open-loop of system I:
\[ G_I(s) = \frac{5}{s \cdot (5 \cdot s + 1)} \]

Open-loop of systems II and III:
\[ G(s) = \frac{5 \cdot (1 + 0.8s)}{s \cdot (5 \cdot s + 1)} \]
Root locus for the three systems

a) System I

Closed-loop zero

b) System II
Closed-loop zeros:

System I: none
System II: 1+0.8s=0
System III: none

Observations:

- The root locus presents the *closed loop poles* but gives no information about *closed-loop zeros*.
- Two system with same root locus (same closed-loop poles) may have *different responses due to different closed-loop zeros*. 
Unit-step response curves for systems I, II and III:

The closed-loop transfer function of System III is

$$\frac{C_{III}(s)}{R(s)} = \frac{1}{(s + 0.5 + j0.866)(s + 0.5 - j0.866)}$$

For a unit-impulse input,

$$C_{III}(s) = \frac{j0.577}{s + 0.5 + j0.866} + \frac{-j0.577}{s + 0.5 - j0.866}$$

- The unit-step response of system II is the fastest of the three.
- This is due to the fact that derivative control responds to the rate of change of the error signal. Thus, it can produce a correction signal before the error becomes large. This leads to a faster response.
**Conditionally Stable Systems**

System which can be stable or unstable depending on the value of gain $K$.

**Minimum Phase Systems**

All poles and zeros are in the left half plane.
Frequency Response Methods

\[ x(t) = X \sin(\omega t) \]
\[ X(s) = \frac{\omega X}{s^2 + \omega^2} \]

\[ Y(s) = G(s) \cdot X(s) = G(s) \cdot \frac{\omega X}{s^2 + \omega^2} = \frac{a}{s + j\omega} + \frac{\bar{a}}{s - j\omega} + \sum_i \frac{b_i}{s + s_i} \]

\[ y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + \sum_i b_i e^{-s_i t} \]

stable system \iff \text{Re}(-s_i) < 0 \text{ for all } i

for \( t \to \infty \Rightarrow y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} \)

\[ a = G(s) \cdot \frac{\omega X}{s^2 + \omega^2} \bigg|_{(s = -j\omega)} = -\frac{X G(-j\omega)}{2j} \]

\[ \bar{a} = G(s) \cdot \frac{\omega X}{s^2 + \omega^2} \bigg|_{(s = j\omega)} = \frac{X G(j\omega)}{2j} \]
\[ G(j\omega) = |G(j\omega)| \cdot e^{j\varphi} \]

\[ \varphi = \tan^{-1}\left( \frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))} \right) \]

and

\[ G(-j\omega) = |G(j\omega)| \cdot e^{-j\varphi} \]

\[ y(t) = X \cdot |G(j\omega)| \cdot \frac{e^{j(\omega t + \varphi)} - e^{-j(\omega t + \varphi)}}{2j} = Y \cdot \sin(\omega t + \varphi) \]

\[ \varphi > 0 \text{ phase lag} \]

\[ \varphi < 0 \text{ phase lead} \]

\[ G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \]

\[ |G(j\omega)| = \left| \frac{Y(j\omega)}{X(j\omega)} \right| \quad \text{Magnitude response} \]

\[ \varphi = \angle(G(j\omega)) = \angle \left( \frac{Y(j\omega)}{X(j\omega)} \right) \quad \text{Phase response} \]
Connection between pole locations and Frequency Response

\[ G(s) = \frac{K(s+z)}{s(s+p)} \]

\[ |G(j\omega)| = \frac{|K| \cdot |j\omega + z|}{|j\omega| \cdot |j\omega + p|} \]

\[ \angle G(j\omega) = \phi - \theta_1 - \theta_2 \]

Frequency Response Plots

- Bode Diagrams
- Polar Plots (Nyquist Plots)
- Log-Magnitude-Versus-Phase Plots (Nichols Plots)
Bode Diagrams

• Magnitude response \[ |G(j\omega)| \]
  \[ 20 \log |G(j\omega)| \text{ in dB} \]

• Phase response \[ \angle G(j\omega) \text{ in degrees} \]

Basic factors of G(j\omega):

• Gain K
• Integral or derivative factors \((j\omega)^\pm I\)
• First-order factors \((1 + j\omega T)^\pm I\)
• Quadratic factors \[ \left(1 + 2j\frac{\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right)^\pm I \]

1. Gain Factor K

Horizontal straight line at magnitude \(20 \log(K) \text{ dB}\)

Phase is zero
2. **Integral or derivative factors** $(j\omega)^{\pm 1}$

- $(j\omega)^{-1}$

  \[
  20 \log \left| \frac{1}{j\omega} \right| = -20 \log \omega
  \]
  \[
  \text{magnitude: straight line with slope } -20 \text{ dB/decade}
  \]
  \[
  \text{phase: } -90^\circ
  \]

- $(j\omega)$

  \[
  20 \log |j\omega| = 20 \log \omega
  \]
  \[
  \text{magnitude: straight line with slope } 20 \text{ dB/decade}
  \]
  \[
  \text{phase: } +90^\circ
  \]

(a) Bode diagram of $G(j\omega) = 1/j\omega$

(b) Bode diagram of $G(j\omega) = j\omega$. 
3. **First order factors** \((1 + j\omega T)^{\pm 1}\)

- \((1 + j\omega T)^{-1}\)

**Magnitude:**

\[
20 \log \left| \frac{1}{1 + j\omega T} \right| = -20 \log \sqrt{1 + \omega^2 T^2} \text{ dB}
\]

for \(\omega << T^{-1} \Rightarrow 0 \text{ dB magnitude}\)

for \(\omega >> T^{-1} \Rightarrow -20 \log(\omega T) \text{ dB magnitude}\)

**Approximation of the magnitude:**

- for \(\omega\) between 0 and \(\omega = \frac{1}{T}\) 0 dB
- for \(\omega >> \frac{1}{T}\) straight line with slope –20 dB /decade

**Phase:**

\[
\angle (1 + j\omega T)^{-1} = -\tan^{-1}(\omega T)
\]

for \(\omega = 0\) \(\varphi = 0^\circ\)

for \(\omega = \frac{1}{T} \Rightarrow -\tan^{-1}(\frac{T}{T}) = 1\) \(\varphi = -45^\circ\)

for \(\omega = \infty\) \(\varphi = -90^\circ\)
• \((1 + j\omega T)^+ 1\)

Using \(20 \log |1 + j\omega T| = -20 \log \left| \frac{1}{1 + j\omega T} \right|\)

\[ \angle (1 + j\omega T) = \tan^{-1}(\omega T) = -\angle \left( \frac{1}{1 + j\omega T} \right) \]
4. Quadratic Factors

\[ G(j\omega) = \frac{1}{1 + 2\zeta j \left( \frac{\omega}{\omega_n} \right) + \left( \frac{j\omega}{\omega_n} \right)^2} \quad 0 < \zeta < 1 \]

**Magnitude:**

\[ 20 \log |G(j\omega)| = -20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 - \left(2 j \frac{\omega}{\omega_n}\right)^2} \]

for \( \omega << \omega_n \Rightarrow 0 \text{ dB} \)

for \( \omega >> \omega_n \Rightarrow -20 \log \left( \frac{\omega^2}{\omega_n^2} \right) = -40 \log \left( \frac{\omega}{\omega_n} \right) \text{ dB} \)

**Phase:**

\[ \varphi = \tan^{-1} \angle G(j\omega) = -\tan^{-1} \left[ \frac{2\zeta \frac{\omega}{\omega_n}}{1 - \left( \frac{\omega}{\omega_n} \right)^2} \right] \]

**Resonant Frequency:**

\[ \omega_r = \omega_n \sqrt{1 - 2\zeta^2} \]

**Resonant Peak Value:**

\[ M_r = \left| G(j\omega) \right|_{\text{max}} = \frac{1}{2\zeta \sqrt{1 - \zeta^2}} \]
Example:

\[ G(j\omega) = \frac{10(j\omega + 3)}{(j\omega)(j\omega + 2)((j\omega)^2 + j\omega + 2)} \]

\[ G(j\omega) = 7.5 \cdot \left( \frac{(j\omega)}{3} + 1 \right) \]

\[ G(j\omega) = \frac{j\omega}{2 + 1} \left( \frac{(j\omega)^2}{2} + \frac{j\omega}{2} + 1 \right) \]
Bode diagram of the system considered in Example 8–1.
**Frequency Response of non-Minimum Phase systems**

Minimum phase systems have all poles and zeros in the left half s-plane and were discussed before.

Consider

\[
A_1(s) = 1 + Ts \quad A_2(s) = 1 - Ts \quad A_3(s) = Ts - 1
\]

then

\[
|A_1(j\omega)| = |A_2(j\omega)| = |A_3(j\omega)|
\]

\[
\angle A_2(j\omega) = -\angle A_1(j\omega)
\]

\[
\angle A_3(j\omega) = 180^\circ - \angle A_1(j\omega)
\]

We know that

- phase of \( A_1(j\omega) \) from \( 0^\circ \) to \( +90^\circ \)
- phase of \( A_2(j\omega) \) from \( 0^\circ \) to \( -90^\circ \)
- phase of \( A_3(j\omega) \) from \( 180^\circ \) to \( +90^\circ \)
Phase-angle characteristic of the two systems $G_1(s)$ and $G_2(s)$ having the same magnitude response but $G_1(s)$ is minimum phase while $G_2(s)$ is not.

**Frequency Response of Unstable Systems**

Consider

$$G_1(s) = \frac{1}{1 + Ts}, \quad G_2(s) = \frac{1}{1 - Ts}, \quad G_3(s) = \frac{1}{Ts - 1}$$

then

$$|G_1(j\omega)| = |G_2(j\omega)| = |G_3(j\omega)|$$

$$\angle G_2(j\omega) = -\angle G_1(j\omega)$$

$$\angle G_3(j\omega) = -180^\circ - \angle G_1(j\omega)$$

We know that

- phase of $G_1(j\omega)$ from $0^\circ$ to $-90^\circ$
- phase of $G_2(j\omega)$ from $0^\circ$ to $+90^\circ$
- phase of $G_3(j\omega)$ from $-180^\circ$ to $-90^\circ$
Relationship between System Type and Log-Magnitude curve

Type of system determines:
- the slope of the log-magnitude curve at low frequencies
- for minimum phase, also the phase at low frequencies

Type 0

Position Error Coefficient \( K_p \neq 0 \)

- Slope at low frequencies: 0 dB/decade
- Phase at low frequencies (minimum phase): 0°

\[
\lim_{\omega \to 0} G(j\omega) = K_p
\]
Type 1

Velocity Error Coefficient \( K_v \neq 0 \) \( (K_p = \infty) \)

\[
K_v = \lim_{\omega \to 0} j\omega G(j\omega)
\]

\[
G(j\omega) = \frac{K_v}{j\omega} \quad \text{for} \quad \omega << 1
\]

\[
20\log K_v = 20\log \left| \frac{K_v}{j\omega} \right| \quad \text{for} \quad \omega = 1
\]

- Slope at low frequencies: -20 db/decade
- Phase at low frequencies (minimum phase): -90°
**Type 2**

Acceleration Error Coefficient $K_a$

$$K_a = \lim_{\omega \to 0} (j\omega)^2 G(j\omega) \neq 0$$

($K_p = K_v = \infty$)

$$G(j\omega) = \frac{K_a}{(j\omega)^2} \text{ for } \omega << 1$$

$$20\log K_a = 20\log \left| \frac{K_a}{(j\omega)^2} \right| \text{ for } \omega = 1$$

- Slope at low frequencies: -40 dB/decade
- Phase at low frequencies (minimum phase): -180°
**Polar Plots (Nyquist Plots)**

\[ G(j\omega) = |G(j\omega)| \cdot \angle G(j\omega) \]

\[ = \text{Re}[G(j\omega)] + \text{Im}[G(j\omega)] \]

**Advantage over Bode plots:** only one plot

**Disadvantage:** Polar plot of \( G(j\omega) = G_1(j\omega) \cdot G_2(j\omega) \) is more difficult to construct than its Bode plot.

**Basic factors of \( G(j\omega) \):**

**Integral or derivative factors** \((j\omega)^{\pm 1}\)

\[ G(j\omega) = \frac{1}{j\omega} = -j \frac{1}{\omega} = \frac{1}{\omega} \angle -90^\circ \]

\[ G(j\omega) = j\omega = \omega \angle 90^\circ \]
First order factors \((1 + j\omega T)^{\pm 1}\)

\[
G(j\omega) = \frac{1}{1+j\omega T} = X + jY
\]

\[
X = \frac{1}{1 + \omega^2 T^2}, \quad Y = \frac{-\omega T}{1 + \omega^2 T^2}
\]

It can be show that \((X - 0.5)^2 + Y^2 = (0.5)^2\)

\[\Rightarrow\] Polar plot is a circle with
- Center \((1/2, 0)\) and
- Radius 0.5.
**Quadratic Factors**

\[
G(j\omega) = \frac{1}{1 + 2\zeta j \left( \frac{\omega}{\omega_n} \right) + \left( \frac{j \omega}{\omega_n} \right)^2}
\]

\[1 > \zeta > 0\]

\[
\lim_{\omega \to 0} G(j\omega) = 1 \angle 0^\circ
\]

\[
\lim_{\omega \to \infty} G(j\omega) = 0 \angle -180^\circ
\]

\[
G(j\omega_n) = \frac{1}{j2\zeta} \angle -90^\circ
\]

---

![Diagram](image)
\[
\left(1 + 2\zeta \left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2\right)^{-1}
\]

\[
\lim_{\omega \to 0} G(j\omega) = 1 \angle 0^\circ
\]

\[
\lim_{\omega \to \infty} G(j\omega) = \infty \angle 180^\circ
\]
General shapes of polar plots

\[ G(j\omega) = \frac{K(1 + j\omega T_1) \cdots (1 + j\omega T_m)}{(j\omega)^\lambda (1 + j\omega T_{\lambda+1}) \cdots (1 + j\omega T_n)} \]

n = order of the system (denominator)
λ = type of system \( \lambda > 0 \)
m = order of numerator \( n > m \)

For low frequencies: The phase at \( \omega \to 0 \) is \( \lambda (-90^\circ) \)

For system type 1, the low frequency asymptote is obtained by taking:

\[ \text{Re}[G(j\omega)] \text{ for } \omega \to 0 \]

For high frequencies: The phase is: \( (n - m) (-90^\circ) \)
Log-Magnitude-Versus-Phase Plots (Nichols Plots)
Example:

Frequency Response of a quadratic factor

The same information presented in three different ways:

- Bode Diagram
- Polar Plot
- Log-Magnitude-Versus-Phase Plots

Three representations of the frequency response of
\[
\frac{1}{1 + 2\zeta \left( \frac{\omega}{\omega_0} \right) + \left( \frac{\omega}{\omega_0} \right)^2}, \text{ for } \zeta > 0.
\]

(a) Bode diagram; (b) polar plot; (c) log-magnitude versus phase plot.
Nyquist Stability Criterion

The Nyquist stability criterion relates the stability of the \textit{closed loop} system to the frequency response of the \textit{open-loop} system.

\[
\text{Open-loop: } \quad G(s) \cdot H(s)
\]

\[
\text{Closed-loop: } \quad \frac{G(s)}{1 + G(s) \cdot H(s)}
\]

\textbf{Advantages of the Nyquist Stability Criterion:}

- Simple graphical procedure to determine whether a system is stable or not
- The degree of stability can be easily obtained
- Easy for compensator design
- The response for steady-state sinusoidal inputs can be easily obtained from measurements
**Preview**

**Mathematical Background**

- Mapping theorem
- Nyquist path

**Nyquist stability criterion**

\[ Z = N + P \]

\( Z \): Number of zeros of \((1 + H(s)G(s))\) in the right half plane = number of unstable poles of the closed-loop system

\( N \): Number of clockwise encirclements of the point \(-1 + j0\)

\( P \): Number of poles of \(G(s)H(s)\) in the right half plane

**Application of the Stability Criterion**

- Sketch the Nyquist plot for \( \omega \in (0^+, \infty) \)
- Extend to \( \omega \in (-\infty, +\infty) \)
- Apply the stability criterion (find \( N \) and \( P \) and compute \( Z \)).
Mapping Theorem

The total number \( N \) of clockwise encirclements of the origin of the \( F(s) \) plane, as a representative point \( s \) traces out the entire contour in the clockwise direction, is equal to \( Z - P \).

\[
F(s) = \frac{A(s)}{Q(s)}
\]

\( P: \) Number of poles, \( Q(s) = 0 \)
\( Z: \) Number of zeros, \( A(s) = 0 \)
Mapping for $F(s) = s/(s+0.5)$, $(Z = P = 1)$

Example of Mapping theorem $(Z - P = 2)$.

Example of Mapping theorem $(Z - P = -1)$. 
Application of the mapping theorem to stability analysis

Mapping theorem: The number of clockwise encirclements of the origin is equal to the difference between the zeros and poles of \( F(s) = 1 + G(s)H(s) \).

Zeros of \( F(s) \) = poles of closed-loop system
Poles of \( F(s) \) = poles of open-loop system
Frequency response of open-loop system: $G(j\omega)H(j\omega)$

Frequency response of a type 1 system
Nyquist stability criterion

Consider

\[ R(s) + \frac{G(s)}{1+H(s)G(s)} H(s) \rightarrow C(s) \]

The Nyquist stability criterion states that:

\[ Z = N + P \]

- **Z**: Number of zeros of \( 1+H(s)G(s) \) in the right half s-plane = number of poles of closed-loop system in right half s-plane.
- **N**: Number of clockwise encirclements of the point \(-1+j0\) (when tracing from \( \omega = -\infty \) to \( \omega = +\infty \)).
- **P**: Number of poles of \( G(s)H(s) \) in the right half s-plane

Thus:

- if \( Z = 0 \) \( \rightarrow \) closed-loop system is stable
- if \( Z > 0 \) \( \rightarrow \) closed-loop system has \( Z \) unstable poles
- if \( Z < 0 \) \( \rightarrow \) impossible, a mistake has been made
Alternative form for the Nyquist stability criterion:

If the open-loops system $G(s)H(s)$ has $k$ poles in the right half $s$-plane, then the closed-loop system is stable if and only if the $G(s)H(s)$ locus for a representative point $s$ tracing the modified Nyquist path, encircles the $-1+j0$ point $k$ times in the counterclockwise direction.

**Frequency Response of $G(j\omega)H(j\omega)$**

*for $\omega : (-\infty, +\infty)$*

a) $\omega : (0^+, +\infty)$ : using the rules discussed earlier

b) $\omega : (0^-, -\infty)$ : $G(-j\omega)H(-j\omega)$ is symmetric with $G(j\omega)H(j\omega)$ (real axis is symmetry axis)

c) $\omega : (0^-, 0^+)$ : next page
Poles at the origin for $G(s)H(s)$:

$$G(s) \cdot H(s) = \frac{(\ldots)}{s^\lambda (\ldots)}$$

If $G(s)H(s)$ involves a factor $\frac{1}{s^\lambda}$, then the plot of $G(j\omega)H(j\omega)$, for $\omega$ between $0^-$ and $0^+$, has $\lambda$ clockwise semicircles of infinite radius about the origin in the $GH$ plane. These semicircles correspond to a representative point $s$ moving along the Nyquist path with a semicircle of radius $\varepsilon$ around the origin in the $s$ plane.
Relative Stability

Consider a modified Nyquist path which ensures that the closed-loop system has no poles with real part larger than $-\sigma_0$:

Another possible modified Nyquist path:

Figure 8–69
Modified Nyquist path.
Phase and Gain Margins

A measure for relative stability of the closed-loop system is how close \( G(j\omega) \), the frequency response of the open-loop system, comes to \(-1+j0\) point. This is represented by phase and gain margins.

**Phase margin**: The amount of additional phase lag at the Gain Crossover Frequency \( \omega_o \) required to bring the system to the verge of instability.

Gain Crossover Frequency: \( \omega_o \) for which \( |G(j\omega_o)| = 1 \)
Phase margin: \( \gamma = 180^\circ + \angle G(j\omega_o) = 180^\circ + \phi \)

**Gain margin**: The reciprocal of the magnitude \( G(j\omega) \) at the Phase Crossover Frequency \( \omega_1 \) required to bring the system to the verge of instability.

Phase Crossover Frequency: \( \omega_1 \) where \( \angle G(j\omega_1) = -180^\circ \)
Gain margin: \( K_g = \frac{1}{|G(j\omega_1)|} \)

Gain margin in dB: \( K_g \text{ in dB} = -20 \log |G(j\omega_1)| \)

\( K_g \text{ in dB} > 0 = \text{stable (for minimum phase systems)} \)
\( K_g \text{ in dB} < 0 = \text{unstable (for minimum phase systems)} \)
Figure: Phase and gain margins of stable and unstable systems (a) Bode diagrams; (b) Polar plots; (c) Log-magnitude-versus-phase plots.
If the open-loop system is minimum phase and has both phase and gain margins positive,

→ then the closed-loop system is stable.

- For good relative stability both margins are required to be positive.

- Good values for minimum phase system:
  - Phase margin: 30° – 60°
  - Gain margin: above 6 dB
Correlation between damping ratio and frequency response for 2nd order systems

\[
\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

\[
\frac{C(j\omega)}{R(j\omega)} = M(\omega) \cdot e^{j\alpha(\omega)}
\]

Phase margin:

\[
\gamma = 180^\circ + \angle G(j\omega)
\]

\(G(j\omega)\): open loop transfer function

\[
|G(j\omega)| = \left| \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)} \right| \quad \text{becomes unity for} \quad \omega_1 = \omega_n \sqrt{1 + 4\zeta^4 - 2\zeta^2}
\]

and

\[
\gamma = \tan^{-1}\left[\frac{2\zeta\omega_n}{\omega_1}\right] = \tan^{-1}\left[\frac{2\zeta}{\sqrt{1 + \zeta^4 - 2\zeta^2}}\right]
\]

\[\Rightarrow \gamma \text{ depends only on } \zeta\]
Performance specifications in the frequency domain:

\[ 0 \leq \omega \leq \omega_c : \text{Bandwidth} \]

Slope of log-magnitude curve: \textit{Cutoff Rate}

- ability to distinguish between signal and noise

\[ \omega_r : \text{Resonant Frequency} \]

- indicative of transient response speed
- \( \omega_r \rightarrow \) increase, transient response faster (dominant complex conjugate poles assumed)

\[ M_r = \max |G(j\omega)| : \text{Resonant Peak} \]
Closed-Loop Frequency Response

Open-loop system: \( G(s) \)

Stable closed-loop system: \[
\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}
\]
Closed-Loop Frequency Response:

\[
\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)} = M \cdot e^{j\alpha}
\]

Constant Magnitude Loci:

\[
G(j\omega) = X + jY
\]

\[
M = \frac{|X + jY|}{|1 + X + jY|} = \text{const}
\]

\[
\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{(M^2 - 1)^2}
\]

Constant Phase-Angle Loci

\[
G(j\omega) = X + jY \rightarrow \angle e^{j\alpha} = \angle \frac{X + jY}{1 + X + jY} = \text{const}
\]

\[
\left(X + \frac{1}{2}\right)^2 + \left(Y + \frac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2, \quad N = \tan \alpha
\]
Figure: A family of constant M circles.
Figure:
(a) $G(j\omega)$ locus superimposed on a family of M circles;
(b) $G(j\omega)$ locus superimposed on a family of N circles;
(c) Closed-loop frequency-response curves
Experimental Determination of Transfer Function

- Derivation of mathematical model is often difficult and may involve errors.

- Frequency response can be obtained using sinusoidal signal generators.

  Measure the output and obtain:
  - Magnitudes (quite accurate)
  - Phase (not as accurate)

Use the Magnitude data and asymptotes to find:

- Type and error coefficients
- Corner frequencies
- Orders of numerator and denominator
- If second order terms are involved, $\zeta$ is obtained from the resonant peak.

Use phase to determine if system is minimum phase or not:

- Minimum phase: $\omega \to \infty$ phase = -90 (n - m)
  (n-m) difference in the order of denominator and numerator.