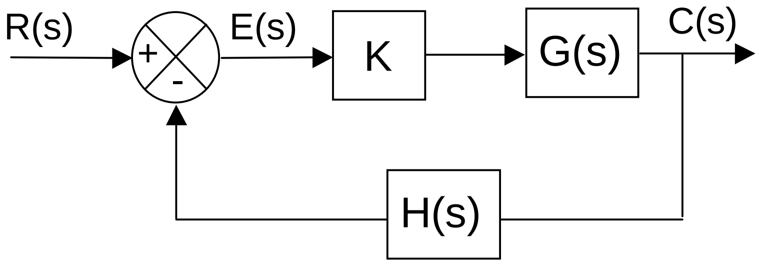
# ROOT LOCUS

Consider the system



$$\frac{C(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K \cdot G(s) \cdot H(s)}$$

**Root locus** presents the *poles of the closed-loop system*

when the gain K changes from 0 to ∞

$$1 + K \cdot G(s) \cdot H(s) = 0 \Rightarrow \begin{cases} \left| K \cdot G(s) \cdot H(s) \right| = 1 & \textit{Magnitude Condition} \\ \angle G(s) \cdot H(s) = \pm 180^\circ \cdot (2k+1) & \textit{Angle Condition} \end{cases}$$

$$k = 0, 1, 2, \ldots$$

## Example:

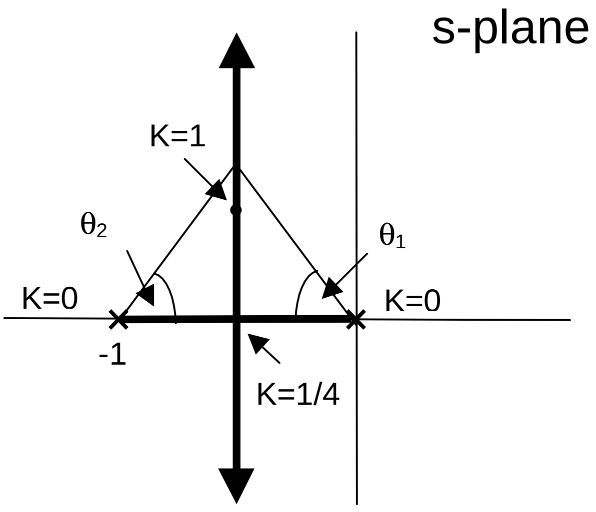
$$K \cdot G(s) \cdot H(s) = \frac{K}{s \cdot (s+1)}$$

$$1 + \mathrm{K} \cdot \mathrm{G(s)} \cdot \mathrm{H(s)} = 0 \Rightarrow s^2 + s + K = 0$$

$$s_{1,2} = -\frac{1}{2} \pm \frac{1}{2} \cdot \sqrt{1 - 4 \cdot K}$$

## Angle condition:

$$\angle\left(\frac{K}{s \cdot (s+1)}\right) = -\angle s - \angle s+1 = -(180 - \theta_1) - \theta_2 = \pm 180^\circ$$



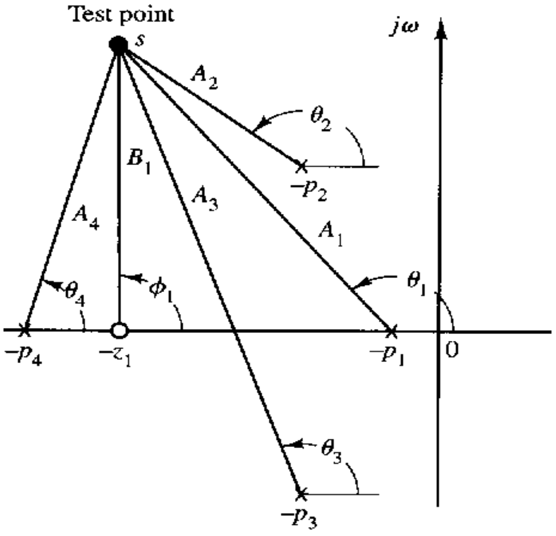
# Magnitude and Angle Conditions

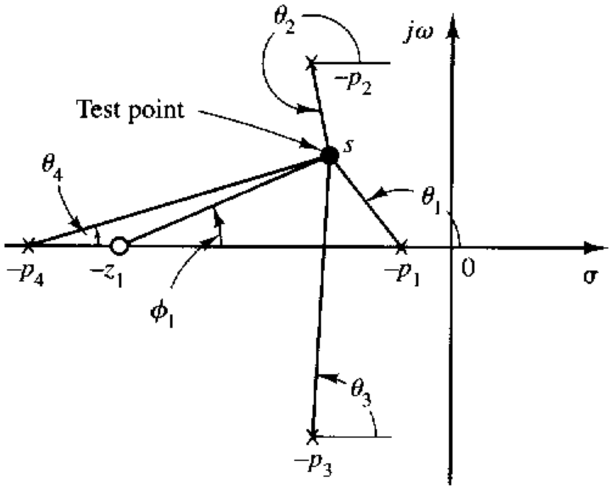
$$K\cdot G(s)\cdot H(s) \quad = \quad \frac{K\cdot(s+z_1)}{(s+p_1)\cdot(s+p_2)\cdot(s+p_3)\cdot(s+p_4)}$$

$$\left|K\cdot G(s)\cdot H(s)\right| \quad = \quad \frac{K\cdot B_1}{A_1\cdot A_2\cdot A_3\cdot A_4} \quad = \quad 1$$

$$\angle G(s)\cdot H(s) \quad = \quad \varphi_1-\theta_1-\theta_2-\theta_3-\theta_4 \quad = \quad \pm 180^\circ\cdot(2\cdot k+1)$$

*for k = 0, 1, 2, …*





# Construction Rules for Root Locus

Open-loop transfer function:

$$K\,H(s)\cdot G(s) \quad = \quad K\,\frac{B(s)}{A(s)}$$

m:  order of open-loop numerator polynomial

n:   order of open-loop denominator polynomial

## Rule 1:  Number of branches

The number of branches is equal to the number of poles of the open-loop transfer function.

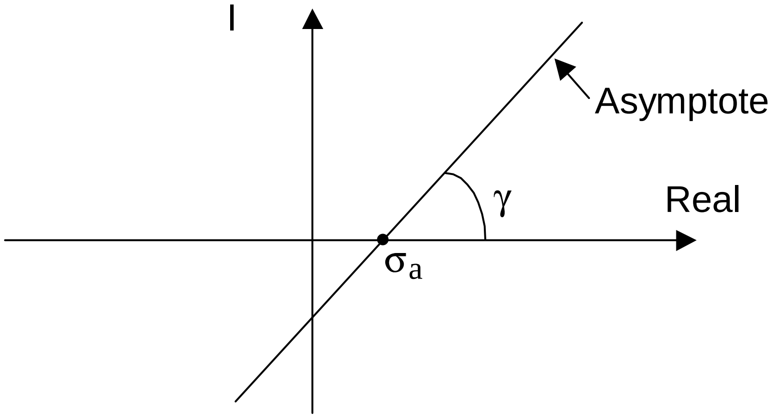
## Rule 2:  Real-axis root locus

If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

## Rule 3:  Root locus end-points

The locus starting point ( K=0 ) are at the open-loop poles and the locus ending points ( K=∞ ) are at the open loop zeros and n-m branches terminate at infinity.

## **Rule 4:** Slope of asymptotes of root locus as s approaches infinity



$$\gamma = \frac{\pm 180^\circ \cdot (2k+1)}{n-m}, \quad k = 0, 1, 2, \ldots$$

## **Rule 5:** Abscissa of the intersection between asymptotes of root locus and real-axis

$$\sigma_a = \frac{\sum_{i=1}^{n}(-p_i) - \sum_{i=1}^{m}(-z_i)}{n-m}$$

$(-p_i)$ = poles of open-loop transfer function
$(-z_i)$ = zeros of open-loop transfer function

## **Rule 6:** Break-away and break-in points

From the characteristic equation

$$f(s) = A(s) + K \cdot B(s) = 0$$

the break-away and -in points can be found from:

$$\frac{dK}{ds} = -\frac{A^{'}(s) \cdot B(s) - A(s) \cdot B^{'}(s)}{B^{2}(s)} = 0$$

## **Rule 7:** Angle of departure from complex poles or zeros

Subtract from 180° the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

## **Rule 8:** Imaginary-axis crossing points

Find these points by solving the characteristic equation for $s=j\omega$ or by using the Routh’s table.

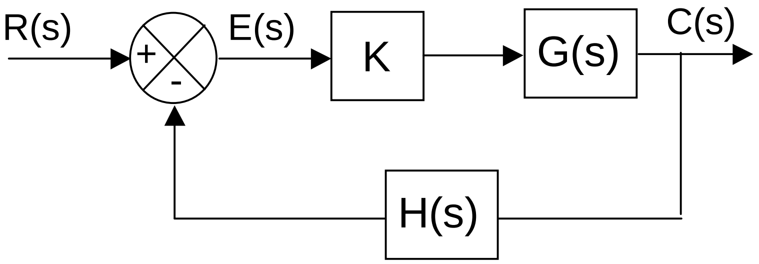
## Rule 9: Conservation of the sum of the system roots

If the *order of numerator is lower than the order of denominator by two or more*, then the sum of the roots of the characteristic equation is constant.

Therefore, if some of the roots more towards the left as K is increased, the other roots must more toward the right as K in increased.

# Discussion of Root Locus Construction Rules

Consider:



$$K \cdot H(s) \cdot G(s) = K \cdot \frac{B(s)}{A(s)} = K \cdot \frac{\sum_{i=0}^{m} b_i \cdot s^{m-i}}{\sum_{i=0}^{n} \alpha_i \cdot s^{n-i}}$$

m: number of zeros of open-loop *KH(s)G(s)*

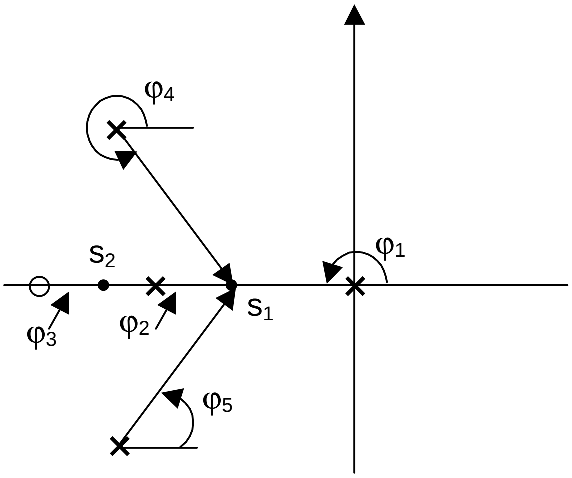
n: number of poles of open-loop *KH(s)G(s)*

Characteristic Equation: $f(s) = A(s) + K \cdot B(s) = 0$

## Rule 1: Number of branches

The characteristic equation has n zeros ⇒
the root locus has n branches

## Rule 2: Real-axis root locus



Consider two points $s_1$ and $s_2$ :

$$s_1 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 0 = \varphi_3\,, \quad \varphi_4 + \varphi_5 = 180 \cdot 2 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 3 \cdot 180 \end{cases}$$

$$s_2 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 180, \quad \varphi_3 = 0\,, \quad \varphi_4 + \varphi_5 = 360 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 4 \cdot 180 \end{cases}$$

Therefore, $s_1$ is on the root locus; $s_2$ is not.

## **Rule 3:** Root locus end-points

Magnitude condition:

$$\left|\frac{B(s)}{A(s)}\right| = \frac{1}{K} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

K=0 open loop poles

K=∞ m open loop zeros

m-n branches approach infinity

## **Rule 4:** Slope of asymptotes of root locus as s approaches infinity

$$K \cdot \frac{B(s)}{A(s)} = -1$$

$$\lim_{s \to \infty} \frac{K \cdot B(s)}{A(s)} = \lim_{s \to \infty} \frac{K}{s^{n-m}} = -1$$

$$s^{n-m} = -K \ \ for \ s \to \infty$$

Using the angle condition:

$$\angle s^{n-m} = \angle -K = \pm 180^\circ \cdot (2 \cdot k + 1) \text{ , } k = 1, 2, 3, \ldots$$

or

$$(n-m)\cdot\angle s = \pm 180^\circ\cdot(2\cdot k+1)$$

leading to

$$\angle s = \quad \gamma \quad = \frac{\pm 180^\circ\cdot(2\cdot k+1)}{n-m}$$

## **Rule 5:** Abscissa of the intersection between asymptotes of root-locus and real axis

$$\frac{A(s)}{B(s)} \quad = \quad \frac{s^n + s^{n-1}\cdot\sum_{i=1}^{n} p_i + \ldots + \prod_{i=1}^{m} p_i}{s^m + s^{m-1}\cdot\sum_{i=1}^{m} z_i + \ldots + \prod_{i=1}^{n} z_i} \quad = \quad -K$$

Dividing numerator by denominator yields:

$$s^{n-m} - \left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right)\cdot s^{n-m-1} + \ldots \quad = \quad -K$$

For large values of s this can be approximated by:

$$\left(s - \frac{\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i}{n-m}\right)^{n-m} \quad = \quad -K$$

The equation for the asymptote (for $s\rightarrow\propto$) was found in Rule 4 as

$$s^{n-m} \;=\; -K$$

this implies $\sigma_a = -\dfrac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} = \dfrac{\sum_{i=1}^{n} -p_i - \sum_{i=1}^{m} -z_i}{n-m}$

**<u>Rule 6</u>: Break-away and break-in points**

At break-away (and break-in) points the characteristic equation:

$$f(s) \;=\; A(s) + K \cdot B(s) \;=\; 0$$

has multiple roots such that:

$$\frac{df(s)}{ds} \;=\; 0 \;\Rightarrow\; A'(s) + K \cdot B'(s) \;=\; 0 \qquad \left( A'(s) \;=\; \frac{dA(s)}{ds} \right)$$

$\Rightarrow$ for $K = -\dfrac{A'(s)}{B'(s)}$ , *f(s)* has multiple roots

Substituting the above equation into f(s) gives:

$$A(s) \cdot B'(s) - A'(s) \cdot B(s) \;=\; 0$$

Another approach is using:

$$K = -\frac{A(s)}{B(s)} \quad \text{from} \quad f(s) = 0$$

This gives:

$$\frac{dK}{ds} = -\frac{A'(s) \cdot B(s) - A(s) \cdot B'(s)}{B^2(s)}$$

and break-away, break-in points are obtained from:

$$\frac{dK}{ds} = 0$$

**Extended Rule 6:**

Consider

$$f(s) = A(s) + K \cdot B(s) = 0$$

and

$$K = -\frac{A(s)}{B(s)}$$

If the first *(y-1)* derivatives of A(s)/B(s) vanish at a given point on the root locus, then there will be *y* branches approaching and *y* branches leaving this point.

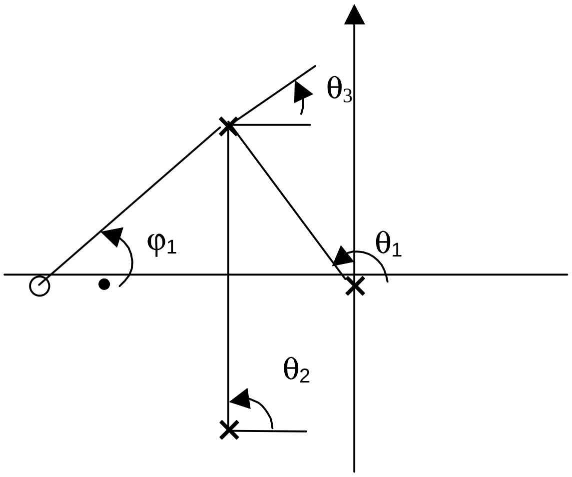
The angle between two adjacent approaching branches is given by:

$$\theta_y = \pm\frac{360^\circ}{y}$$

The angle between a leaving branch and an adjacent approaching branch is:

$$\theta_y \quad = \quad \pm \frac{180^\circ}{y}$$

## **Rule 7:** Angle of departure from complex pole or zero



$$\theta_2 \quad = \quad 90^\circ$$

$$\theta_3 \quad = \quad 180^\circ - (\theta_1 + \theta_2 - \varphi_1)$$

## **Rule 8:** Imaginary-axis crossing points

Example: $f(s) \quad = \quad s^3 + b \cdot s^2 + c \cdot s + K \cdot d = 0$

| | | |
|---|---|---|
| $s^3$ | 1 | c |
| $s^2$ | b | Kd |
| $s^1$ | (bc-Kd)/b | |
| $s^0$ | Kd | |

For crossing points on the Imaginary axis:

$$b \cdot c - K \cdot d = 0 \quad \Rightarrow \quad K \;=\; \frac{bc}{d}$$

Further, $b \cdot s^2 + K \cdot d = 0$ leading to

$$s_{1,2} = \pm j \cdot \sqrt{\frac{K \cdot d}{b}} \quad = \quad \pm j\omega$$

The same result is obtained by solving $f(j\omega) = 0$ .

## **Rule 9:** Conservation of the sum of the system roots

From

$$A(s) + K \cdot B(s) = \prod_{i=1}^{n} (s + r_i)$$

we have

$$\prod_{i=1}^{n} (s + p_i) + K \cdot \prod_{i=1}^{m} (s + r_i) = \prod_{i=1}^{n} (s + r_i)$$

with

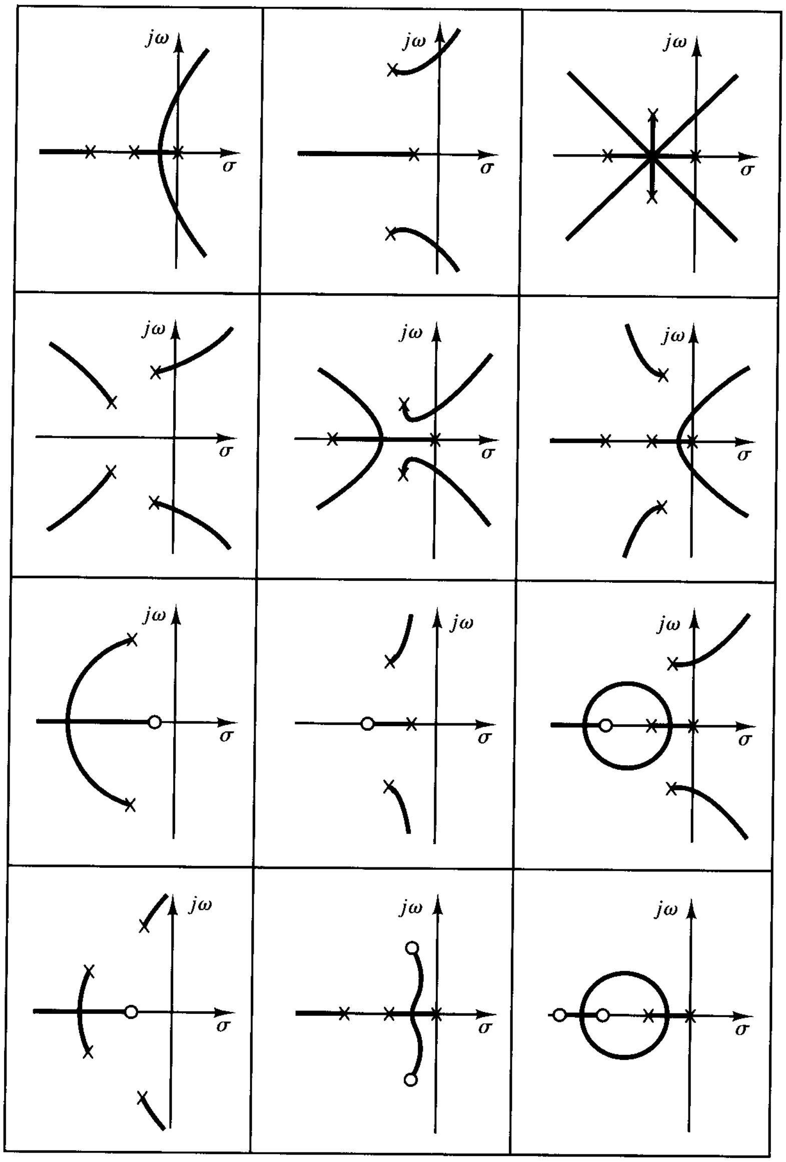
$$A(s) = \prod_{i=1}^{n} (s + p_i) \quad \text{and} \quad B(s) = \prod_{i=1}^{m} (s + z_i)$$

By equating coefficients of $s^{n-1}$ for $n \geq m + 2$, we obtain the following:

$$\text{Sum of open-loop poles} \qquad \sum_{i=1}^{n} - p_i = \sum_{i=1}^{n} - r_i \qquad \text{Sum of closed-loop poles}$$

i.e. the sum of closed-loop poles is independent of K !

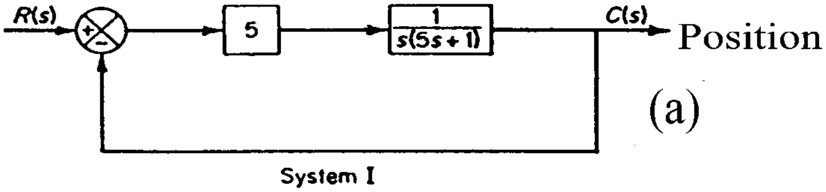
**Table 6–1** Open-Loop Pole–Zero Configurations and the Corresponding Root Loci



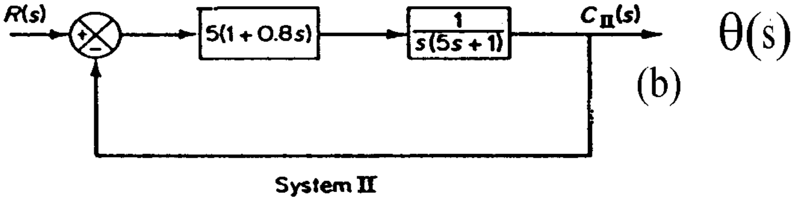
# Effect of Derivative Control and Velocity Feedback

Consider the following three systems:

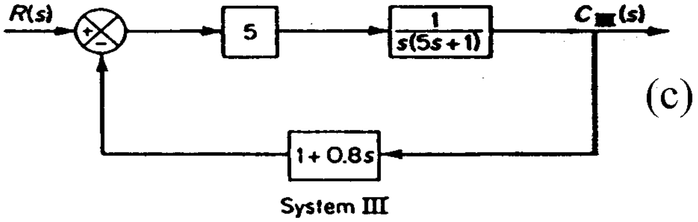
Positional servo.
Closed-loop poles:
$s = -0.1 \pm j \cdot 0.995$



Positional servo with derivative control.
Closed-loop poles:
$s = -0.5 \pm j \cdot 0.866$



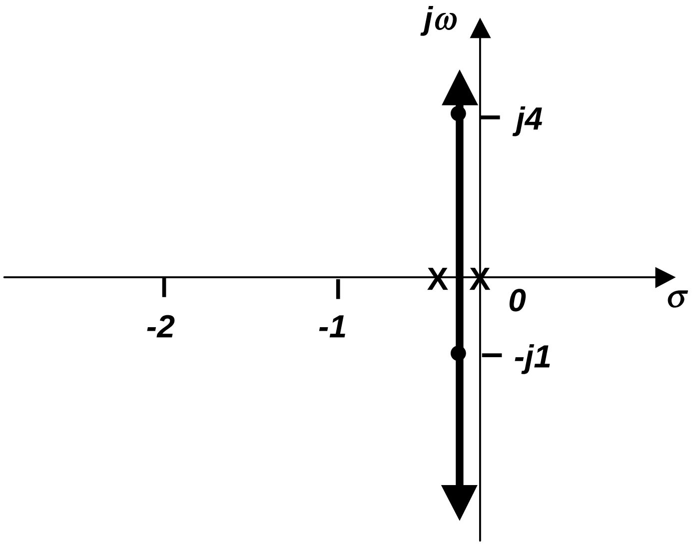
Positional servo with velocity feedback.
Closed-loop poles:
$s = -0.5 \pm j \cdot 0.866$



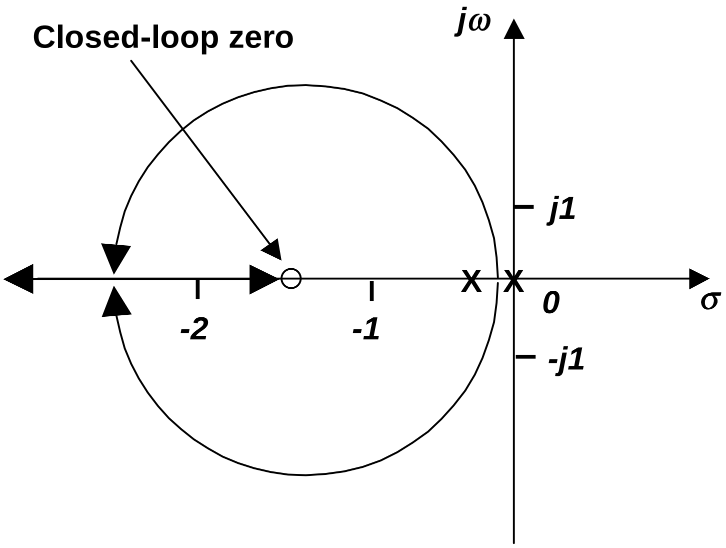
Open-loop of system I: $G_I(s) = \dfrac{5}{s \cdot (5 \cdot s + 1)}$

Open-loop of systems II and III: $G(s) = \dfrac{5 \cdot (1 + 0.8s)}{s \cdot (5 \cdot s + 1)}$

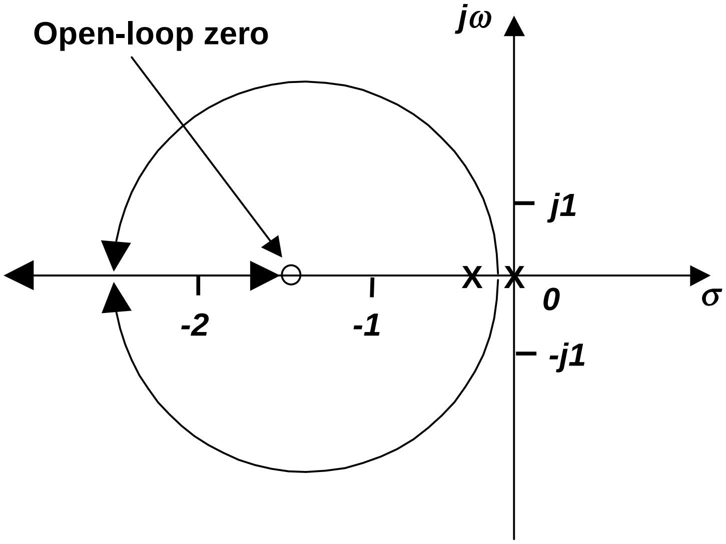
# Root locus for the three systems



a) System I



b) System II



c) System III

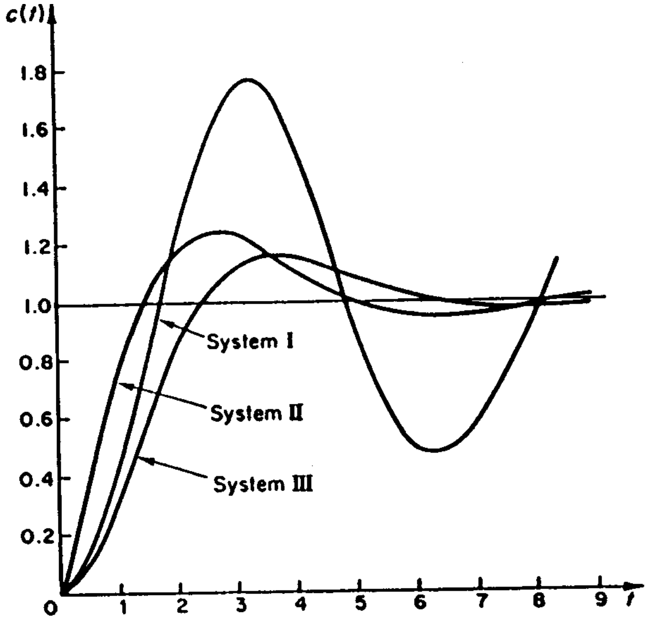
## Closed-loop zeros:

System I: none  
System II: 1+0.8s=0  
System III: none

## Observations:

- The root locus presents the <u>*closed loop poles*</u> but gives no information about <u>*closed-loop zeros*</u>.
- Two system with same root locus (same closed-loop poles) may have *different responses due to different closed-loop zeros*.

Unit-step response curves for systems I,II and III :



The closed-loop transfer function of System III is

$$\frac{C_{III}(s)}{R(s)} = \frac{1}{(s+0.5+j0.866)(s+0.5-j0.866)}$$

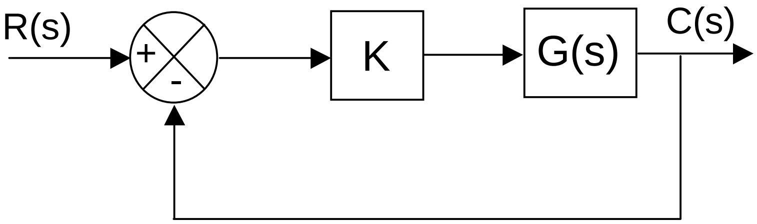
For a unit-impulse input,

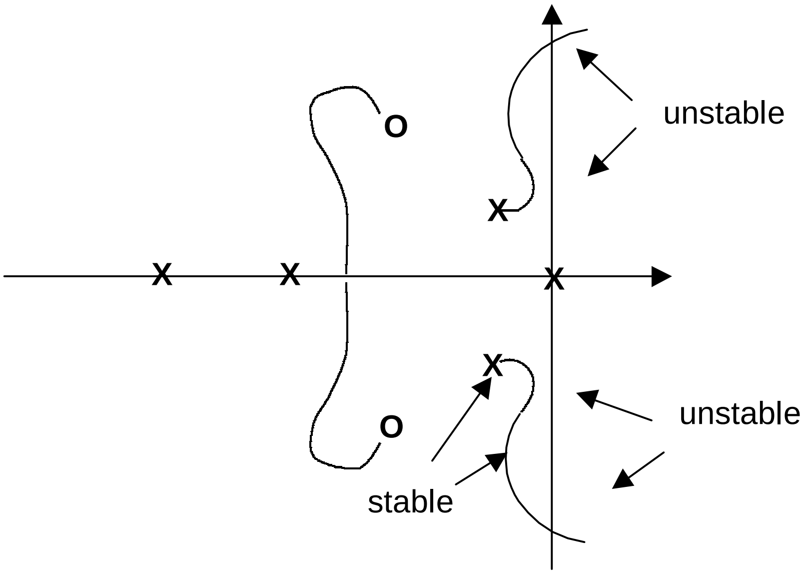
$$C_{III}(s) = \frac{j0.577}{s+0.5+j0.866} + \frac{-j0.577}{s+0.5-j0.866}$$

- The unit-step response of system II is the fastest of the three.
- This is due to the fact that derivative control responds to the rate of change of the error signal. Thus, it can produce a correction signal before the error becomes large. This leads to a faster response.

# Conditionally Stable Systems

System which can be stable or unstable depending on the value of gain K.

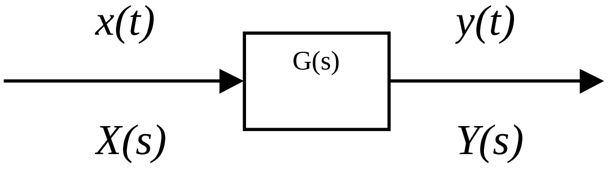




# Minimum Phase Systems

All poles and zeros are in the left half plane.

# Frequency Response Methods



$$x(t) = X\sin(\omega t) \qquad X(s) = \frac{\omega X}{s^2+\omega^2}$$

$$Y(s) \;=\; G(s)\cdot X(s) \;=\; G(s)\cdot\frac{\omega X}{s^2+\omega^2} \;=\; \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \sum_i \frac{b_i}{s+s_i}$$

$$y(t) \;=\; ae^{-j\omega t} + \bar{a}e^{j\omega t} + \sum_i b_i\, e^{-s_i\, t}$$

stable system $\Leftrightarrow$ $Re(-s_i) < 0$ for all i

for $t \rightarrow \infty \qquad \Rightarrow \quad y(t) \;=\; ae^{-j\omega t} + \bar{a}e^{j\omega t}$

$$a \;=\; G(s)\cdot\frac{\omega X}{s^2+\omega^2}\cdot(s+j\omega)\Big|_{(s=-j\omega)} \;=\; -\frac{X\,G(-j\omega)}{2j}$$

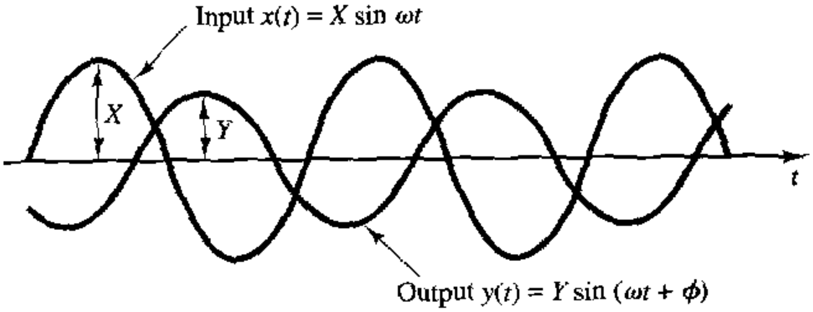
$$\bar{a} \;=\; G(s)\cdot\frac{\omega X}{s^2+\omega^2}(s-j\omega)\Big|_{(s=j\omega)} \;=\; \frac{X\,G(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)| \cdot e^{j\varphi} \qquad \varphi = \tan^{-1}\left(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$$

and

$$G(-j\omega) = |G(j\omega)| \cdot e^{-j\varphi}$$

$$y(t) = X \cdot |G(j\omega)| \cdot \frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j} = Y \cdot \sin(\omega t + \varphi$$



$\varphi > 0$ phase lag

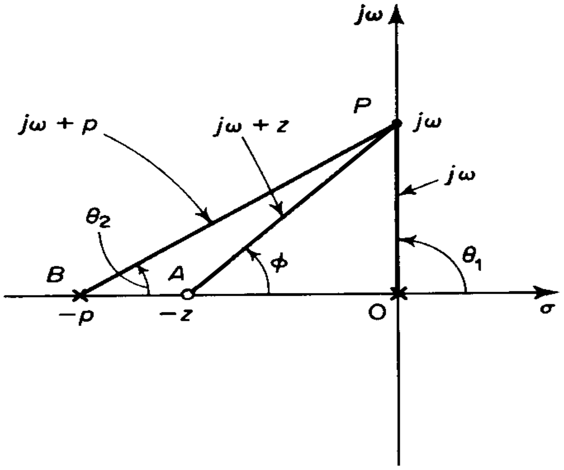
$\varphi < 0$ phase lead

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right|$$ Magnitude response

$$\varphi = \angle(G(j\omega)) = \angle\left(\frac{Y(j\omega)}{X(j\omega)}\right)$$ Phase response

## Connection between pole locations and Frequency Response



Determination of the frequency response in the complex plane.

$$G(s) = \frac{K(s+z)}{s(s+p)}$$

$$|G(j\omega)| = \frac{|K|\cdot|j\omega+z|}{|j\omega|\cdot|j\omega+p|}$$

$$\angle G(j\omega) = \varphi - \theta_1 - \theta_2$$

## Frequency Response Plots

- Bode Diagrams
- Polar Plots (Nyquist Plots)
- Log-Magnitude-Versus-Phase Plots (Nichols Plots)

# Bode Diagrams

- Magnitude response $|G(j\omega)|$

  $20\log|G(j\omega)|$ in dB

- Phase response $\angle G(j\omega)$ in degrees

## Basic factors of G(jω):

- Gain K
- Integral or derivative factors $(j\omega)^{\pm 1}$
- First-order factors $(1+j\omega T)^{\pm 1}$
- Quadratic factors $\left(1+2j\frac{j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right)^{\pm 1}$

## 1. Gain Factor K

Horizontal straight line at magnitude $20\log(K)\ dB$

Phase is zero

## 2. Integral or derivative factors $(j\omega)^{\pm 1}$

- $(j\omega)^{-1}$

$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega$$

magnitude: straight line with slope –20 dB/decade

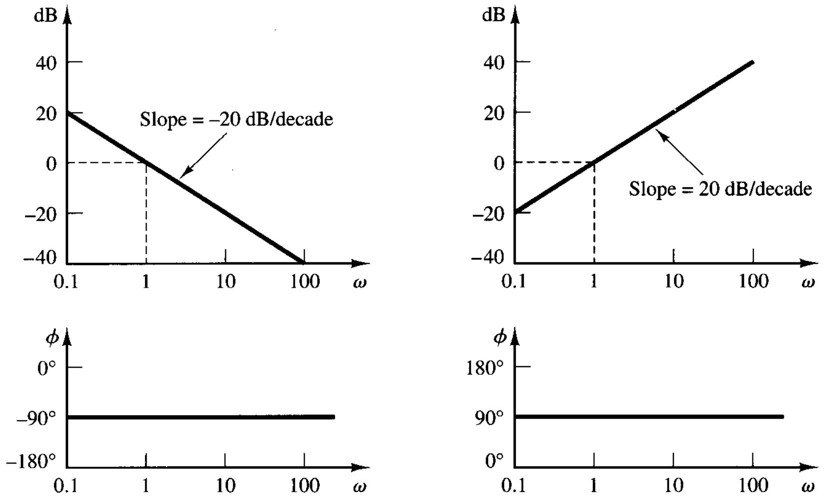
phase: $-90^{\circ}$

- $(j\omega)$

$$20\log\left|j\omega\right| = 20\log\omega$$

magnitude: straight line with slope 20 dB/decade

phase: $+90^{\circ}$



Bode diagram of $G(j\omega) = 1/j\omega$

(a)

Bode diagram of $G(j\omega) = j\omega$

(b)

(a) Bode diagram of $G(j\omega) = 1/j\omega$; (b) Bode diagram of $G(j\omega) = j\omega$.

## 3. First order factors $(1+j\omega T)^{\pm 1}$

- $(1+j\omega T)^{-1}$

*Magnitude:*

$$20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2 T^2}\ dB$$

for $\omega << T^{-1} \Rightarrow 0\ dB$ magnitude

for $\omega >> T^{-1} \Rightarrow -20\log(\omega T)\ dB$ magnitude

*Approximation of the magnitude:*

for ω between 0 and $\omega = \frac{1}{T}$ 0 dB

for $\omega >> \frac{1}{T}$ straight line with slope –20 dB /decade

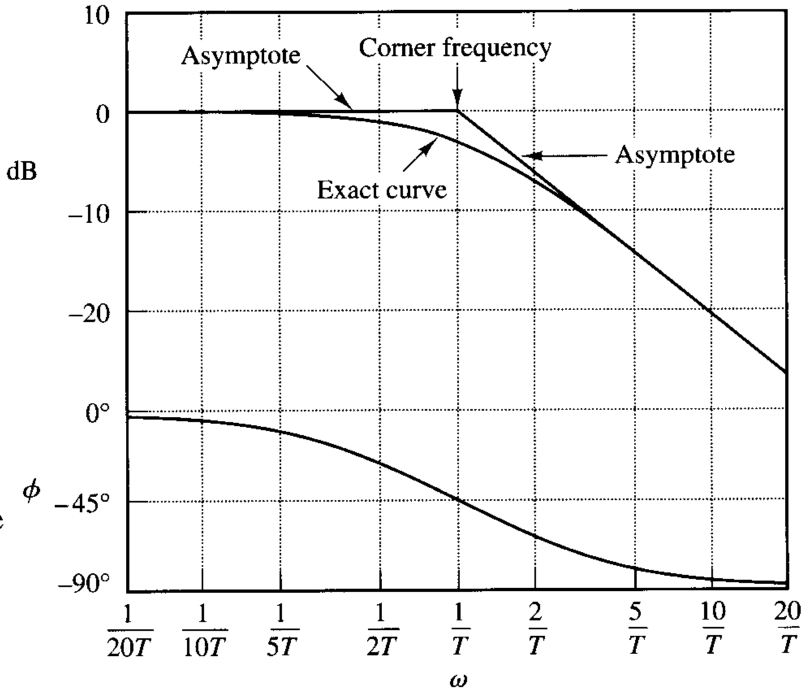
*Phase:*

$$\angle(1+j\omega T)^{-1} = -\tan^{-1}(\omega T)$$

for $\omega = 0$ $\varphi = 0^\circ$

for $\omega = \frac{1}{T} \Rightarrow -\tan^{-1}\left(\frac{T}{T}\right) = 1$ $\varphi = -45^\circ$

for $\omega = \infty$ $\varphi = -90^\circ$

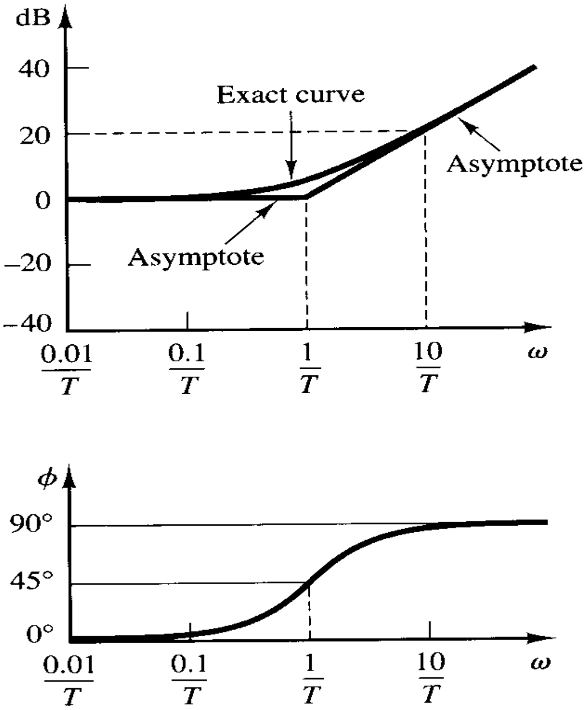


Log-magnitude curve together with the asymptotes and phase angle curve of $1/(1 + j\omega T)$.

- $(1+ j\omega T)^{+1}$

Using $$20\log|1+j\omega T| = -20\log\left|\frac{1}{1+j\omega T}\right|$$

$$\angle(1+j\omega T) = \tan^{-1}(\omega T) = -\angle\left(\frac{1}{1+j\omega T}\right)$$



Log-magnitude curve together with the asymptotes and phase-angle curve for $1 + j\omega T$.

## 4. Quadratic Factors

$$G(j\omega)=\frac{1}{1+2\varsigma\, j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2} \qquad 0<\varsigma<1$$

*Magnitude:*

$$20\log\left|G(j\omega)\right| \;=\; -20\log\sqrt{\left(1-\frac{\omega^2}{\omega^2{}_n}\right)^2-\left(2\,j\left(\frac{\omega}{\omega_n}\right)\right)^2}$$

for

$$\omega \ll \omega_n \quad \Rightarrow \quad 0 \;\; dB$$

for

$$\omega \;\gg\; \omega_n \quad \Rightarrow \quad -20\log\left(\frac{\omega^2}{\omega^2{}_n}\right) \;=\; -40\log\left(\frac{\omega}{\omega_n}\right) \; dB$$

*Phase:*

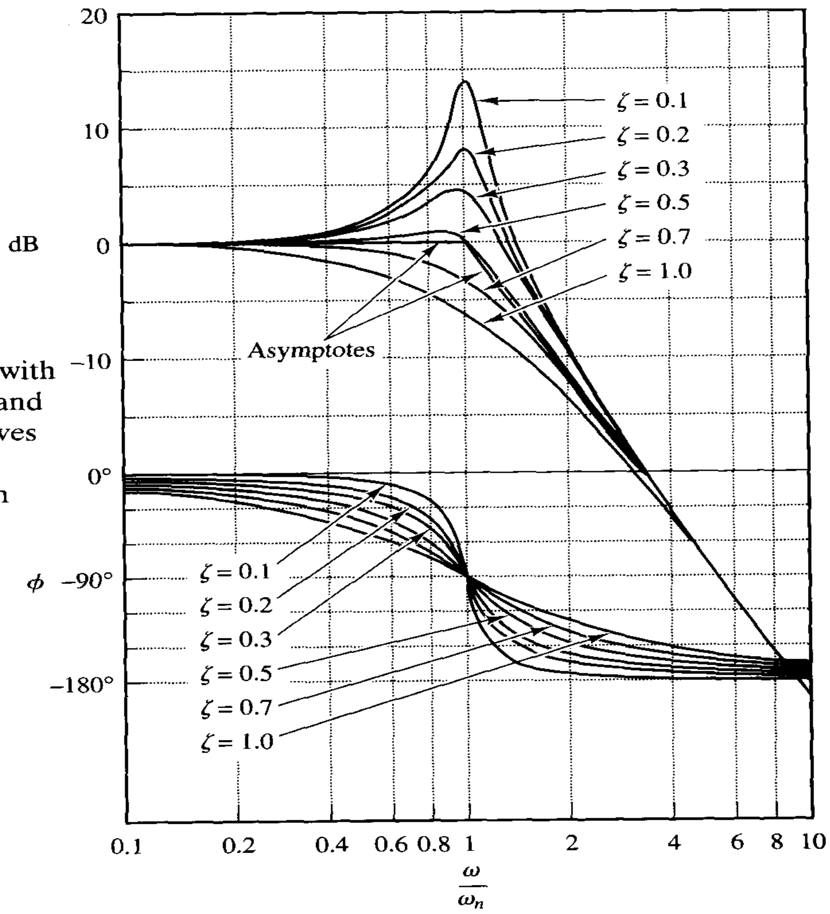
$$\varphi=\tan^{-1}\angle G(j\omega)=-\tan^{-1}\left[\frac{2\varsigma\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

*Resonant Frequency:*

$$\omega_T=\omega_n\sqrt{1-2\varsigma^2}$$

*Resonant Peak Value:*

$$M_T=\left|G(j\omega)\right|_{\max}=\frac{1}{2\xi\varsigma\sqrt{1-\varsigma^2}}$$

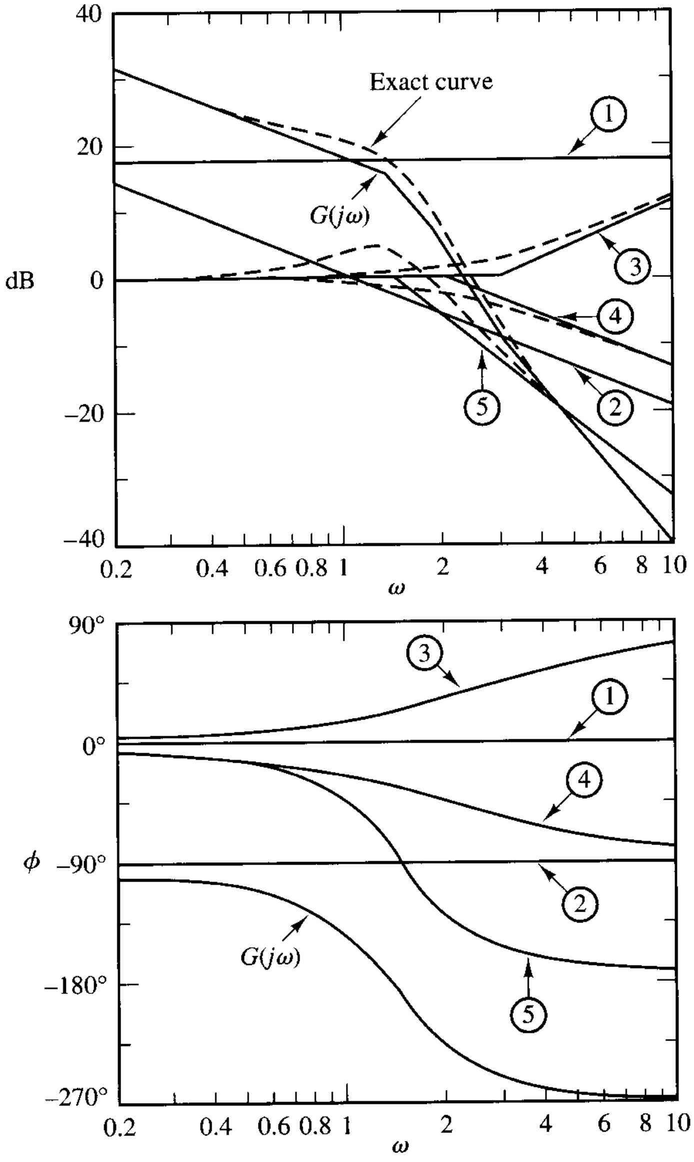


Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic transfer function

Example:

$$G(j\omega) = \frac{10(j\omega+3)}{(j\omega)\cdot(j\omega+2)\cdot\left((j\omega)^2+j\omega+2\right)}$$

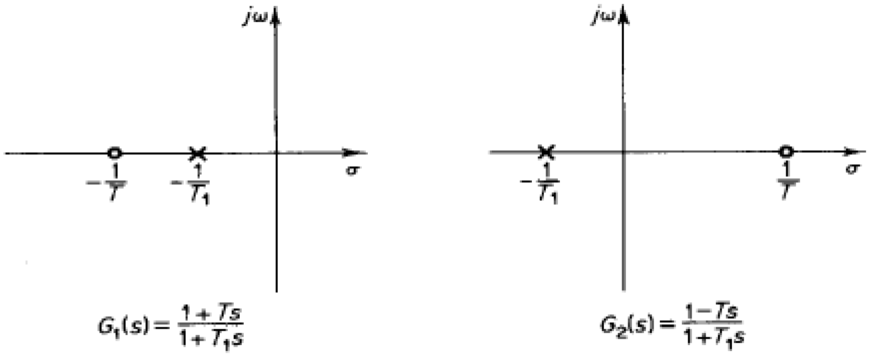
$$G(j\omega)=\frac{7.5\cdot\left(\frac{(j\omega)}{3}+1\right)}{j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{(j\omega)^2}{2}+\frac{j\omega}{2}+1\right)}$$



Bode diagram of the system considered in Example 8-1.

# Frequency Response of non-Minimum Phase systems

Minimum phase systems have all poles and zeros in the left half s-plane and were discussed before



Pole-zero configurations of a minimum phase system $G_1(s)$ and nonminimum phase system $G_2(s)$.

Consider

$$A_1(s) = 1 + Ts \quad A_2(s) = 1 - Ts \quad A_3(s) = Ts - 1$$

then

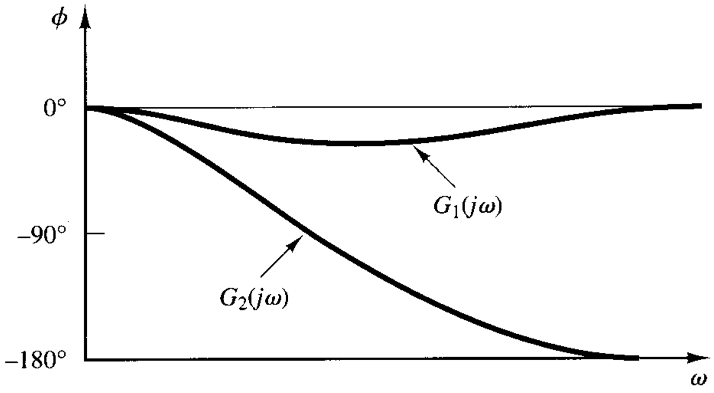
$$|A_1(j\omega)| = |A_2(j\omega)| = |A_3(j\omega)|$$

$$\angle A_2(j\omega) = -\angle A_1(j\omega)$$

$$\angle A_3(j\omega) = 180 - \angle A_1(j\omega)$$

| | | | | |
|---|---|---|---|---|
| We know that | phase of | $A_1(j\omega)$ | from | $0^o$ to $+90^o$ |
| → | phase of | $A_2(j\omega)$ | from | $0^o$ to $-90^o$ |
| | phase of | $A_3(j\omega)$ | from | $180^o$ to $+90^o$ |

Phase-angle characteristics of the systems $G_1(s)$ and $G_2(s)$



Phase-angle characteristic of the two systems $G_1(s)$ and $G_2(s)$ having the same magnitude response but $G_1(s)$ is minimum phase while $G_2(s)$ is not.

## Frequency Response of Unstable Systems

Consider

$$G_1(s) = \frac{1}{1+Ts} \qquad G_2(s) = \frac{1}{1-Ts} \qquad G_3(s) = \frac{1}{Ts-1}$$

then

$$\left|G_1(j\omega)\right| = \left|G_2(j\omega)\right| = \left|G_3(j\omega)\right|$$

$$\angle G_2(j\omega) = -\angle G_1(j\omega)$$

$$\angle G_3(j\omega) = -180^o - \angle G_1(j\omega)$$

| | | | | | |
|---|---|---|---|---|---|
| We know that | phase of | $G_1(j\omega)$ | from | $0^o$ to | $-90^o$ |
| → | phase of | $G_2(j\omega)$ | from | $0^o$ to | $+90^o$ |
| | phase of | $G_3(j\omega)$ | from | $-180^o$ to | $-90^o$ |

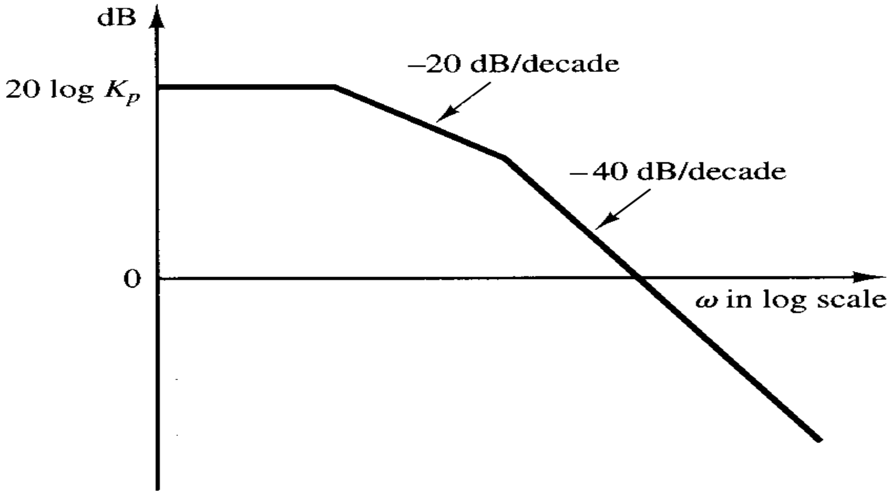
# Relationship between System Type and Log-Magintude curve

Type of system determines:

- the slope of the log-magnitude curve at low frequencies
- for minimum phase, also the phase at low frequencies

## Type 0

Position Error Coefficient $K_p \neq 0$



- Slope at low frequencies: 0 db/decade
- Phase at low frequencies (minimum phase): $0^o$

$$\lim_{\omega \to 0} G(j\omega) = K_p$$

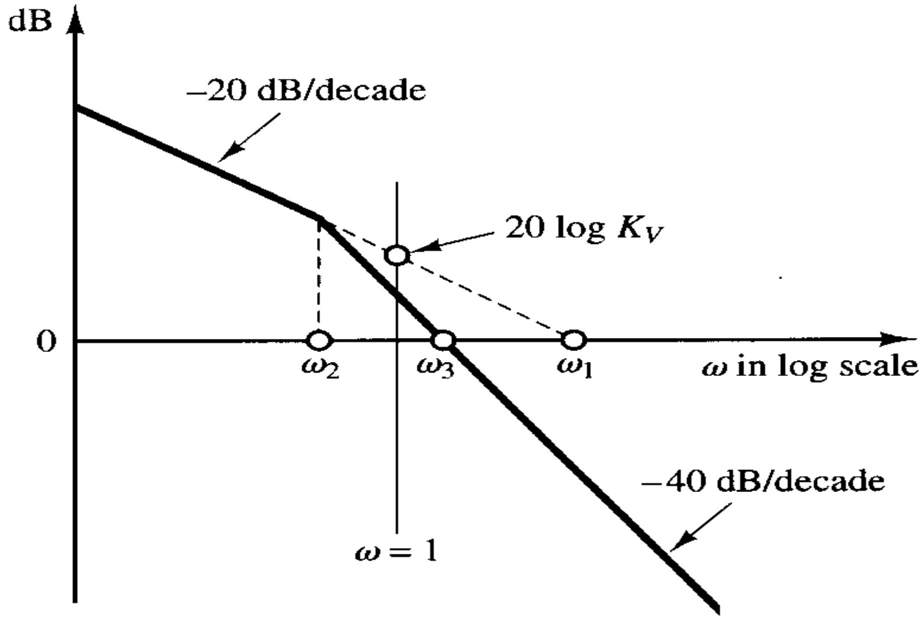
## Type 1

Velocity Error Coefficient $K_v \neq 0 \quad (K_p = \infty)$

$$K_v = \lim_{\omega \to 0} j\omega G(j\omega)$$

$$G(j\omega) = \frac{K_v}{j\omega} \qquad \text{for} \qquad \omega << 1$$

$$20 \log K_v = 20 \log \left| \frac{K_v}{j\omega} \right| \quad for \quad \omega = 1$$



- Slope at low frequencies: -20 db/decade
- Phase at low frequencies (minimum phase): -90°

# Type 2

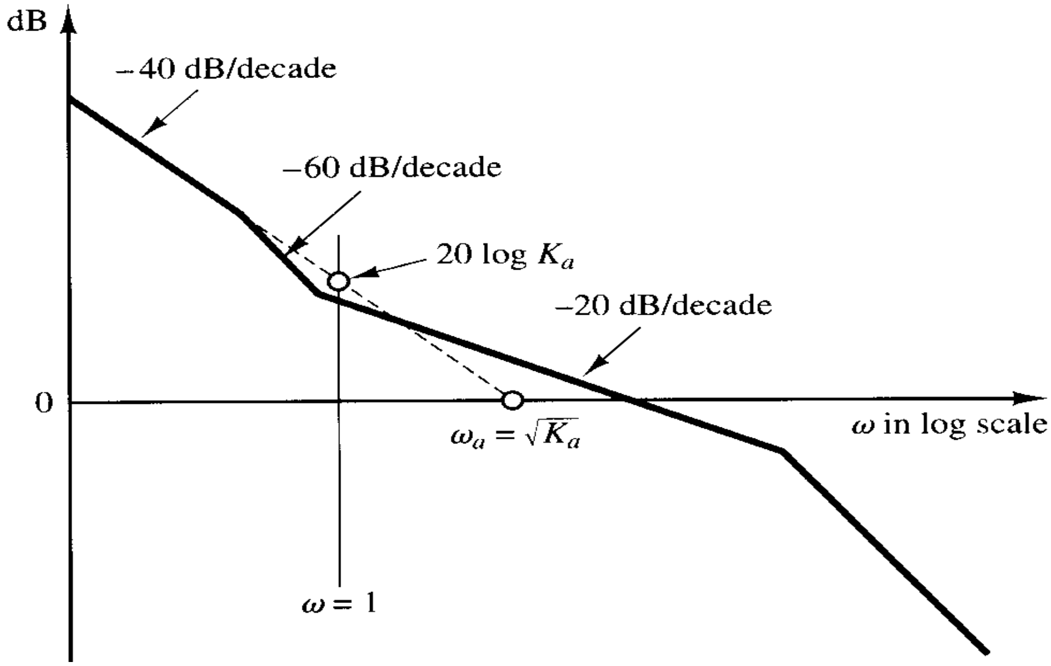
Acceleration Error Coefficient $K_a$

$$K_a = \lim_{\omega \to 0} (j\omega)^2 G(j\omega) \neq 0$$

$$(K_p = K_v = \infty)$$

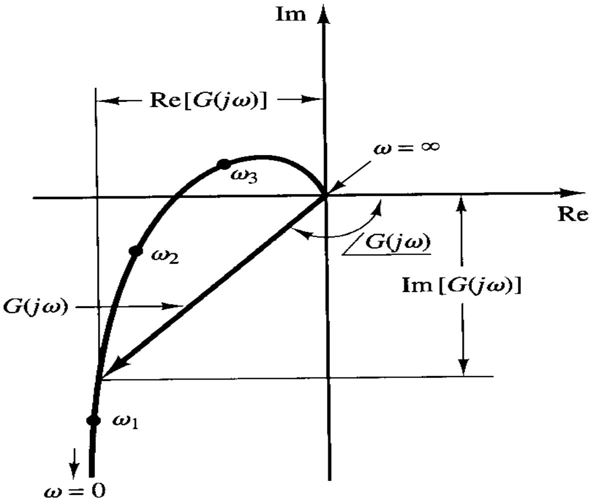
$$G(j\omega) = \frac{K_a}{(j\omega)^2} \quad \text{for} \quad \omega \ll 1$$

$$20 \log K_a = 20 \log \left| \frac{K_a}{(j\omega)^2} \right| \quad for \quad \omega = 1$$



- Slope at low frequencies: -40 db/decade
- Phase at low frequencies (minimum phase): $-180^{\circ}$

# Polar Plots (Nyquist Plots)



$$G(j\omega) = |G(j\omega)| * \angle G(j\omega)$$

$$= \mathrm{Re}[G(j\omega)] + \mathrm{Im}[G(j\omega)]$$

**Advantage over Bode plots:** only one plot

**Disadvantage :** Polar plot of $G(j\omega) = G_1(j\omega) \cdot G_2(j\omega)$ is more difficult to construct than its Bode plot.

## Basic factors of G(jω):

### Integral or derivative factors $(j\omega)^{\pm 1}$

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ$$

$$G(j\omega) = j\omega = \omega \angle 90^\circ$$

## **First order factors** $(1+j\omega T)^{\pm 1}$

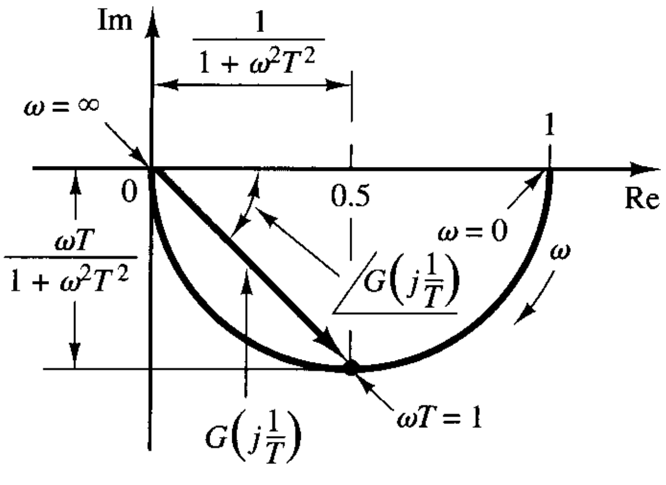
$$G(j\omega) = \frac{1}{1+j\omega T} = X + jY$$

$$X = \frac{1}{1+\omega^2 T^2}, \quad Y = \frac{-\omega T}{1+\omega^2 T^2}$$

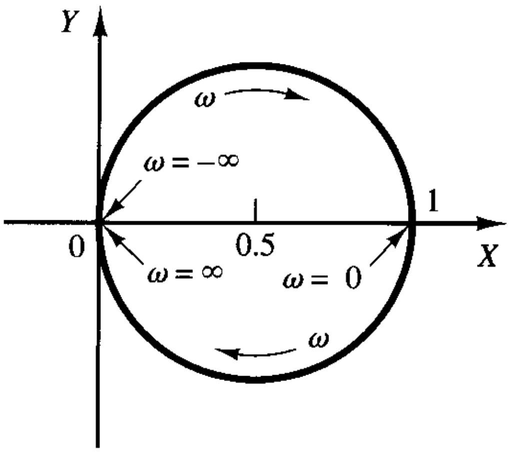
It can be show that $(X-0.5)^2 + Y^2 = (0.5)^2$

$\Rightarrow$ Polar plot is a circle with

- Center (1/2, 0) and
- Radius 0.5.



(a)



(b)

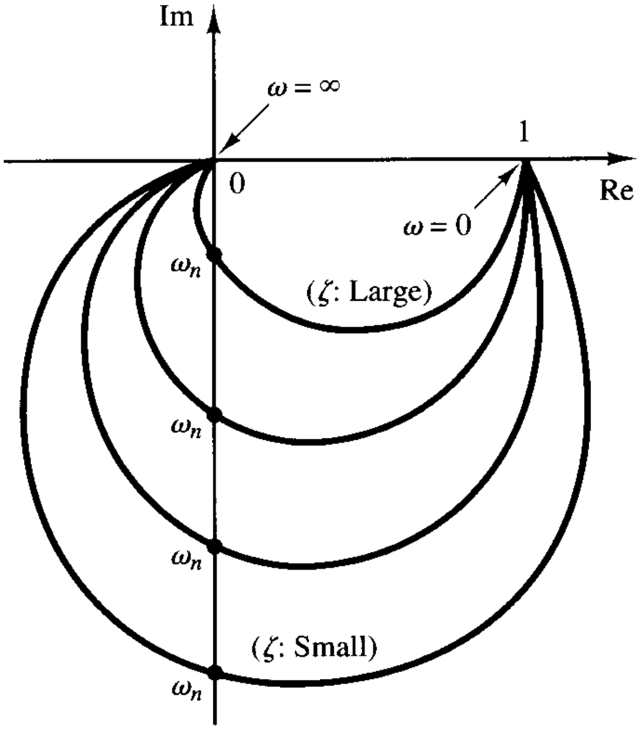
# Quadratic Factors

$$G(j\omega) = \frac{1}{1 + 2\zeta\, j\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2} \qquad 1 > \zeta > 0$$

$$\lim_{\omega \to 0} G(j\omega) = 1\angle 0^\circ$$

$$\lim_{\omega \to \infty} G(j\omega) = 0\angle -180^\circ$$

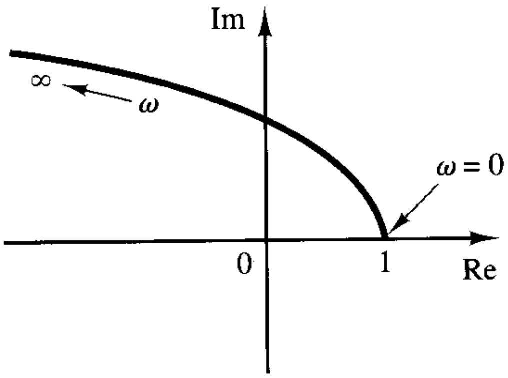
$$G(j\omega_n) = \frac{1}{j2\varsigma}\angle -90^\circ$$



$$\left(1+2\varsigma\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right)^{+1}$$

$$\lim_{\omega \to 0} G(j\omega\ ) \quad = \quad 1\angle 0^{\circ}$$

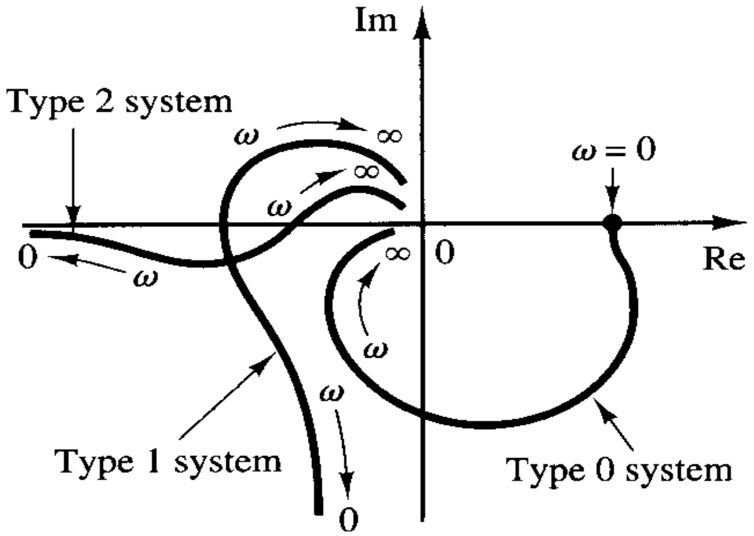
$$\lim_{\omega\to\infty} G(j\omega) = \infty\angle 180^{\circ}$$



# General shapes of polar plots

$$G(j\omega) = \frac{K(1+j\omega\overline{T}_1)\ldots(1+j\omega\overline{T}_m)}{(j\omega)^{\lambda}(1+j\omega T_{\lambda+1})\ldots(1+j\omega T_n)}$$

n = order of the system (denominator)
λ = type of system $\lambda > 0$
m = order of numerator n > m

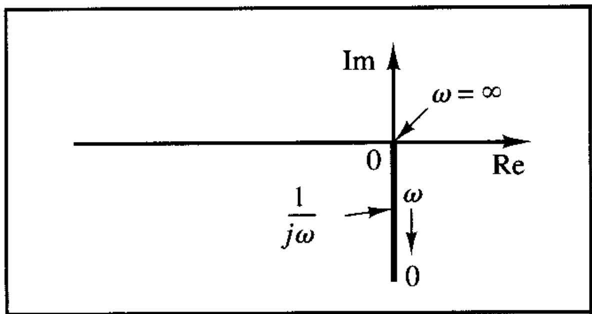


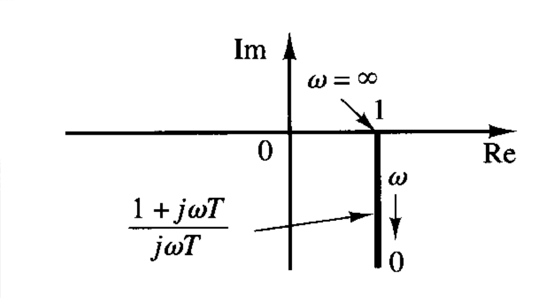
**For low frequencies:** The phase at $\omega \to 0$ is $\lambda\left(-90^\circ\right)$

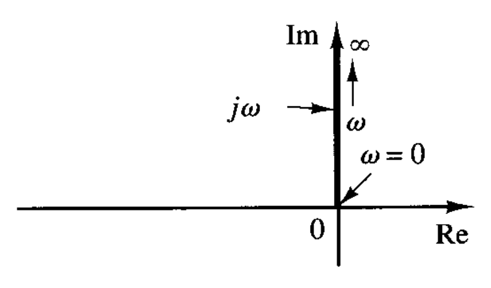
For system type 1, the low frequency asymptote is obtained by taking:

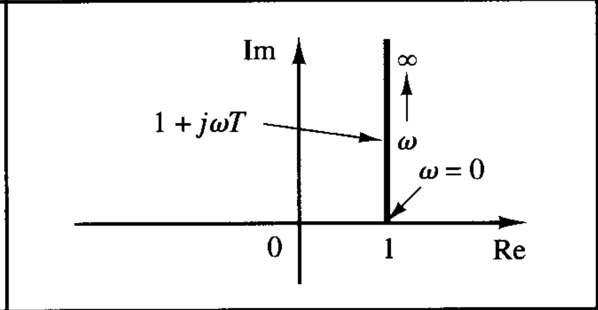
$\mathrm{Re}\left[G(j\omega)\right]$ for $\omega \to 0$

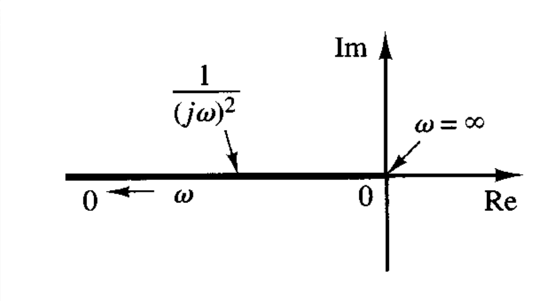
**For high frequencies**: The phase is: (n - m) (- $90^\circ$ )

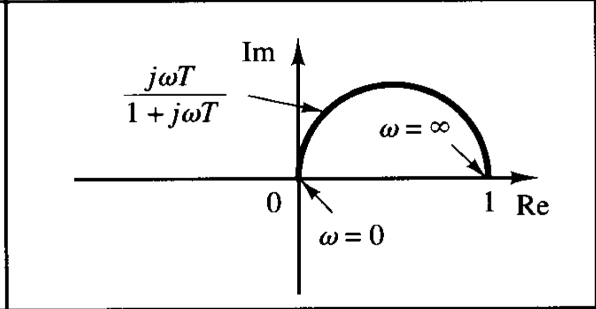

Im
$\omega = \infty$
0
Re
$\frac{1}{j\omega}$
$\omega$
0

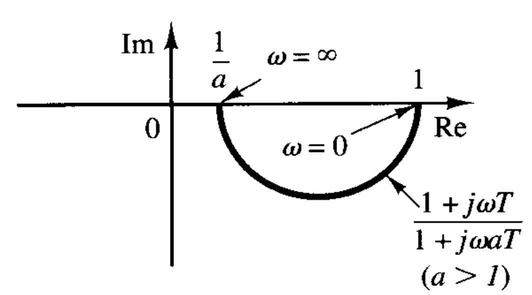

Im
$\omega = \infty$
1
0
Re
$\frac{1 + j\omega T}{j\omega T}$
$\omega$
0

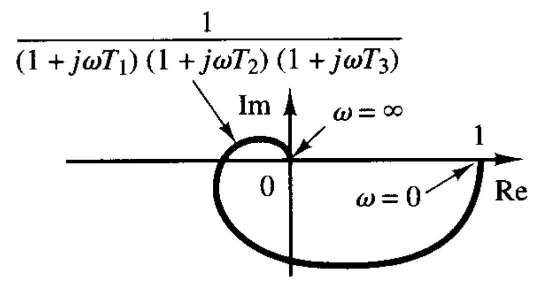

Im
$\infty$
$j\omega$
$\omega$
$\omega = 0$
0
Re

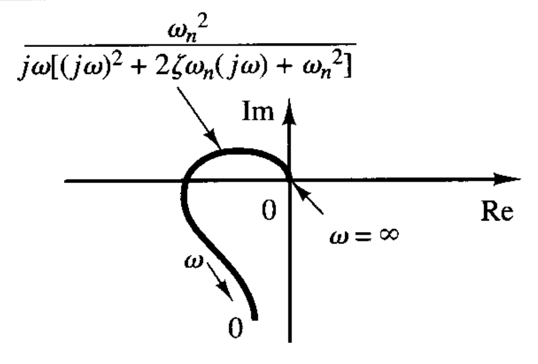

Im
$\infty$
$1 + j\omega T$
$\omega$
$\omega = 0$
0
1
Re

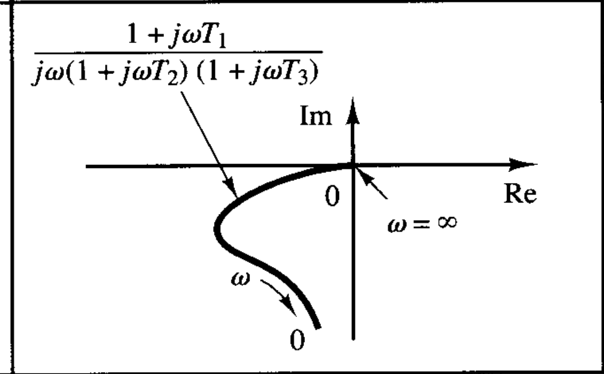

Im
$\frac{1}{(j\omega)^2}$
$\omega = \infty$
0
$\omega$
0
Re


Im
$\frac{j\omega T}{1 + j\omega T}$
$\omega = \infty$
0
1
Re
$\omega = 0$

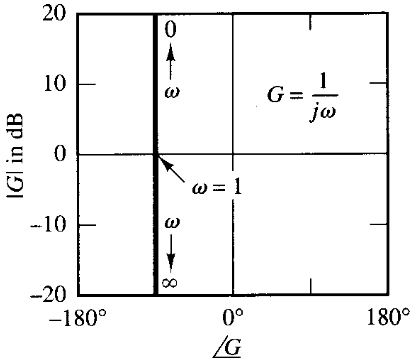

Im
$\frac{1}{a}$
$\omega = \infty$
1
0
Re
$\omega = 0$
$\frac{1 + j\omega T}{1 + j\omega aT}$
$(a > 1)$

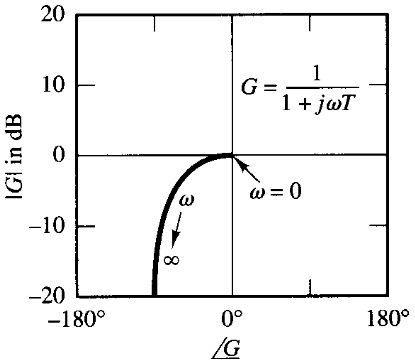

$\frac{1}{(1 + j\omega T_1)(1 + j\omega T_2)(1 + j\omega T_3)}$
Im
$\omega = \infty$
1
0
$\omega = 0$
Re

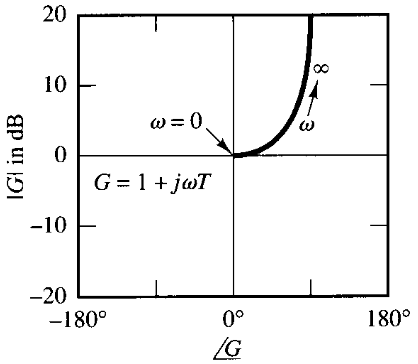

$\frac{\omega_n^2}{j\omega[(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2]}$
Im
0
Re
$\omega = \infty$
$\omega$
0

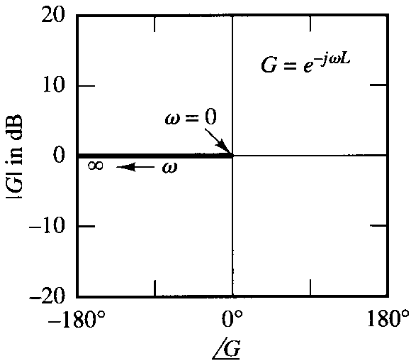

$\frac{1 + j\omega T_1}{j\omega(1 + j\omega T_2)(1 + j\omega T_3)}$
Im
0
Re
$\omega = \infty$
$\omega$
0

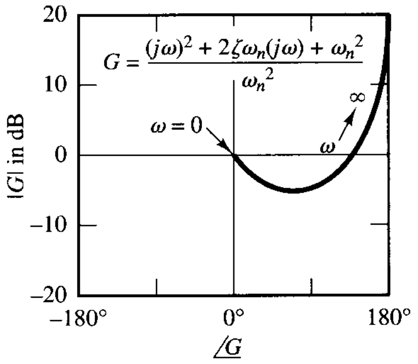
# Log-Magnitude-Versus-Phase Plots (Nichols Plots)

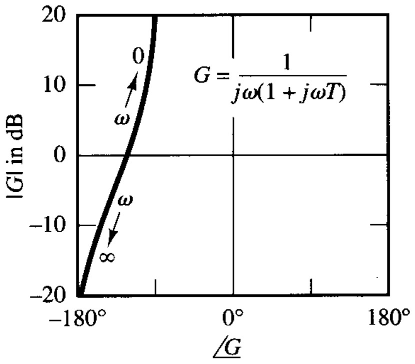










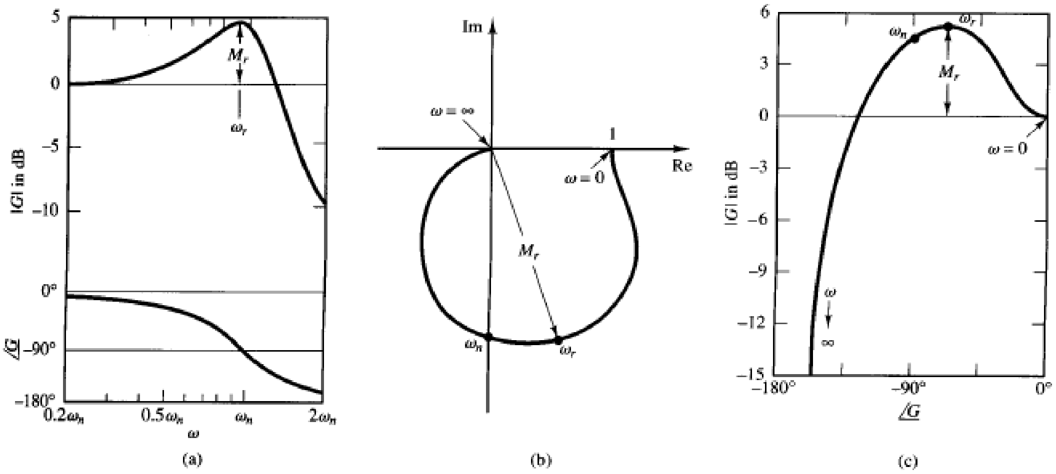


# **Example:**

Frequency Response of a quadratic factor

The same information presented in three different ways:

- Bode Diagram
- Polar Plot
- Log-Magnitude-Versus-Phase Plots

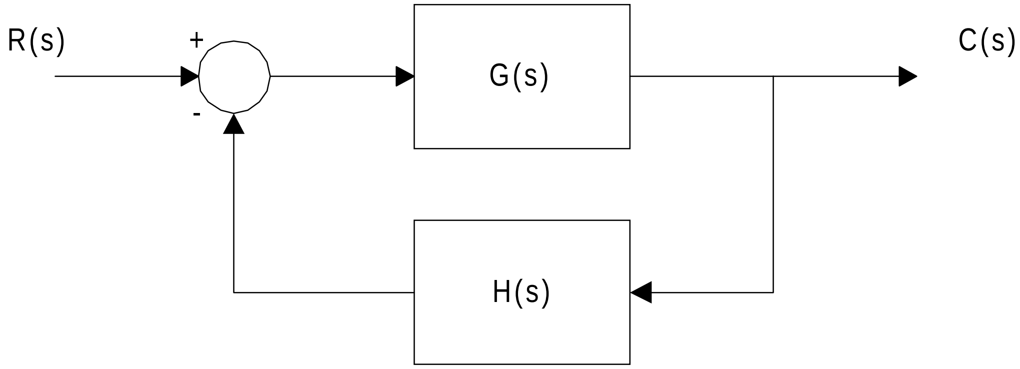


Three representations of the frequency response of $\dfrac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2}$, for $\zeta > 0$.

(a) Bode diagram; (b) polar plot; (c) log-magnitude versus phase plot.

# Nyquist Stability Criterion

The Nyquist stability criterion relates the stability of the ***closed loop*** system to the frequency response of the ***open-loop*** system.



Open-loop: $G(s) \cdot H(s)$

Closed-loop: $\dfrac{G(s)}{1 + G(s) \cdot H(s)}$

## Advantages of the Nyquist Stability Criterion:

- Simple graphical procedure to determine whether a system is stable or not
- The degree of stability can be easily obtained
- Easy for compensator design
- The response for steady-state sinusoidal inputs can be easily obtained from measurements

# Preview

## Mathematical Background

- Mapping theorem
- Nyquist path

## Nyquist stability criterion

**Z=N+P**

**Z**: Number of zeros of (1+H(s)G(s)) in the right half plane = number of unstable poles of the closed-loop system

**N**: Number of clockwise encirclements of the point –1+j0

**P**: Number of poles of G(s)H(s) in the right half plane

## Application of the Stability Criterion

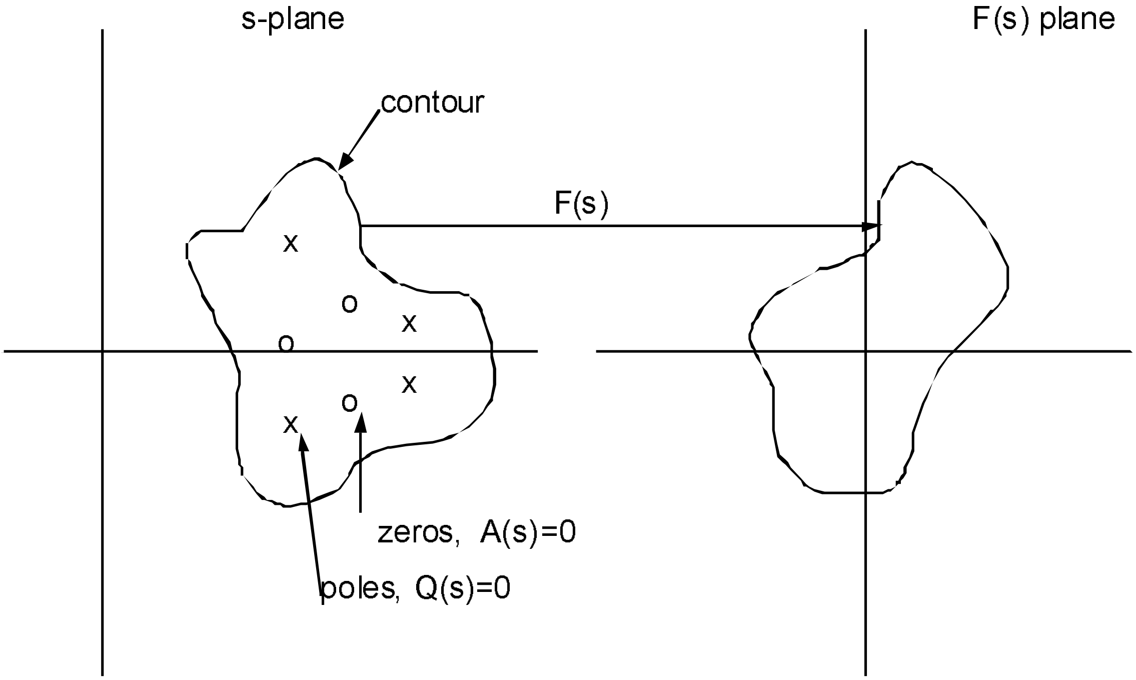
- Sketch the Nyquist plot for $\omega \in (0^+,\infty)$
- Extend to $\omega \in (-\infty,+\infty)$
- Apply the stability criterion (find N and P and compute Z).

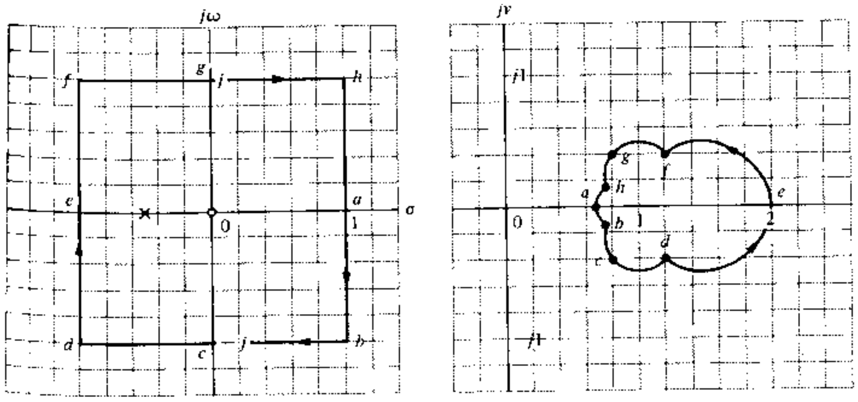
# Mapping Theorem

The total number N of clockwise encirclements of the origin of the F(s) plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to Z – P.

$$F(s) = \frac{A(s)}{Q(s)}$$

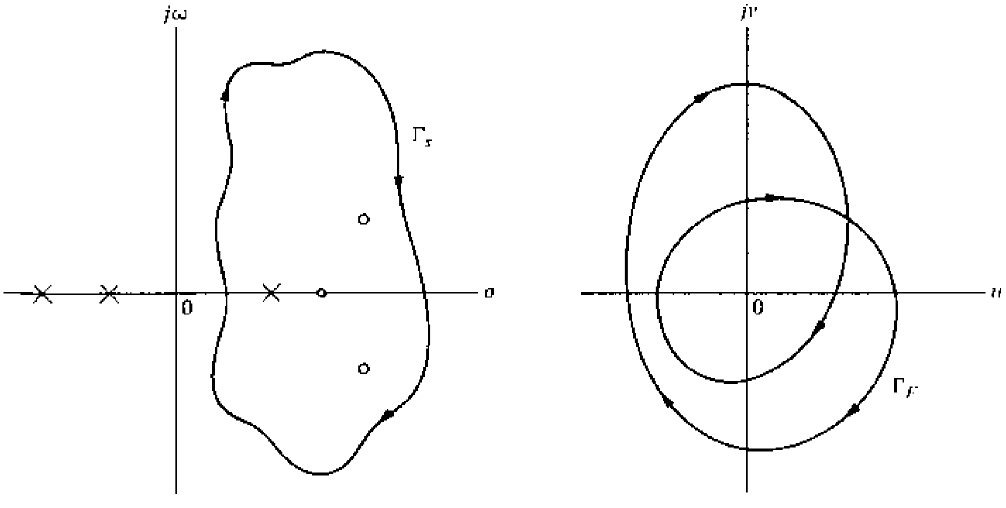
P: Number of poles, Q(s) = 0
Z: Number of zeros, A(s) = 0





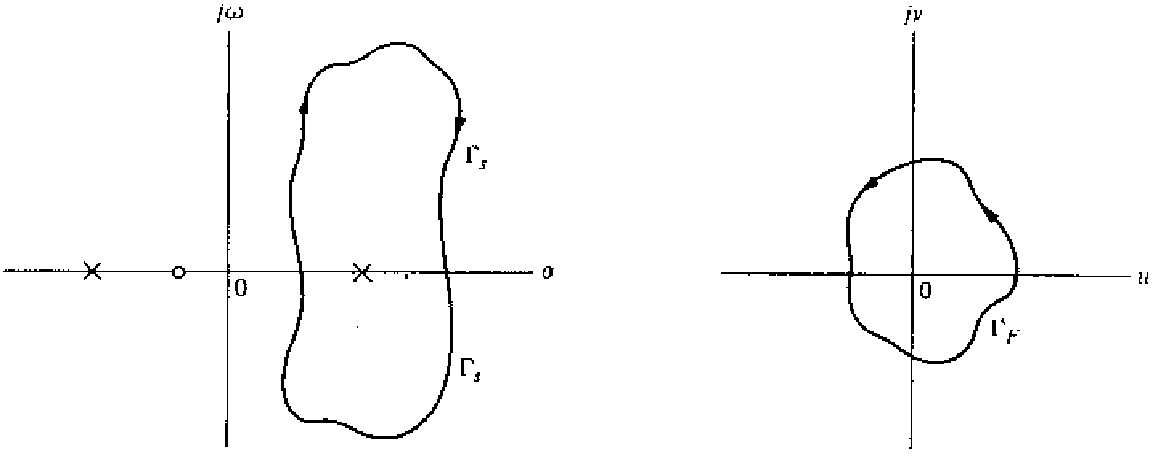
(a) (b)

Mapping for F(s) = s/(s+0.5), (Z = P =1)



(a) (b)

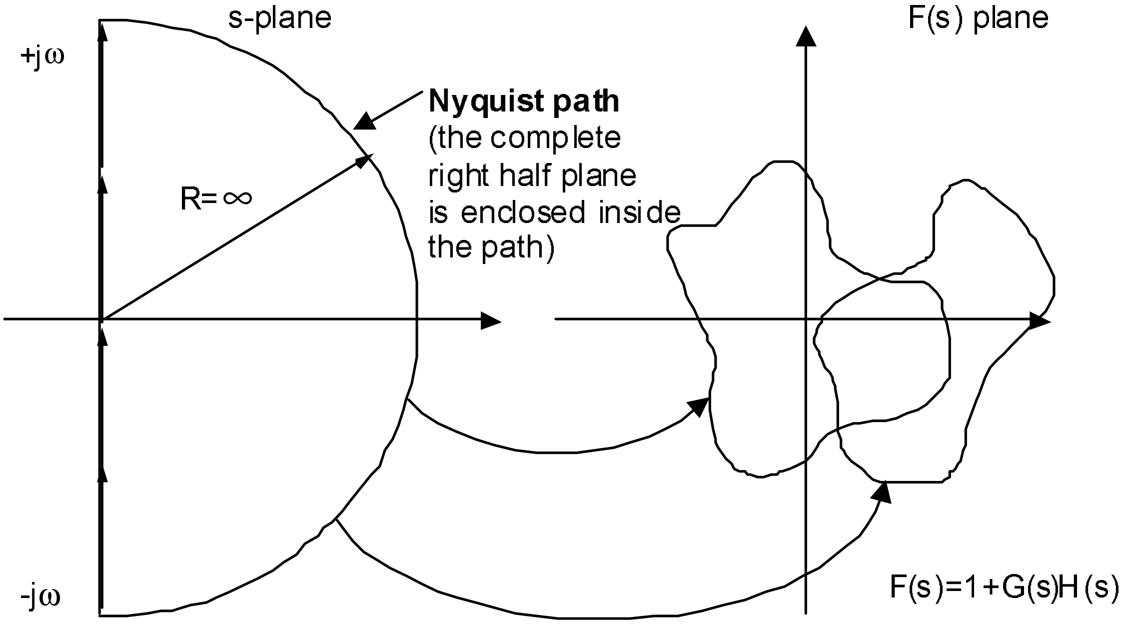
Example of Mapping theorem (Z – P = 2).



(a) (b)

Example of Mapping theorem (Z – P = - 1).

# Application of the mapping theorem to stability analysis

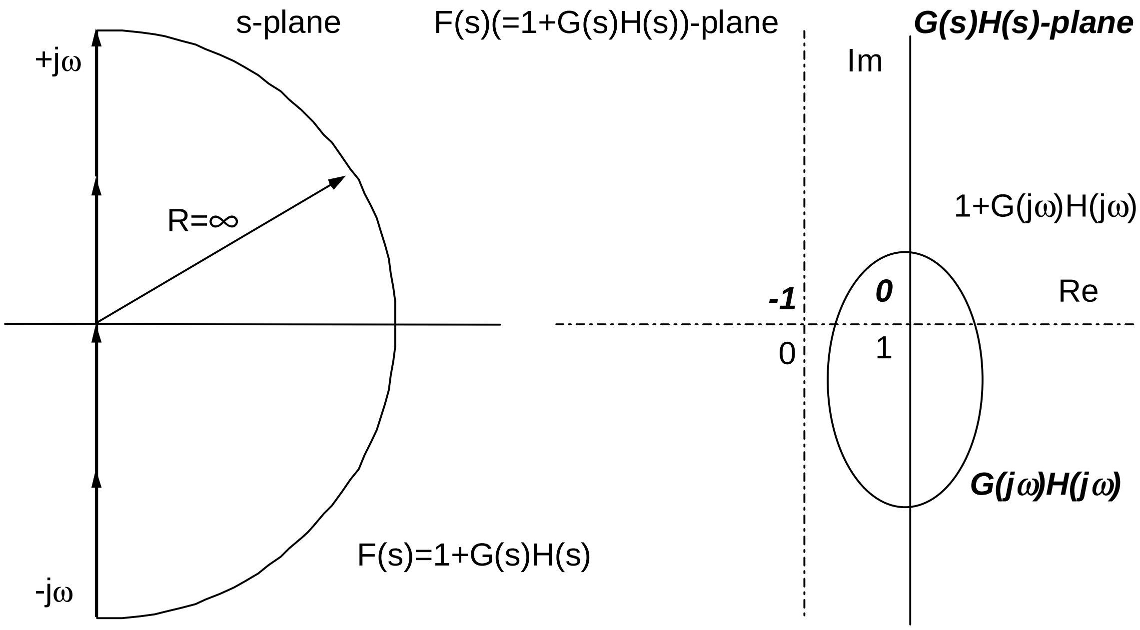


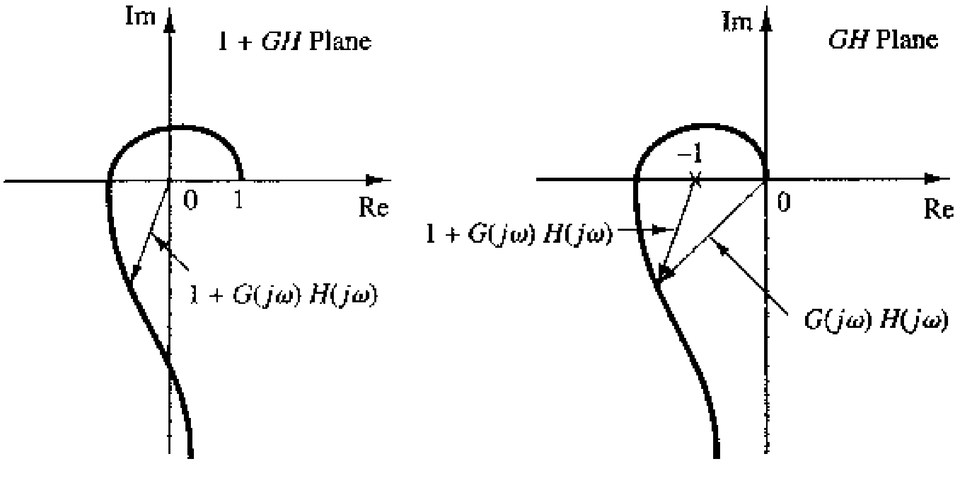
**Mapping theorem**: The number of clockwise encirclements of the origin is equal to the difference between the zeros and poles of F(s) = 1+ G(s)H(s).

Zeros of F(s) = poles of closed-loop system

Poles of F(s) = poles of open-loop system

# Frequency response of open-loop system: G(jω)H(jω)

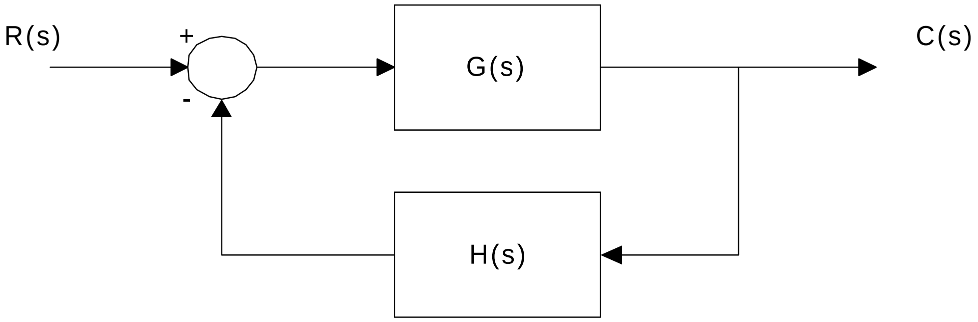




Frequency response of a type 1 system

# Nyquist stability criterion

Consider



The Nyquist stability criterion states that:

$$\underline{Z = N + P}$$

**Z**: Number of zeros of 1+H(s)G(s) in the right half s-plane = number of poles of closed-loop system in right half s-plane.

**N**: Number of clockwise encirclements of the point –1+j0 (when tracing from $\omega = -\infty$ to $\omega = +\infty$).

**P**: Number of poles of G(s)H(s) in the right half s-plane

Thus: if $Z = 0$ → closed-loop system is stable

if $Z > 0$ → closed-loop system has Z unstable poles

if $Z < 0$ → impossible, a mistake has been made

## Alternative form for the Nyquist stability criterion:

If the open-loops system G(s)H(s) has k poles in the right half s-plane, then the closed-loop system is stable if and only if the G(s)H(s) locus for a representative point s tracing the modified Nyquist path, encircles the –1+j0 point k times in the counterclockwise direction.

## Frequency Response of $G(j\omega)H(j\omega)$
## for $\omega : (-\infty,+\infty)$

a) $\omega : (0^+,+\infty)$ : using the rules discussed earlier

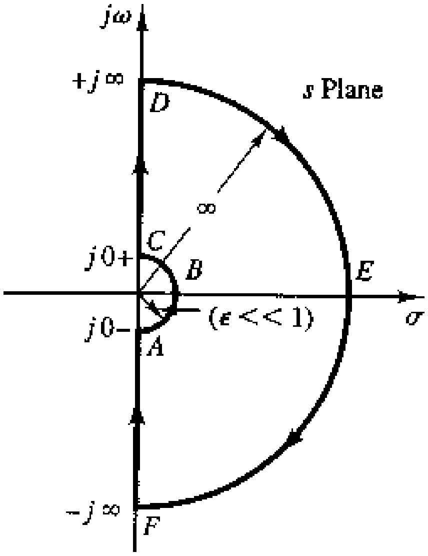
b) $\omega : (0^-,-\infty)$ : $G(-j\omega)H(-j\omega)$ is symmetric with $G(j\omega)H(j\omega)$ (real axis is symmetry axis)

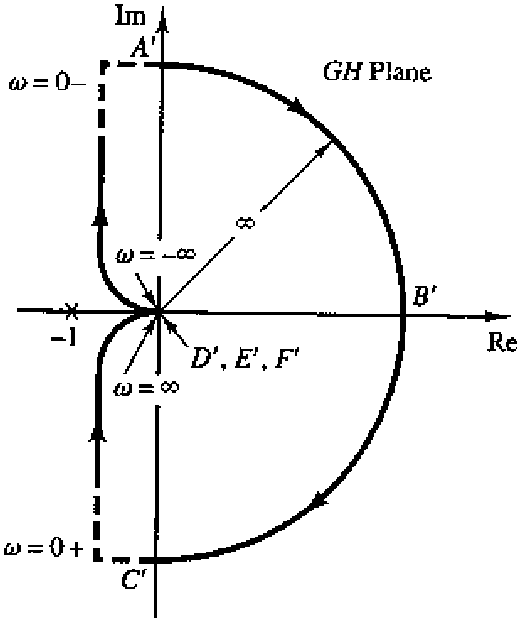
c) $\omega : (0^-,0^+)$ : next page

Poles at the origin for G(s)H(s):

$$G(s)\cdot H(s) = \frac{(\ldots)}{s^{\lambda}(\ldots)}$$

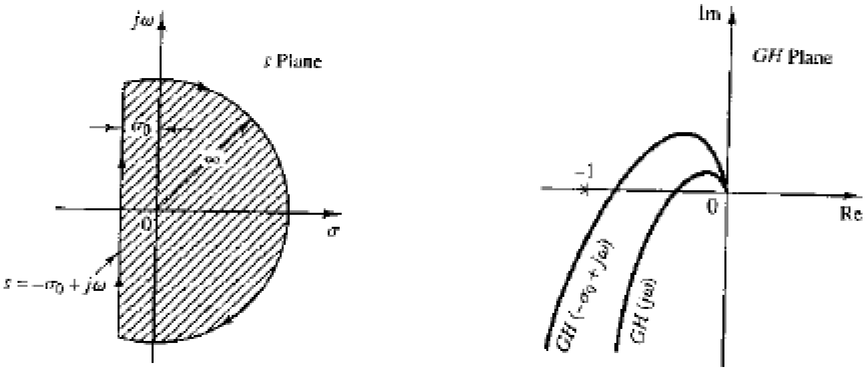
If G(s)H(s) involves a factor $\frac{1}{s^{\lambda}}$, then the plot of G(jω)H(jω), for ω between $0^-$ and $0^+$, has λ clockwise semicircles of infinite radius about the origin in the GH plane. These semicircles correspond to a representative point s moving along the Nyquist path with a semicircle of radius ε around the origin in the s plane.





# Relative Stability

Consider a modified Nyquist path which ensures that the closed-loop system has no poles with real part larger than $-\sigma_0$ :



(a) (b)

Another possible modified Nyquist path:

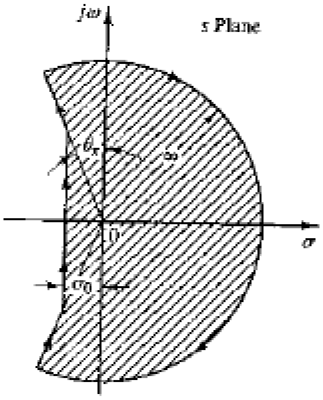


**Figure 8–69**
Modified Nyquist path.

# Phase and Gain Margins

A measure for relative stability of the closed-loop system is how close G(jω), the frequency response of the open-loop system, comes to –1+j0 point. This is represented by phase and gain margins.

**Phase margin**: The amount of additional phase lag at the Gain Crossover Frequency $\omega_o$ required to bring the system to the verge of instability.

Gain Crossover Frequency: $\omega_o$ for which $\left|G\left(j\omega_O\right)\right|=1$

Phase margin: $\gamma = 180^\circ + \angle G(j\omega_o) = 180^\circ + \phi$

**Gain margin:** The reciprocal of the magnitude $\left|G\left(j\omega_1\right)\right|$ at the Phase Crossover Frequency $\omega_1$ required to bring the system to the verge of instability.

Phase Crossover Frequency: $\omega_1$ where $\angle G\left(j\omega_1\right) = -180^\circ$

Gain margin: $K_g = \dfrac{1}{\left|G(j\omega_1)\right|}$

Gain margin in dB: $K_g \ in\ dB = -20\log\left|G\left(j\omega_1\right)\right|$

$K_g$ in dB > 0 = stable (for minimum phase systems)
$K_g$ in dB < 0 = unstable (for minimum phase systems)

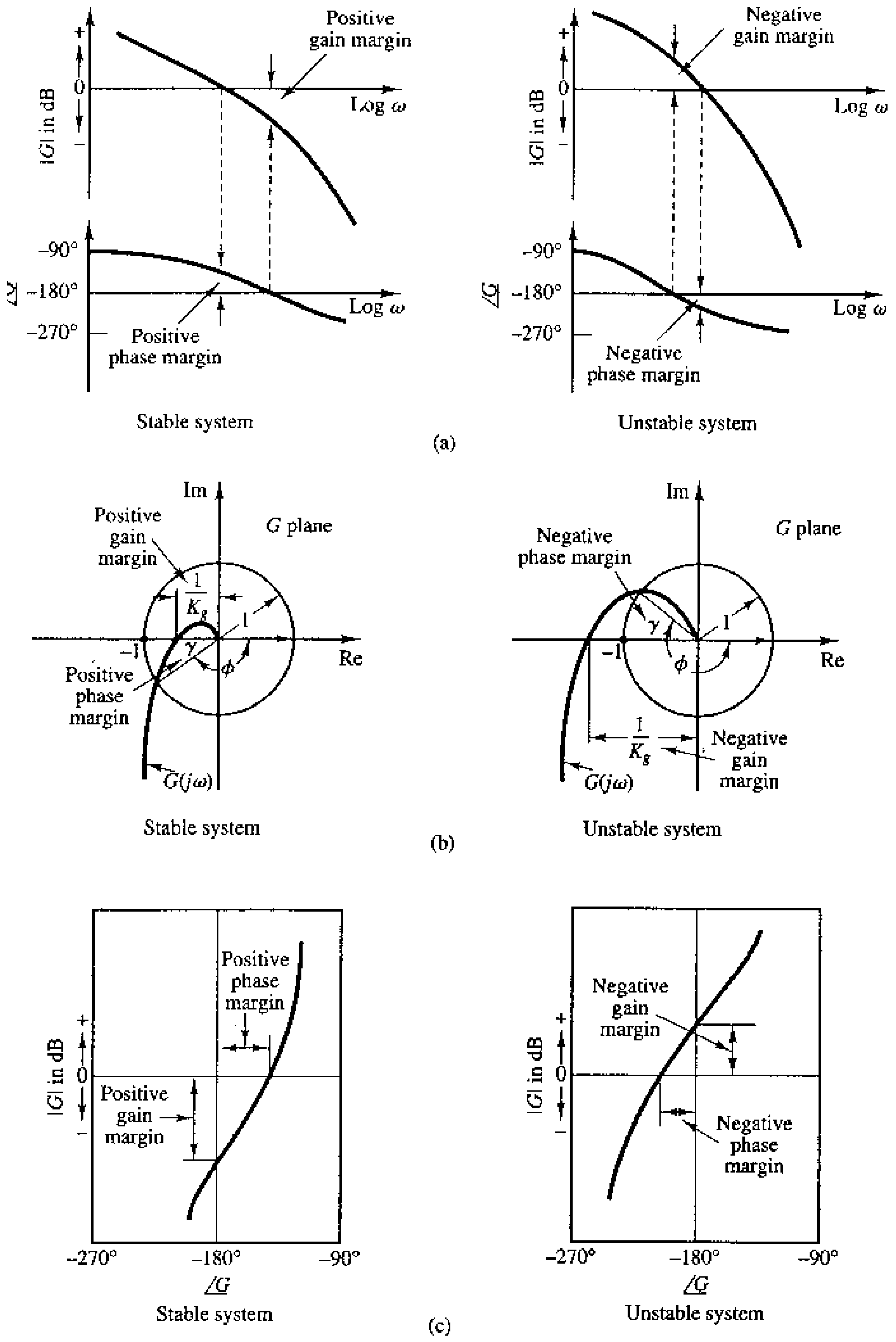


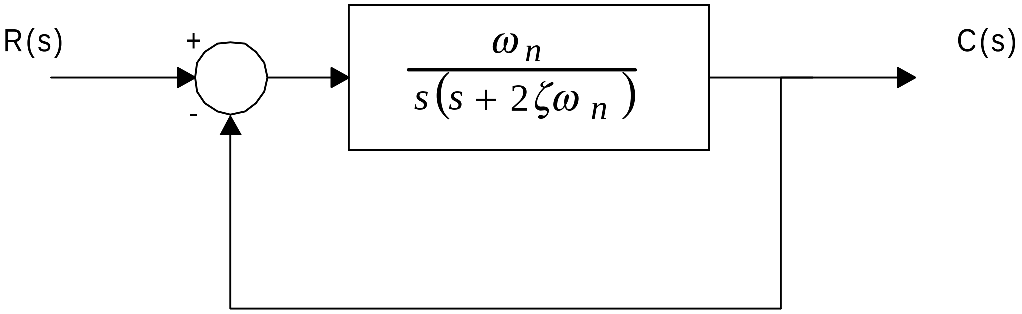
**Figure:** Phase and gain margins of stable and unstable systems (a) Bode diagrams; (b) Polar plots; (c) Log-magnitude-versus-phase plots.

If the open-loop system is minimum phase and has both phase and gain margins positive,

→ then the closed-loop system is stable.

- For good relative stability both margins are required to be positive.
- Good values for minimum phase system:
  - Phase margin : 30° – 60°
  - Gain margin: above 6 dB

# Correlation between damping ratio and frequency response for 2nd order systems



$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\frac{C(j\omega)}{R(j\omega)}=M(\omega)\cdot e^{j\alpha(\omega)}$$

*Phase margin*:

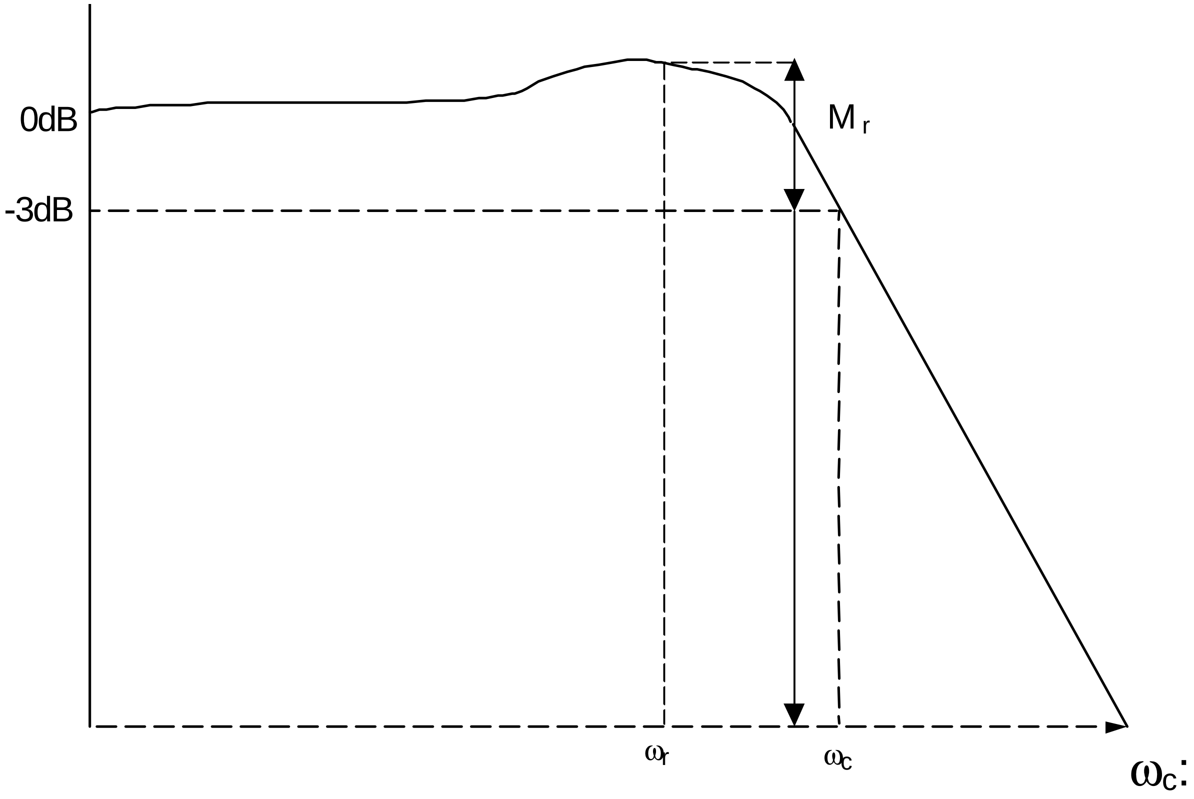
$\gamma = 180° + \angle G(j\omega)$   $G(j\omega)$: open loop transfer function

$\left|G(j\omega)\right| = \left|\frac{\omega_n^2}{j\omega(j\omega+2\zeta\omega_n)}\right|$ becomes unity for $\omega_1 = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$

and $$\gamma = \tan^{-1}\left[\frac{2\zeta\omega_n}{\omega_1}\right] = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+\zeta^4}-2\zeta^2}}\right]$$

→ $\gamma$ depends only on $\zeta$

# Performance specifications in the frequency domain:



$\omega_c$ : *Cutoff Frequency*

$0 \le \omega \le \omega_c$ : *Bandwidth*

Slope of log-magnitude curve: *Cutoff Rate*

- ability to distinguish between signal and noise

$\omega_r$: *Resonant Frequency*

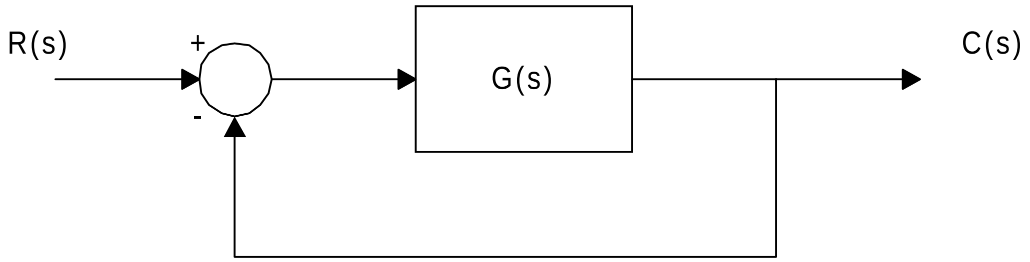
- indicative of transient response speed
- $\omega_r$ → increase, transient response faster (dominant complex conjugate poles assumed)

$M_r = \max|G(j\omega)|$ : *Resonant Peak*

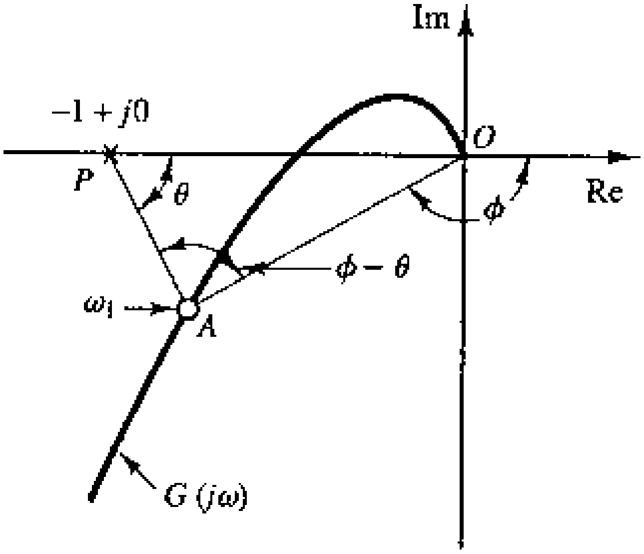
# Closed-Loop Frequency Response

Open-loop system: G(s)

Stable closed-loop system: $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$



$$\frac{|G(s)|}{|1+G(s)|} = \frac{\left|\overrightarrow{OA}\right|}{\left|\overrightarrow{PA}\right|}$$



## Closed-Loop Frequency Response:

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} = M \cdot e^{j\alpha}$$

## Constant Magnitude Loci:

$$G(j\omega) = X + jY$$

$$M = \frac{|X + jY|}{|1 + X + jY|} = const$$

$$\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{\left(M^2 - 1\right)^2}$$

## Constant Phase-Angle Loci

$$G(j\omega) = X + jY \rightarrow \angle e^{j\alpha} = \angle \frac{X + jY}{1 + X + jY} = const$$

$$\left(X + \tfrac{1}{2}\right)^2 + \left(Y + \tfrac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2, \quad N = \tan\alpha$$

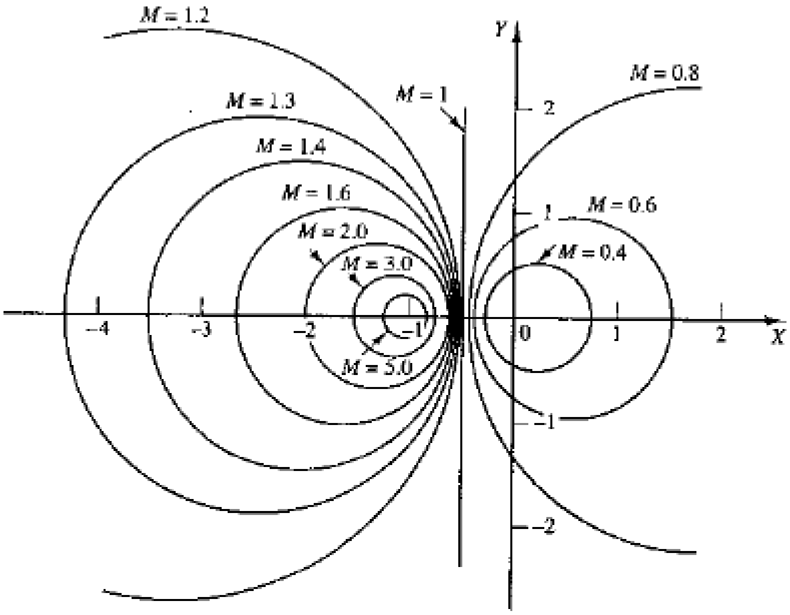
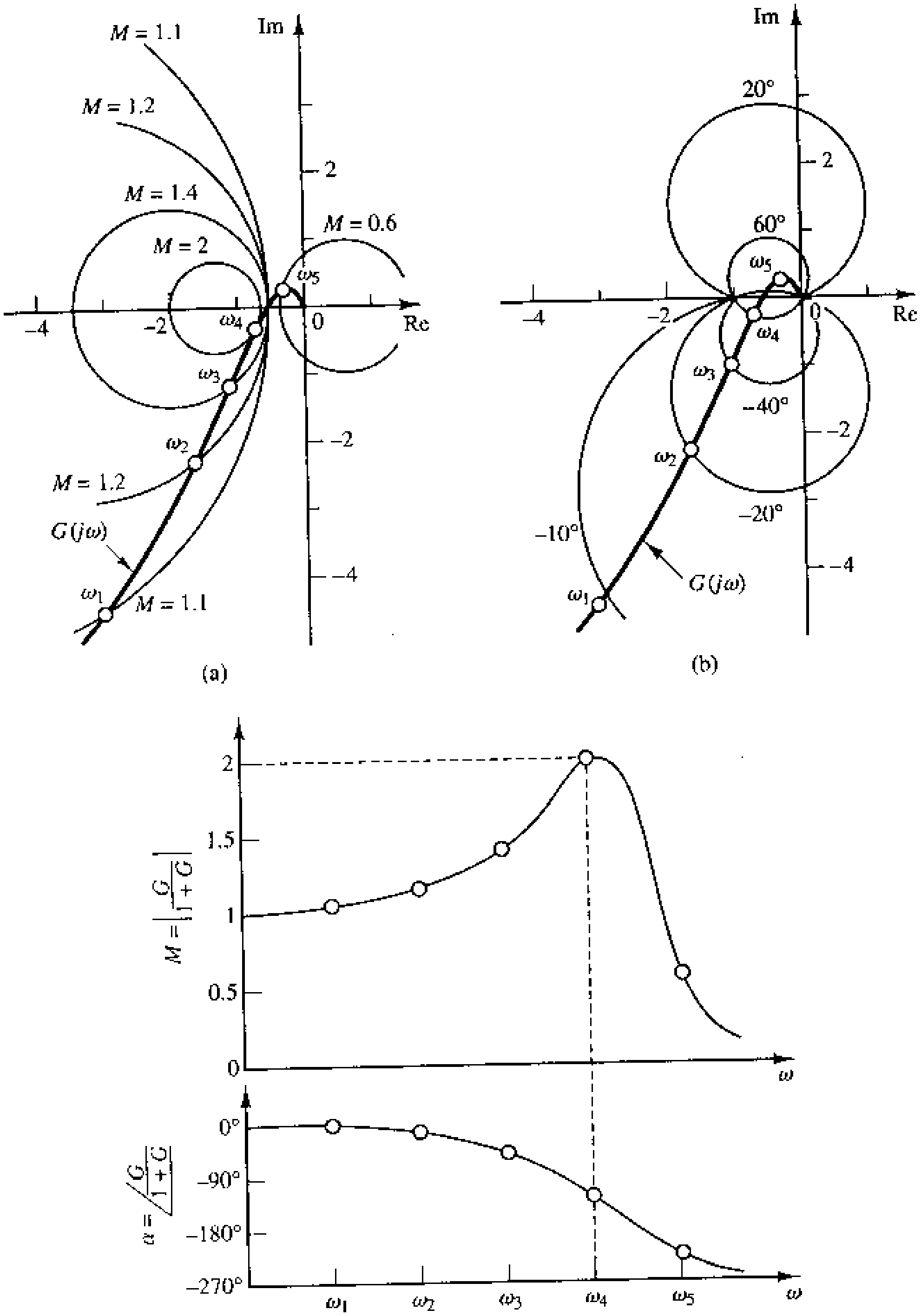


**Figure:** A family of constant M circles.



(a) (b) (c)

**Figure:**
(a) G(jω) locus superimposed on a family of M circles;
(b) G(jω) locus superimposed om a family of N circles;
(c) Closed-loop frequency-response curves

# Experimental Determination of Transfer Function

- Derivation of mathematical model is often difficult and may involve errors.

- Frequency response can be obtained using sinusoidal signal generators.

  Measure the output and obtain:
  - Magnitudes (quite accurate)
  - Phase (not as accurate)

Use the Magnitude data and asymptotes to find:

- Type and error coefficients
- Corner frequencies
- Orders of numerator and denominator
- If second order terms are involved, $\zeta$ is obtained from the resonant peak.

Use phase to determine if system is minimum phase or not:

- Minimum phase: $\omega \rightarrow \infty$ phase = -90 (n - m)
  (n-m) difference in the order of denominator and numerator.