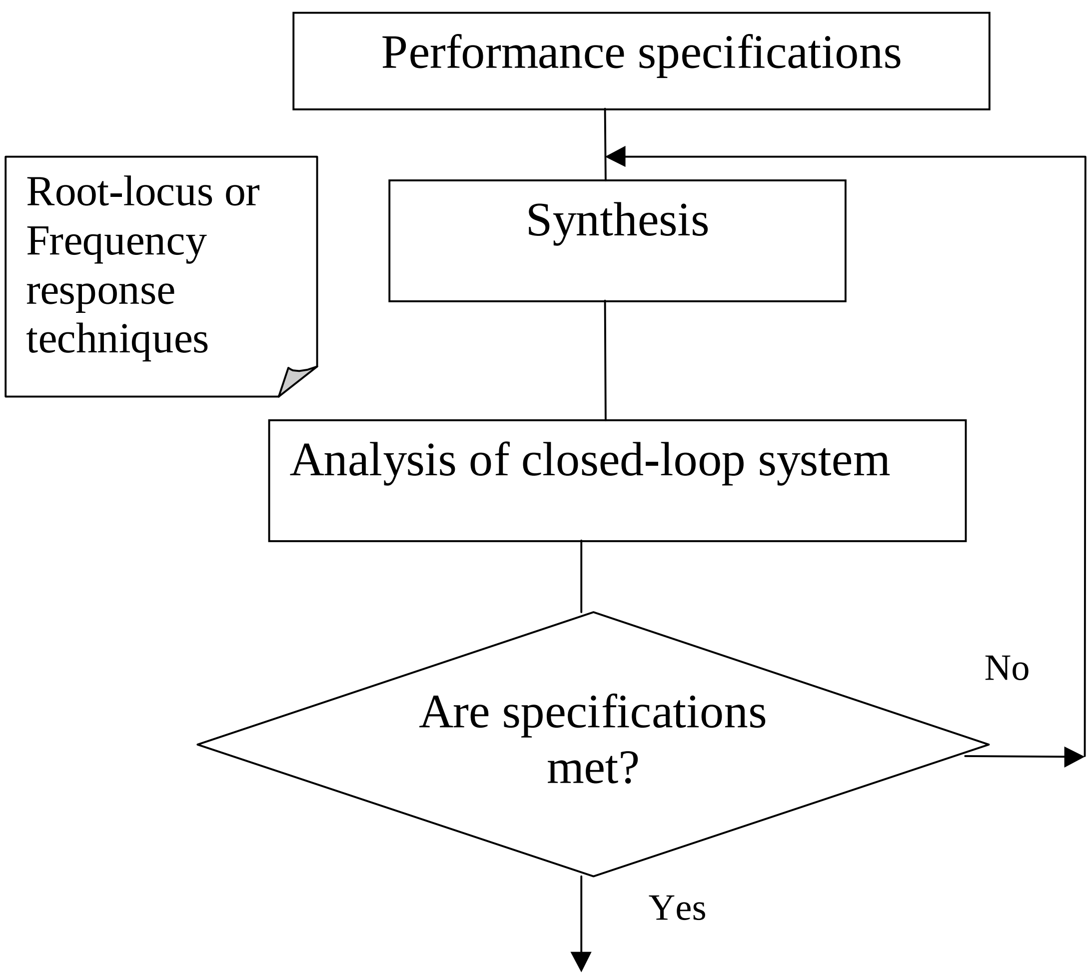
Compensation Techniques

- Performance specifications for the closed-loop system

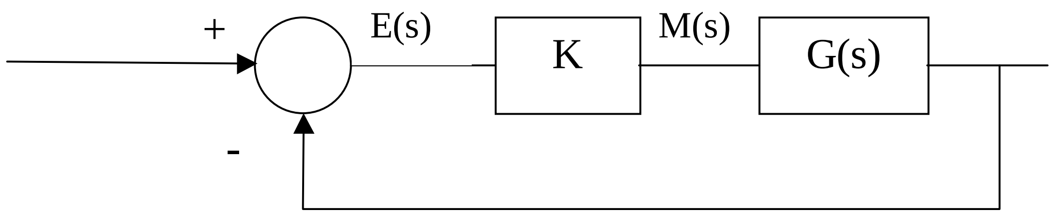
  - Stability
  - Transient response → T_s, M_s (settling time, overshoot) or phase and gain margins
  - Steady-state response → e_{ss} (steady state error)

- Trial and error approach to design



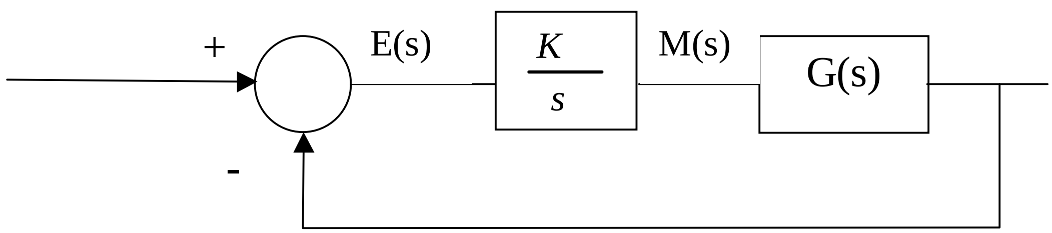
Basic Controls

1. Proportional Control



$$\frac{M(s)}{E(s)} = K \qquad m(t) = K \cdot e(t)$$

2. Integral Control

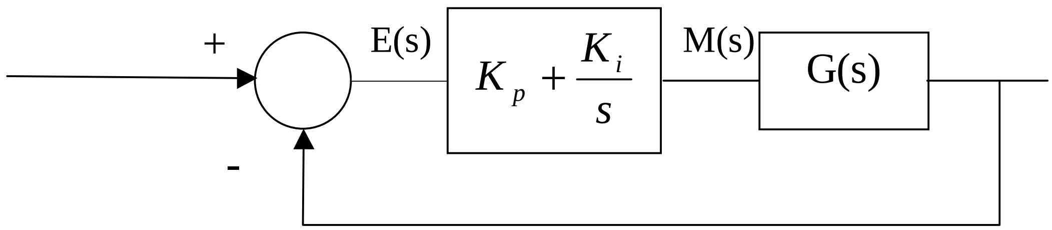


$$\frac{M(s)}{E(s)} = \frac{K}{s} \qquad m(t) = K \int e(t)dt$$

Integral control adds a pole at the origin for the open-loop:

- Type of system increased, better steady-state performance.
- Root-locus is "pulled" to the left tending to lower the system's relative stability.

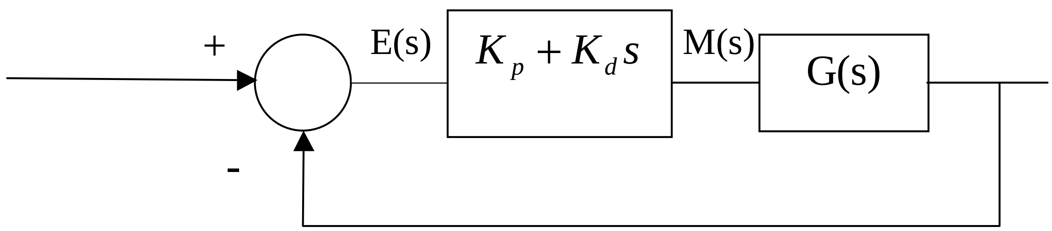
3. Proportional + Integral Control



$$\frac{M(s)}{E(s)} = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \qquad m(t) = K_p e(t) + \int K_i e(t) dt$$

A pole at the origin and a zero at $-\frac{K_i}{K_p}$ are added.

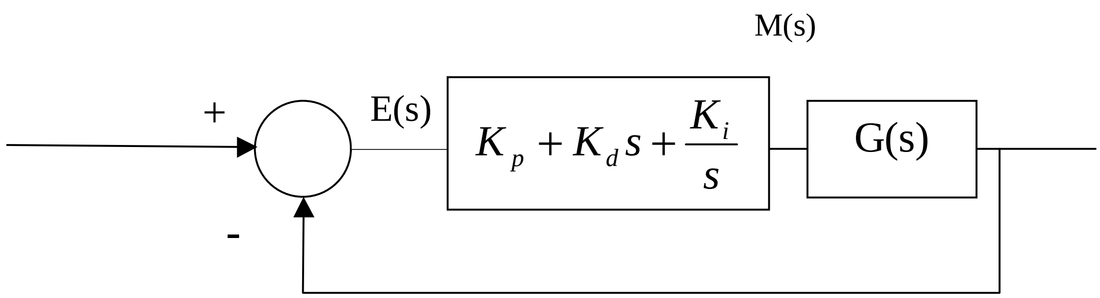
4. Proportional + Derivative Control



$$\frac{M(s)}{E(s)} = K_p + K_d s \qquad m(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

- Root-locus is “pulled” to the left, system becomes more stable and response is sped up.
- Differentiation makes the system sensitive to noise.

5. Proportional + Derivative + Integral (PID) Control

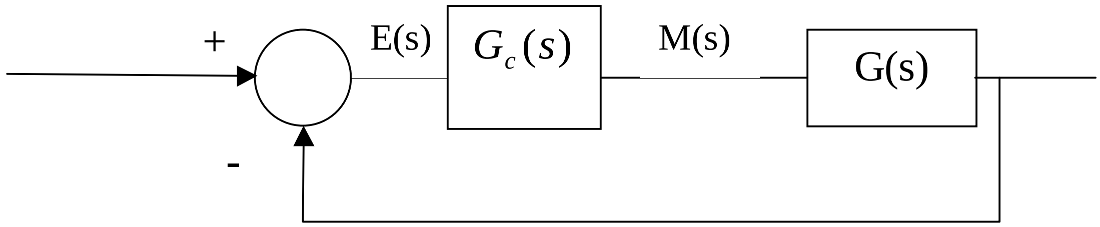


$$\frac{M(s)}{E(s)} = K_p + K_d d + \frac{K_i}{s}$$

$$m(t) = K_p e(t) + K_d \frac{de(t)}{dt} + K_i \int e(t) dt$$

- More than 50% of industrial controls are PID.
- More than 80% in process control industry.
- When G(s) of the system is not known, then initial values for K_p, K_d, K_i can be obtained experimentally and than fine-tuned to give the desired response (Ziegler-Nichols).

6. Feed-forward compensator



Design $G_c(s)$ using Root-Locus or Frequency Response techniques.

Frequency response approach to compensator design

Information about the *performance of the closed-loop system*, obtained from the *open-loop frequency response*:

- *Low frequency* region indicates the steady-state behavior.
- *Medium frequency* (around -1 in polar plot, around gain and phase crossover frequencies in Bode plots) indicates relative stability.
- *High frequency* region indicates complexity.

Requirements on open-loop frequency response

- The gain at low frequency should be large enough to give a high value for error constants.
- At medium frequencies the phase and gain margins should be large enough.
- At high frequencies, the gain should be attenuated as rapidly as possible to minimize noise effects.

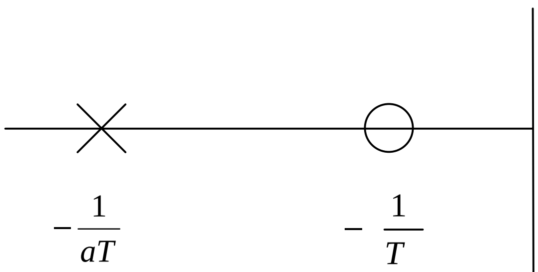
Compensators

- *lead*:improves the transient response.
- *lag*: improves the steady-state performance at the expense of slower settling time.
- *lead-lag*: combines both

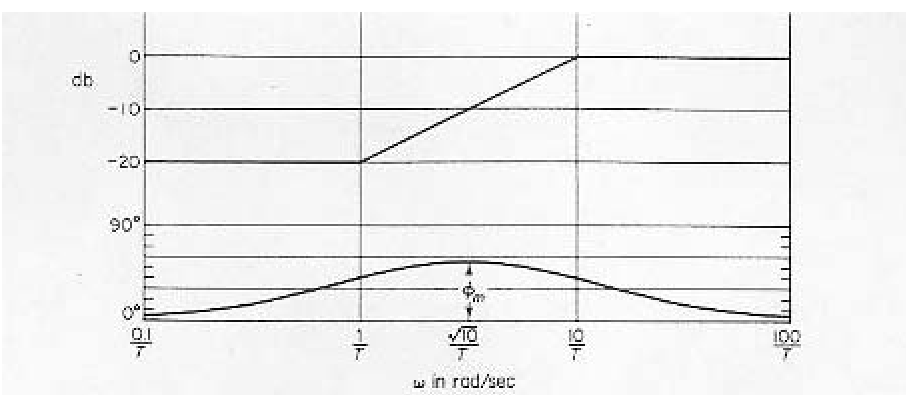
Lead compensators

$$G_c(s) = K_c a \frac{Ts+1}{aTs+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{aT}} \qquad T > 0 \quad \text{and} \quad 0 < \alpha < 1$$

- Poles and zeros of the lead compensator:



- Frequency response of $G_c(j\omega)$:



The maximum phase-lead angle ϕ_m occurs at ω_m , where:

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \qquad \text{and}$$

$$\log\omega_m = \frac{1}{2}\left[\log T + \log\frac{1}{aT}\right] \quad \rightarrow \quad \omega_m = \frac{1}{\sqrt{a}\,T}$$

Since

$$\left|\frac{1+j\omega T}{1+j\omega aT}\right|_{\omega=\omega_m} = \frac{1}{\sqrt{a}}$$

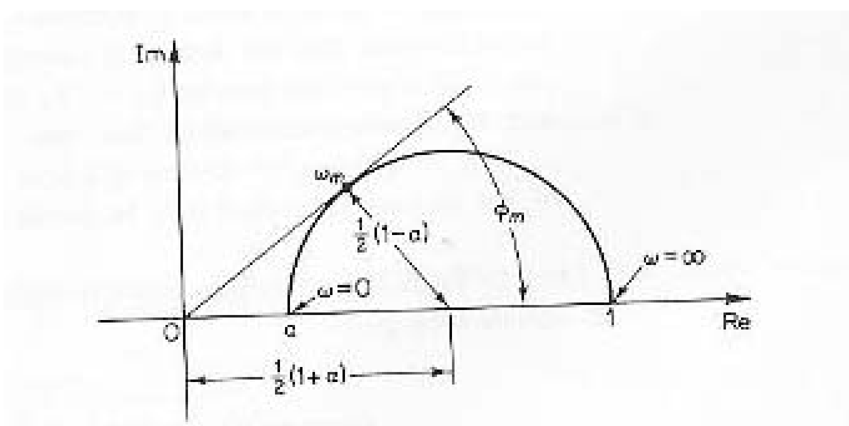
the magnitude of $G_c(j\omega)$ at ω_m is given by:

$$|G_c(j\omega_m)| = K_c\sqrt{a}$$

Polar plot of a lead network

$$\frac{a(j\omega T+1)}{(j\omega aT+1)}$$ where $0 < a < 1$

is given by



Lead compensation based on the frequency response

Procedure:

1. Determine the compensator gain $K_c\alpha$ satisfying the given error constant.
2. Determined the additional phase lead ϕ_m required (+ 10%~15%) for the gain adjusted ($K_c\alpha G(s)$) open-loop system.
3. Obtain α from $\sin\phi_m = \frac{1-a}{1+a}$
4. Find the new gain cross over frequency ω_c from

$$K_c a \left| G(j\omega_c) \right| = 10 \log a$$

5. Find T from ω_c and transfer function of $G_c(s)$

$$T = \frac{1}{\sqrt{a}\,\omega_c}$$ and

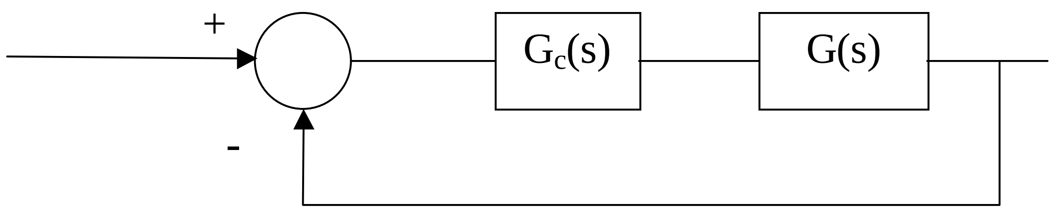
$$G_c(s) = K_c a \frac{Ts+1}{aTs+1}$$

General effect of lead compensator:

- Addition of phase lead near gain crossover frequency.
- Increase of gain at higher frequencies.
- Increase of system bandwidth.

Example:

Consider



where $G(s) = \dfrac{4}{s(s+2)}$

Performance requirements for the system:

Steady-state:	$K_v = 20$
Transient response:	phase margin >50°
	gain margin >10 dB

Analysis of the system with $G_c(s) = K$

For $K_v = 20$ → K = 10

This leads to:	phase margin ≈ 17°
	gain margin ≈ +∞ dB

Design of a lead compensator:

$$G_c(s) = K_c a \frac{Ts+1}{aTs+1}$$

1. $K_v = \lim_{s\to 0} sG_c(s)G(s) = \dfrac{4K_c a}{2} = 2K_c a = 20$ → $K_c\alpha = 10$

2. From the Bode plot of $K_c\alpha G(j\omega)$, we obtain that the additional phase-lead required is: 50° - 17° = 33°.

   We choose 38° (~33° + 15%)

3. $$\sin\phi_m = \sin 38^\circ = \frac{1-a}{1+a}$$ → $\alpha = 0.24$

4. Since for ω_m, the frequency with the maximum phase-lead angle, we have:

$$\left|\frac{1+j\omega_m T}{1+j\omega_m aT}\right| = \frac{1}{\sqrt{a}}$$

We choose ω_c, the new gain crossover frequency so that

$\omega_m = \omega_c$ and $\left|G_c(s)G(s)\right|_{s=j\omega_c} = 1$

This gives that:

$$\left|K_c aG(j\omega_c)\right| = \left|\frac{40}{j\omega_c(j\omega_c+2)}\right|$$

has to be equal:

$$\left(\frac{1}{\sqrt{a}}\right)^{-1} = -6.2dB$$

From the Bode plot of $K_c\alpha G(j\omega)$ we obtain that

$$\left|\frac{40}{j\omega_c(j\omega_c+2)}\right| = -6.2dB$$ at $\omega_c = 9$ rad/sec

5. This implies for T

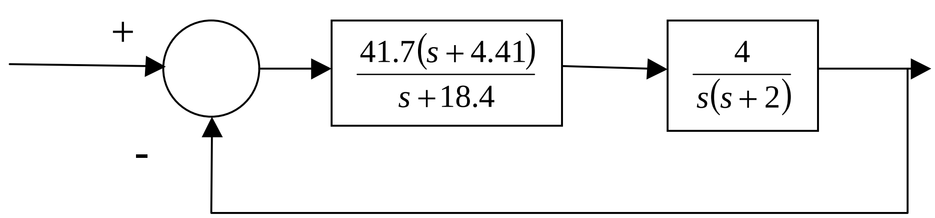
$$\omega_c = \frac{1}{\sqrt{aT}} = \frac{1}{\sqrt{0.24T}} = 9\,rad/\sec \quad \rightarrow \quad \frac{1}{T} = 4.41$$

and

$$K_c = \frac{20}{2a} = 41.7$$

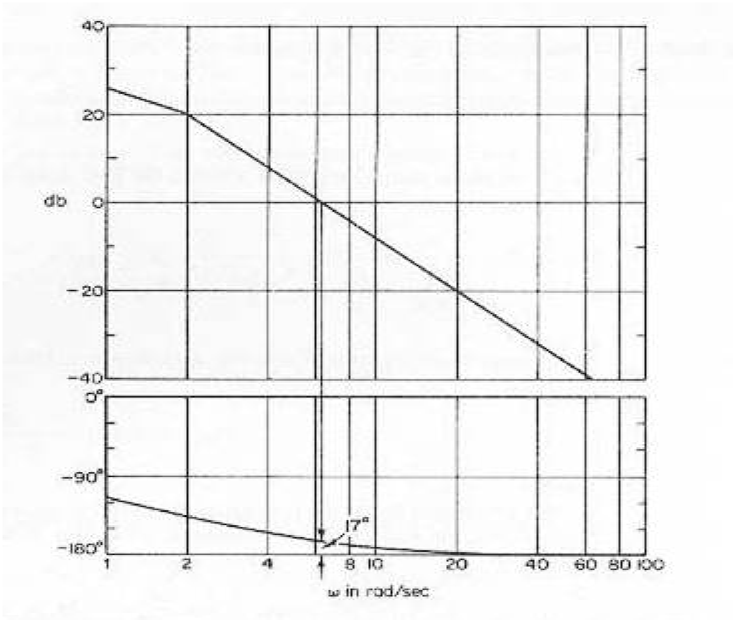
$$G_c(s) = 41.7\,\frac{s + 4.41}{s + 18.4}$$

The compensated system is given by:

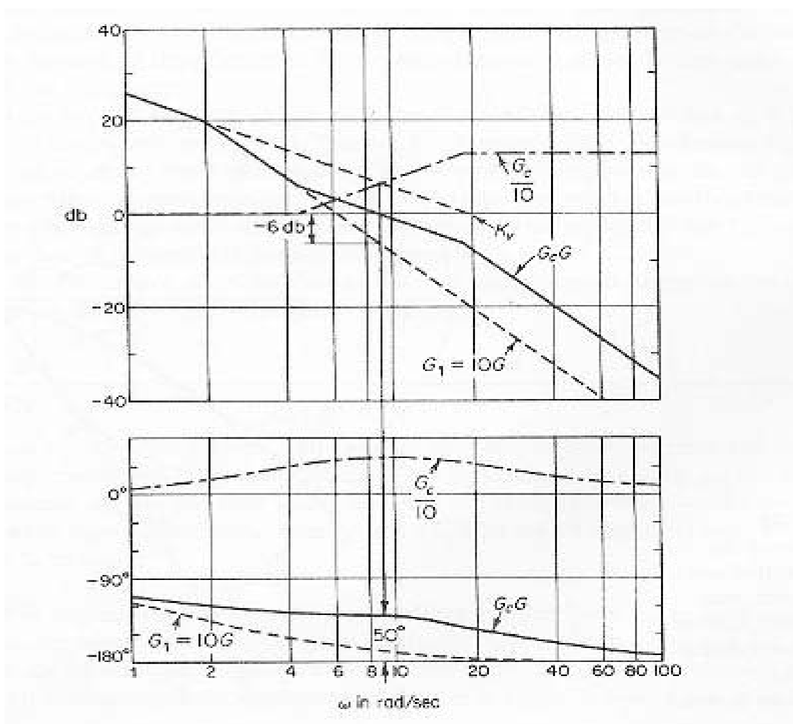


The effect of the lead compensator is:

- Phase margin: from 17° to 50° → better transient response with less overshoot.
- ω_c : from 6.3rad/sec to 9 rad/sec → the system response is faster.
- Gain margin remains ∞ .
- K_v is 20, as required → acceptable steady-state response.



Bode diagram for $K_c \cdot a \cdot G(j\omega) = \dfrac{40}{j\omega(j\omega+2)}$



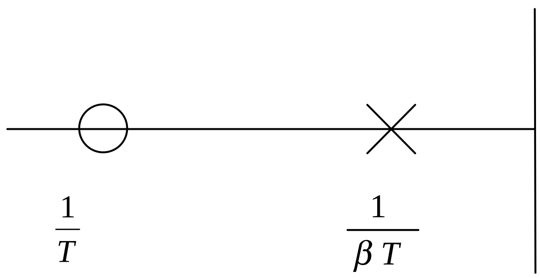
Bode diagram for the compensated system

$$G_c(j\omega)G(j\omega) = 41.7\,\frac{j\omega + 4.41}{j\omega + 18.4} \cdot \frac{4}{j\omega(j\omega + 2)}$$

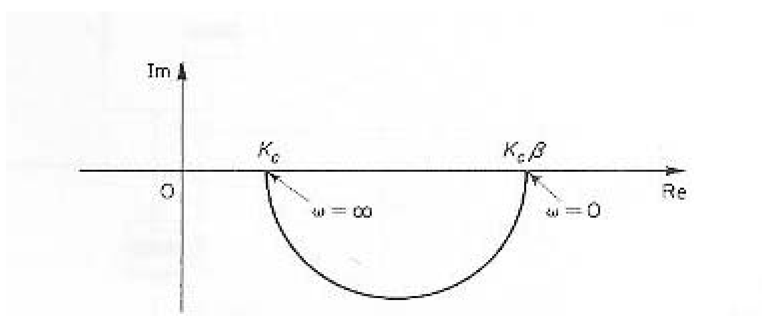
Lag compensators

$$G_c(s) = K_c\beta \frac{Ts+1}{\beta Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad T > 0\,,\ \beta > 1$$

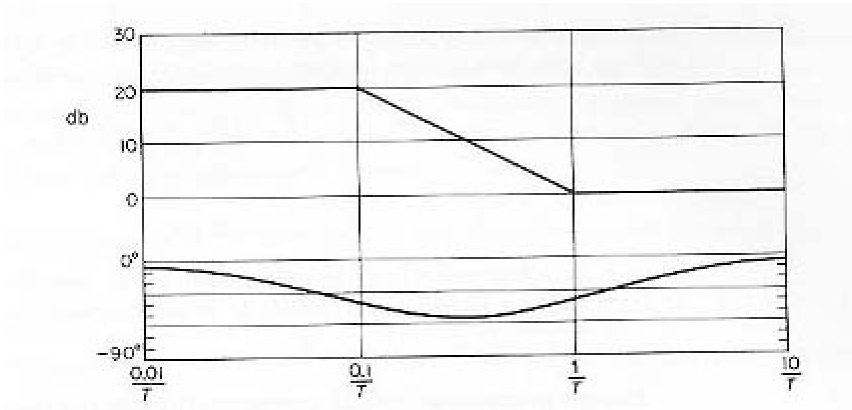
Poles and zeros:



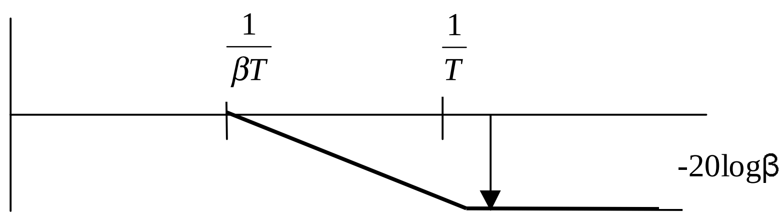
Frequency response:



Polar plot of a lag compensator $K_c\beta(j\omega T+1)/(j\omega\beta T+1)$



Bode diagram of a lag compensator with K_c=1, β = 10



Magnitude of *(jωT+1)/(jωβT+1)*

Lag compensation based on the frequency response

Procedure:

1. Determine the compensator gain $K_c\beta$ to satisfy the requirement for the given error constant.
2. Find the frequency point where the phase of the gain adjusted open-loop system ($K_c\beta G(s)$) is equal to -180° + the required phase margin + 5°~ 12°.

   This will be the new gain crossover frequency ω_c.
3. Choose the zero of the compensator $\omega = 1/T$ at about 1 octave to 1 decade below ω_c .
4. Determine the attenuation necessary to bring the magnitude curve down to 0dB at the new gain crossover frequency

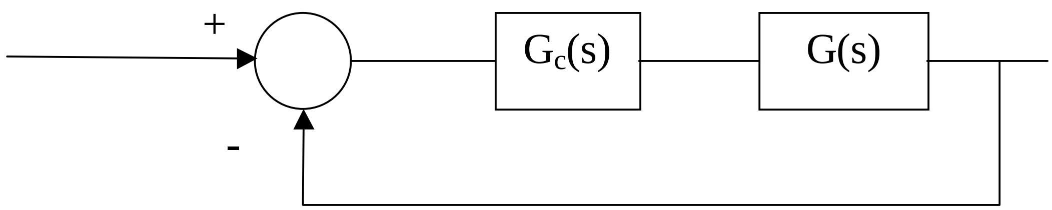
   $$-K_c\beta\left|G\left(j\omega_c\right)\right| = -20\ \log\ \beta \quad \rightarrow \ \beta$$
5. Find the transfer function $G_c(s)$.

General effect of lag compensation:

- Decrease gain at high frequencies.
- Move the gain crossover frequency lower to obtain the desired phase margin.

Example:

Consider



where

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

Performance requirements for the system:

Steady state:	$K_v = 5$
Transient response:	Phase margin > 40°
	Gain margin > 10 dB

Analysis of the system with $G_c(s) = K$

$$K_v = \lim_{s \to 0} KG(s) = K = 5$$

for K = 5, the closed-loop system is unstable

Design of a lag compensator:

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts + 1}{\beta Ts + 1}$$

1. $K_v = \lim_{s \to 0} G_c(s)G(s) = K_c\beta = 5$
2. Phase margin of the system 5G(s) is -13°

   → the closed-loop system is unstable.

   From the Bode diagram of 5G(jω) we obtain that the additional required phase margin of 40° + 12° = 52° is obtained at ω = 0.5 rad/sec.

   The new gain crossover frequency will be:

   $\omega_c = 0.5$ rad/sec

3. Place the zero of the lag compensator at ω = 1/T = 0.1 rad/sec(at about 1/5 of ω_c).

4. The magnitude of 5G(jω) at the new gain crossover frequency ω_c =0.5 rad/sec is 20 dB. In order to have ω_c as the new gain crossover frequency, the lag compensator must give an attenuation of -20db at ω_c.

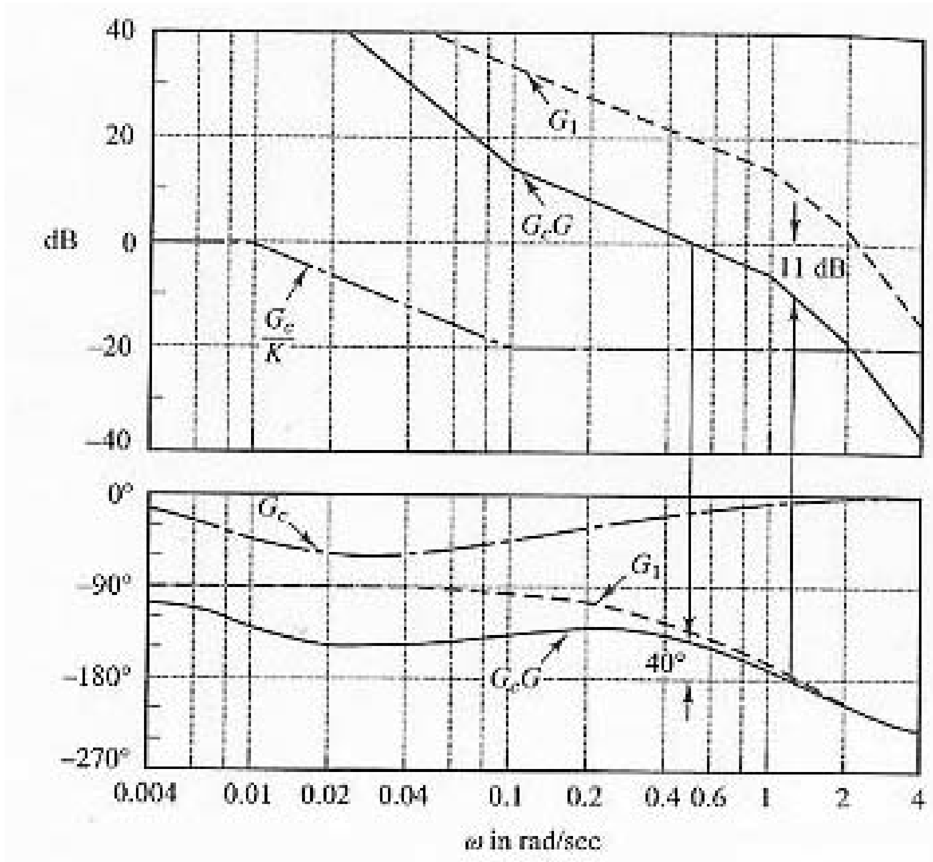
   Therefore

   $$-20\log\beta = -20 \text{ dB} \quad \rightarrow \quad \beta = 10$$

5. $K_c = \frac{5}{\beta} = 0.5$, *pole* : $\frac{1}{\beta T} = 0.01$

   and

   $$G_c(s) = 0.5\frac{s + 0.1}{s + 0.01}$$



Bode diagrams for:

- $G_1(j\omega) = 5G(j\omega)$ (gain-adjusted $K_c\beta G(j\omega)$ open-loop transfer function),
- $G_c(j\omega)/K = G_c(j\omega)/5$ (compensator divided by gain $K_c\beta = 5$),
- $G_c(j\omega)G(j\omega)$ (compensated open-loop transfer function)

The effect of the lag compensator is:

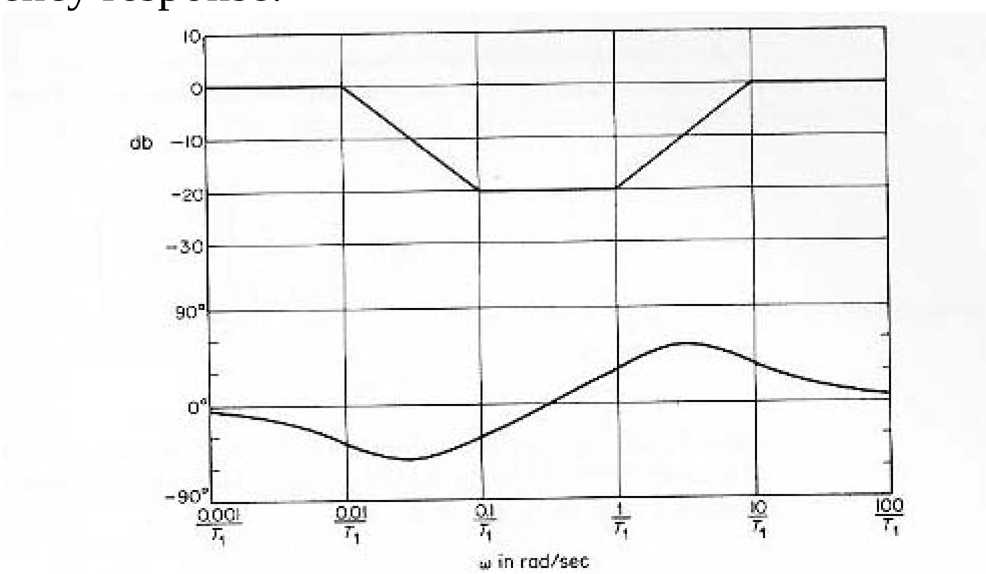
- The original unstable closed-loop system is now stable.
- The phase margin ≈ 40° → acceptable transient response.
- The gain margin ≈ 11dB → acceptable transient response.
- K_v is 5 as required → acceptable steady-state response.
- The gain at high frequencies has been decreased.

Lead-lag compensators

$$G_c(s) = K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{aT_1}} \cdot \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} = K_c \frac{\beta}{\gamma} \frac{sT_1+1}{s\frac{T_1}{\gamma}+1} \cdot \frac{sT_2+1}{s\beta T_2+1}$$

$T_1, T_2 > 0$, $\beta > 1$ and $\gamma > 1$

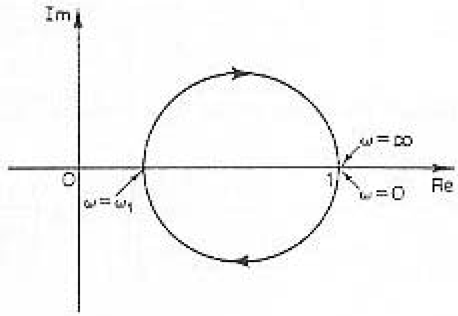
Frequency response:



Bode diagram of a lag-lead compensator given by

$$G_c(s) = K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \right) \left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right)$$

with $K_c = 1$, $\gamma = \beta = 10$ and $T_2 = 10\,T_1$



Polar plot of a lag-lead compensator given by

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

with $K_c = 1$ and $\gamma = \beta$

<u>Comparison between lead and lag compensators</u>

Lead compensator	**Lag** compensator
o High pass	o Low pass
o Approximates derivative plus proportional control	o Approximates integral plus proportional control
o Contributes phase lead	o Attenuation at high frequencies
o Increases the gain crossover frequency	o Moves the gain-crossover frequency lower
o Increases bandwidth	o Reduces bandwidth