ELEC 360 Control Theory and Systems I

Part A: Introduction

Laplace Transforms

Mathematical Modeling of Dynamic Systems

Part B: Transient Response Analysis

Steady State Response Analysis

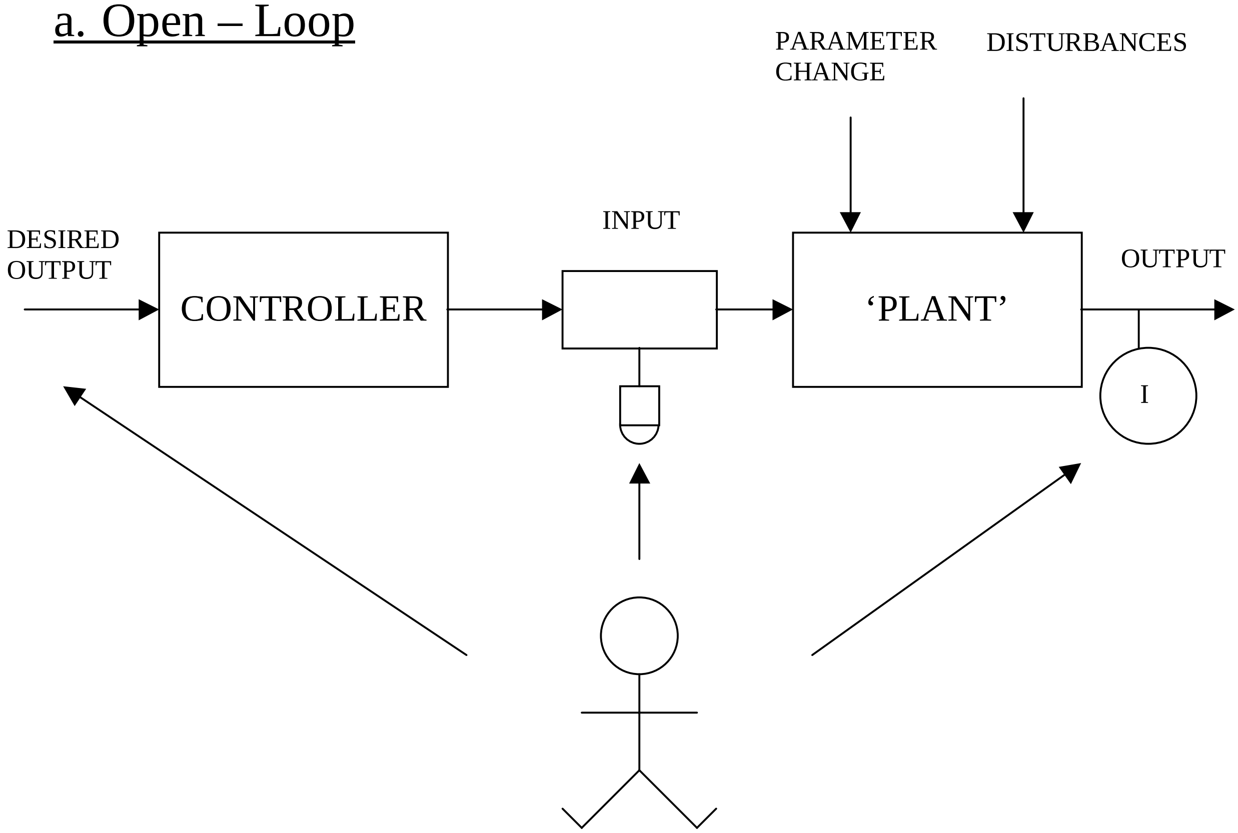
Part C: Root Locus Analysis

Frequency Response Analysis

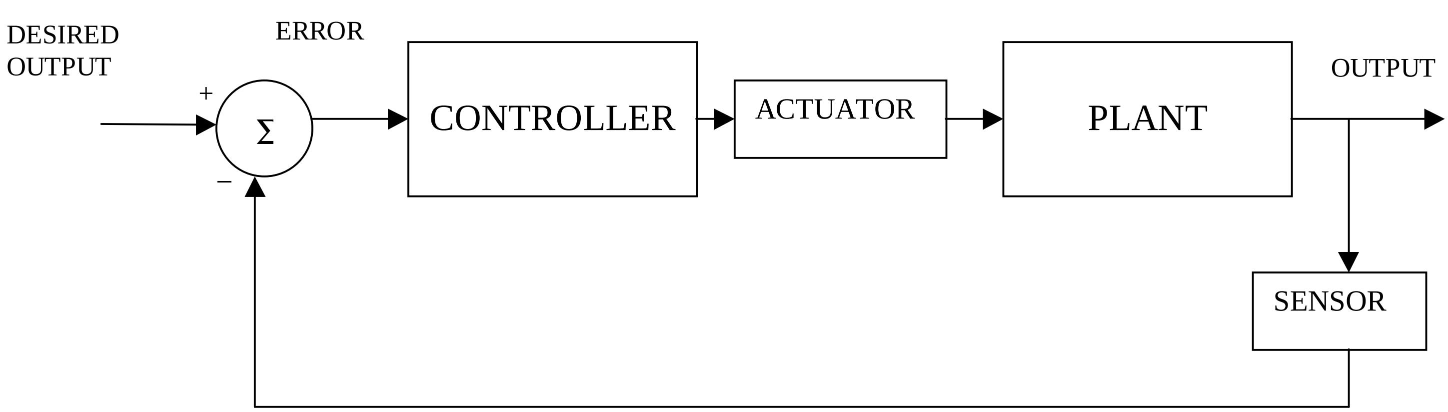
Part D: Control System Design

Open-loop and Closed-loop Control

a. Open – Loop



b. Closed – Loop



Some History

Date	*Technology*	*Problem*	*People*	*Method*
300 – 0 BC		Water clocks Oil lamps	Ktesibios Philon	
		Pneumatica	Heron of Alexandria	
16th – 17th century		Temperature and pressure regulators	Cornelis Drebbel D. Papin	
		Speed regulators	J. Watt I. Polzunov	
19 century	Steam engine	Stability	Maxwell	DE
		Stability	Ruth Hurwitz	
1920	Ship Steering	Stab/design	Minorsky	DE
1927-32	Feedback amps	Stab/design	Bode Nyquist	LT
1930's	Power drives	Stab/design	Brown	LT
1940's	Gun & radar syst.	Stab/design	Many	LT

1950's	Aircraft control	Stab/time resp.	Evans	Root locus
	General theory	'Optimal' control	Wiener Pontryakin	C of V
1960s	Aerospace	Multivariable State space Optimal Control	Kalman Bellman Russian work	SS
1970s	Industrial control	Disturbance rejection Computational methods & many others	Many	SS
1980s	Industrial control	Worst-case design Plant changes Robust control	Many	Operator theory

Comments:

- This is main stream – many other branches (computer control, chemical process control, etc.).
- Recent work has confirmed that the *techniques developed for SISO systems by Bode, Nyquist, Evans & others* are (when intelligently used) capable of producing excellent designs.

The Controller Design Process

Modeling

1. Model the system to be controlled.

2. Simplify the model, if necessary, so that it's tractable.

3. Analyze the model & simulate if necessary; decide what sensors and actuators are needed and where they should be placed.

4. (Usually) identify – by experiment – the values of model parameters.

5. Verify (by simulation and comparison with plant behavior) that the model adequately represents the plant – if not, repeat from 1.

6. Decide on performance specifications.

Design

7. Decide on the type of controller to be used.

8. Design a controller to meet the specs; if impossible or overly complex, repeat from 6.

Verification

9. Simulate the controlled system (computer model – step 1 – or pilot plant); if unsatisfactory, repeat from appropriate point.

Implementation

10. Choose hardware & software; implement and test the controller.

11. Tune the controller on-line if needed; train operators and maintainers to get best use out of the system.

MATHEMATICAL BACKGROUND

Complex Number $s = \sigma + j\omega$

Complex Function $G(s) = G_R + jG_I$

$$\frac{dG(s)}{ds} = \lim_{\Delta s \to 0} \frac{G(s+\Delta s) - G(s)}{\Delta s}$$

- G(s) is *analytic* in a region if G(s) and all its derivatives exist in this region.

- *Rational function*:

$$G(s) = \frac{A(s)}{B(s)} = \frac{K \prod_{i=1}^{m} (s + z_i)}{\prod_{j=1}^{n} (s + p_j)}$$

$-z_i$: zeros, $-p_j$: poles, $m \leq n$

Rational functions are analytic in the s-plane except at isolated points called *singularities*.

Poles are singularities of G(s).

Laplace Transforms

Consider f(t), such that f(t) = 0 for t < 0

$$£\left[f(t)\right] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

$$£^{-1}\left[f(t)\right] = f(t) = \frac{1}{2\pi}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

£[*f(t)*] exists if:

1. f(t) is sectionally continuous in every finite interval in the range t > 0

2. f(t) is of exponential order as t approaches infinity, i.e., there exists a real and positive constant σ such that

$$e^{-\sigma t}\left|f(t)\right| \rightarrow 0 \qquad \text{for } t \rightarrow \infty \text{ and } \sigma > \sigma_c$$

where σ_c is the *abscissa of convergence.*

Remark 1: $f(t) = e^{t^2}$ for $0 \le t \le \infty$

→ does not have a Laplace transform

$f(t) = e^{t^2}$ for $0 \le t \le T < \infty$ and
$f(t) = 0$ for $t > T$

→ does have a Laplace transform

Remark 2:

$$£_{+}\left[f(t)\right]=\int_{0+}^{\infty} f(t)e^{-st}\,dt$$

$$£_{-}\left[f(t)\right]=\int_{0-}^{\infty} f(t)e^{-st}\,dt$$

$$£_{-}\left[f(t)\right]=£_{+}\left[f(t)\right]+\int_{0-}^{0+} f(t)e^{-st}\,dt$$

$£_{+}$ and $£_{-}$ are equal iff

$$\int_{0-}^{0+} e^{-st} f(t)\,dt = 0$$

Remark 3:

$$£_{+}\left[\delta(t)\right]=\int_{0+}^{\infty}\delta(t)e^{-st}\,dt = 0$$

$$£_{-}\left[\delta(t)\right]=\int_{0-}^{\infty}\delta(t)e^{-st}\,dt = 1$$

Laplace Transform Theorems

f(t) is a Laplace transformable function and

F(s) its Laplace trasform

- $£\left[f(t-\alpha)\cdot u(t-\alpha)\right] = e^{-\alpha s}F(s)$ *u(t)* : step
- $£\left[e^{-\alpha t} f(t)\right] = F(s+\alpha)$
- $£\left[f\left(\frac{t}{\alpha}\right)\right] = \alpha F(\alpha s)$
- $£\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0)$

$$\int_0^\infty f(t)e^{-st}dt = f(t)\frac{e^{-st}}{s}\Big|_0^\infty - \int_0^\infty \left(\frac{d}{dt}f(t)\right)\frac{e^{-st}}{-s}dt$$

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}£\left[\frac{d}{dt}f(t)\right]$$

- $£\left[\frac{d^n}{dt^n} f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots sf^{n-2}(0) - f^{n-1}(0)$
- $£\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)dt\right]_{t=0}}{s}$

Final value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt} f(t)$ are Laplace transformable
- $\lim_{t\to\infty} f(t)$ exists
- F(s) analytic in Re(s) ≥ 0 except for a single pole at s = 0

Based on the Laplace Transform theorems:

$$\lim_{s \to 0} \int_0^\infty \left[\frac{d}{dt} f(t) \right] e^{-st}\, dt = \lim_{s \to 0} \left[sF(s) - f(0) \right]$$

since $\lim_{s \to 0} e^{-st} = 1$

→ $\int_0^\infty \frac{d}{dt} f(t) dt = f(\infty) - f(0) = \lim_{s\to 0} \left[sF(s) - f(0) \right]$

→ $f(\infty) = \lim_{s\to 0} sF(s)$

Initial value Theorem

$$f(0+) = \lim_{s \to \infty} sF(s)$$

under the following assumptions:

- f(t) and $\frac{d}{dt} f(t)$ are both Laplace transformable
- $\lim_{s \to \infty} sF(s)$ exists

From the Laplace Transform Theorems:

$$\lim_{s\to\infty} £_+\left[\frac{d}{dt} f(t)\right] = \lim_{s\to\infty} \int_{0+}^{\infty}\left[\frac{d}{dt} f(t)\right] e^{-st} dt = 0$$

$$= \lim_{s\to\infty} sF(s) - f(0+) = 0$$

Convolution

$$f_1(t) \circ - - \bullet F_1(s)$$

$$f_2(t) \circ - - \bullet F_2(s)$$

$$f_3(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau = £^{-1}[F_1($$

Example:

Consider

$$f_1(t) = \begin{cases} 1 & for \quad 0 \le t \le T \\ 0 & else \end{cases}$$

$$f_2(t) = \begin{cases} 1 & for \quad 0 \le t \le \frac{T}{2} \\ 0 & else \end{cases}$$

Find $f_3(t)$

$$f_3(t) = f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) d\tau = £^{-1}[F_1(s) \cdot F_2(s)]$$

$$f_3(t)=\int_0^t d\tau = t \quad \text{for} \quad 0 \le t \le \frac{T}{2}$$

$$f_3(t)=\int_0^{\frac{T}{2}} d\tau = \frac{T}{2} \quad \text{for} \quad \frac{T}{2} \le t \le T$$

$$f_3(t)=\int_{t-T}^{\frac{T}{2}} d\tau = \frac{3T}{2}-t \quad \text{for} \quad T \le t \le \frac{3T}{2}$$

$$f_3(t)=0 \quad \text{else}$$

Using $f_3(t)=\pounds^{-1}\left[F_1(s)\cdot F_2(s)\right]$

$$F_1(s)=\frac{1-e^{-sT}}{s}$$

$$F_2(s)=\frac{1-e^{\frac{-sT}{2}}}{s}$$

$$F_3(s)=\frac{\left(1-e^{-sT}\right)\left(1-e^{\frac{-sT}{2}}\right)}{s^2}$$

$$f_3(t)=t-\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)-(t-T)u(t-T)+\left(t-\frac{3T}{2}\right)u\left(t-\frac{3T}{2}\right)$$

for $t \ge 0$ *u(t): step*

Inverse Laplace transform

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds$$

Consider

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)\ldots(s+z_m)}{(s+p_1)\ldots(s+p_n)}$$

1. All roots are *distinct and real:*

$$F(s) = \frac{a_1}{s+p_1} + \ldots + \frac{a_n}{s+p_n}$$

$$a_k = \left[\frac{B(s)}{A(s)}(s+p_k)\right]_{s=-p_k}$$

$$£^{-1}\left[\frac{a_k}{s+p_k}\right] = a_k e^{-p_k t}$$

2. *Complex conjugate* poles

$$F(s)=\frac{B(s)}{A(s)}=\frac{a_1 s+a_2}{(s+p_1)(s+p_2)}+\frac{a_3}{s+p_3}+\dots$$

$$p_1 = p_2 *$$

$$(a_1 s + a_2)_{s=-p_1} = \left[\frac{B(s)}{A(s)}(s+p_1)(s+p_2)\right]_{s=-p_1}$$

equating real and imaginary parts → α_1, α_2

Obtain *f(t)* using α_1, α_2 and the following table entries:

$$£[e^{-at}\cos\omega t]=\frac{s+a}{(s+a)^2+\omega^2}$$

$$£[e^{-at}\sin\omega t]=\frac{\omega}{(s+a)^2+\omega^2}$$

2. *Multiple real* poles

$$F(s)=\frac{B(s)}{A(s)}=\frac{B(s)}{(s+p_1)^r(s+p_{r+1})\dots}$$

$$=\frac{b_r}{(s+p_1)^r}+\frac{b_{r-1}}{(s+p_1)^{r-1}}+\dots+\frac{b_1}{s+p_1}+\frac{a_{r+1}}{s+p_{r+1}}\dots$$

where

$$b_r = \left[\frac{B(s)}{A(s)} (s + p_1)^r \right]_{s=-p_1}$$

$$b_{r-1} = \left\{ \frac{d}{ds} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

$$b_{r-j} = \frac{1}{j!} \left\{ \frac{d^j}{ds^j} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

$$b_1 = \frac{1}{(r-1)!} \left\{ \frac{d^{r-1}}{ds^{r-1}} \left[\frac{B(s)}{A(s)} (s + p_1)^r \right] \right\}_{s=-p_1}$$

and

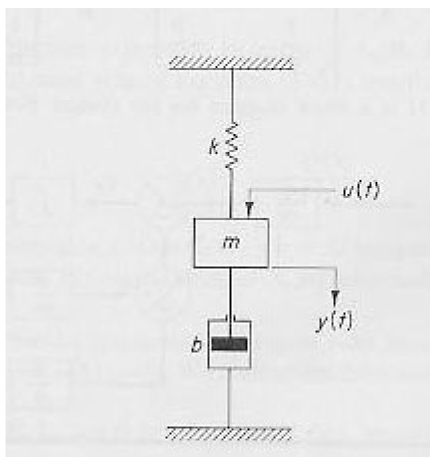
$$£^{-1}\left[\frac{1}{(s+p_1)^n} \right] = \frac{t^{n-1}}{(n-1)!} e^{-p_1 t}$$

Solution of linear differential equations using the Laplace transform

1. Take the Laplace transform of each term and convert the differential equation into an algebraic equation. Obtain the Laplace transform of the dependent variable.

2. The solution is obtained after inverse Laplace transform of the dependent variable.

Example: Simple mechanical system



$m\ddot{y} + ky = u(t)$ $\qquad m, k > 0$

$$m\left[s^2Y(s) - sy(0) - \dot{y}(0)\right] + kY(s) = U(s)$$

$$Y(s) = \frac{U(s)}{ms^2 + k} + \frac{msy(0) + m\dot{y}(0)}{ms^2 + k}$$

for U(s) = 1/s

$$y(t) = \left(\frac{1}{k} - \frac{1}{k}\cos\sqrt{\frac{k}{m}}t\right) + \left(y(0)\cos\sqrt{\frac{k}{m}}t + \dot{y}(0)\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t\right)$$

Transfer function

$$a_0 \overset{n}{y} + \ldots + a_n y = b_0 \overset{m}{u} + \ldots b_m u \qquad m \leq n$$

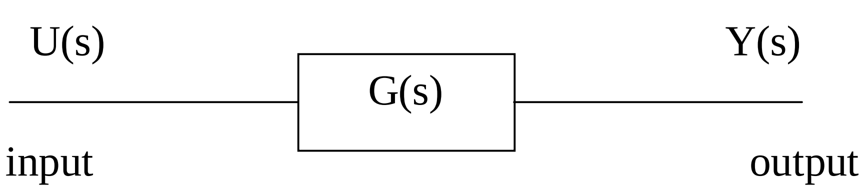
Take the Laplace transforms
(assuming zero initial conditions)

$$\frac{Y(s)}{U(s)} = \frac{B(s)}{A(s)} = \frac{b_0 s^m + \ldots + b_m}{a_0 s^n + \ldots + a_n} = G(s)$$

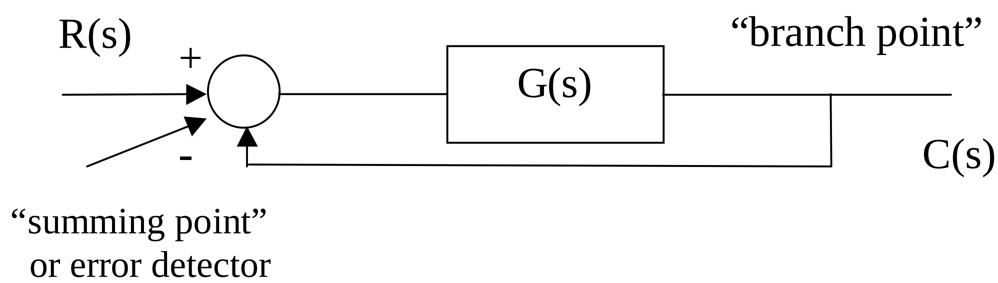
G(s) is the *transfer function* from U(s) to Y(s)

Block diagrams

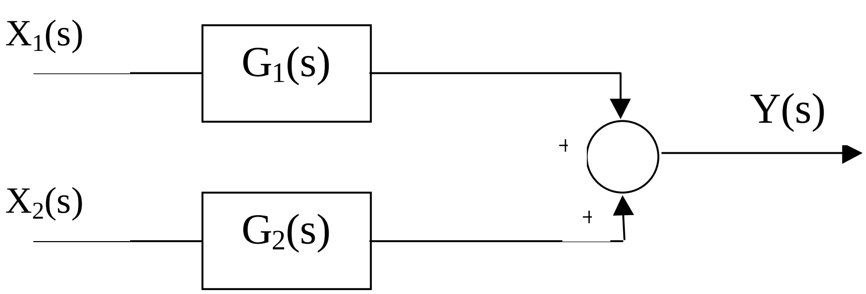
<u>Open-loop</u>



Closed-loop

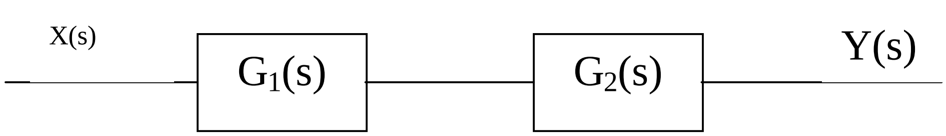


Parallel



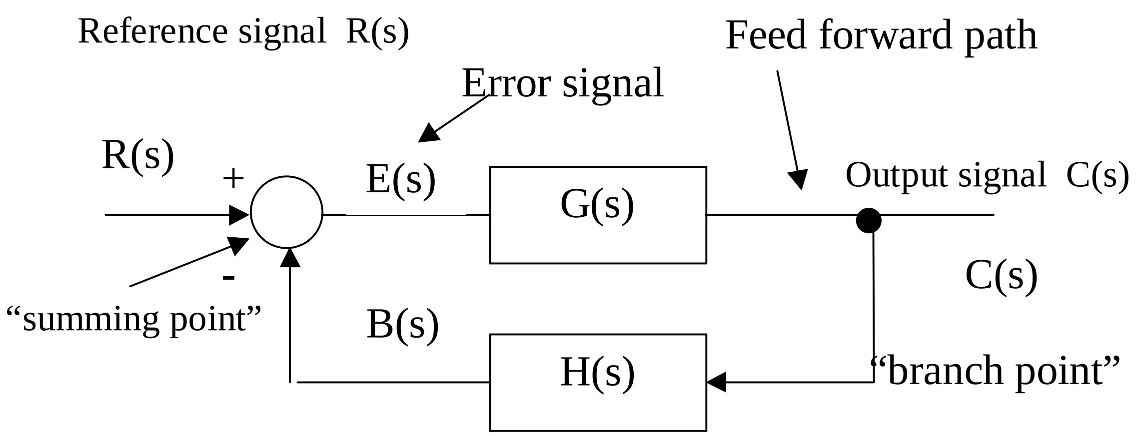
$$Y(s) = G_1(s)X_1(s) + G_2(s)X_2(s)$$

Series (Cascade)



$$Y(s) = G_1(s)G_2(s)X(s)$$

Closed-loop



Feed forward transfer function: $\dfrac{C(s)}{E(s)} = G(s)$

Feed back transfer function: $\dfrac{B(s)}{C(s)} = H(s)$

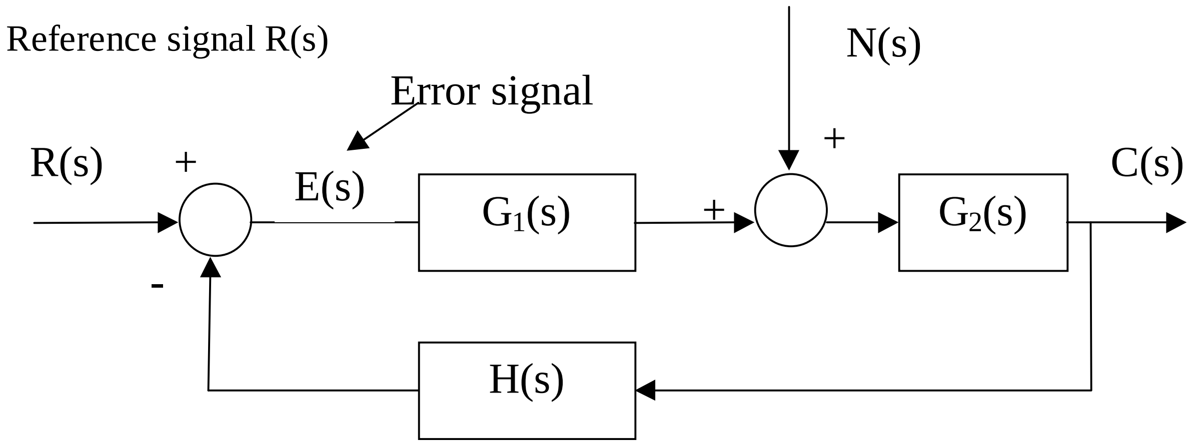
Open-loop transfer function: $\dfrac{B(s)}{E(s)} = G(s)H(s)$

Closed-loop transfer function: $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$

$$C(s) = G(s)E(s) = G(s)[R(s) - B(s)] = G(s)R(s) - G(s)H(s)C(s)$$

$$\rightarrow \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Closed-loop subjected to disturbance



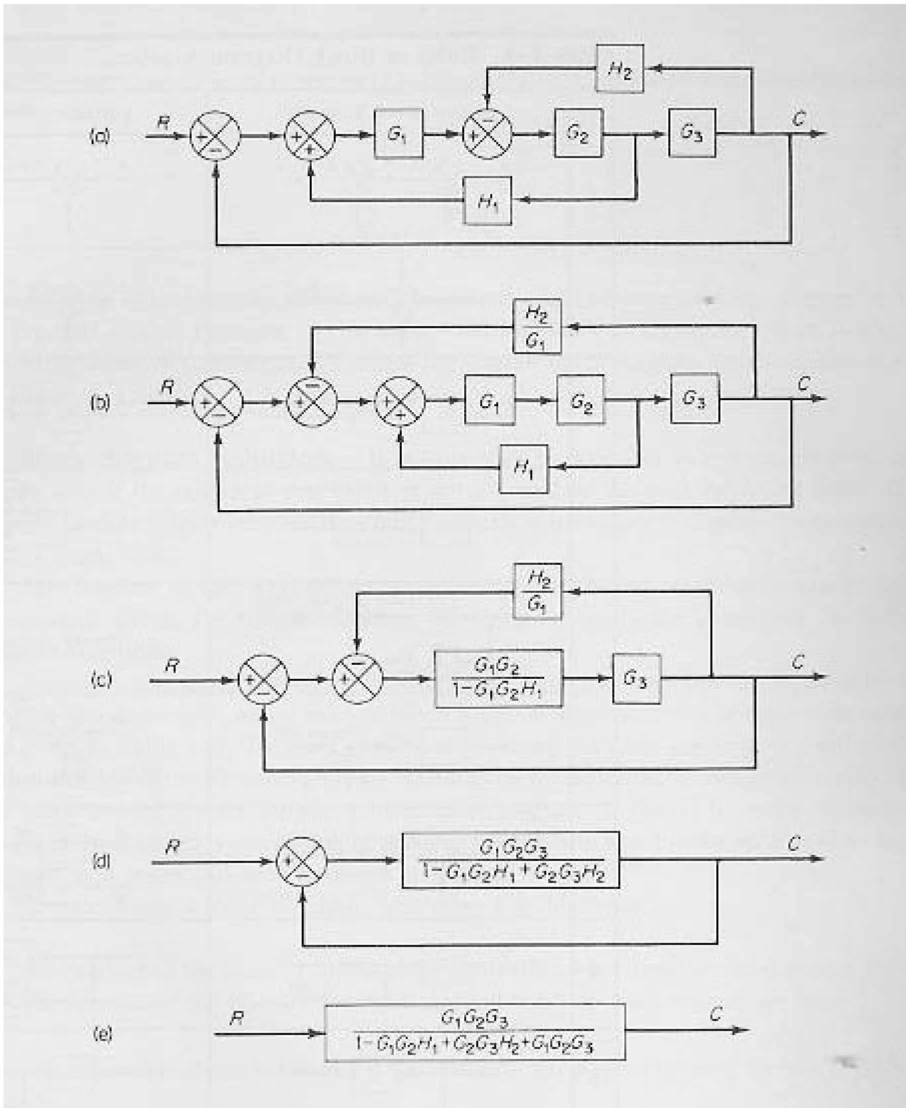
$$\frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$ assuming R(s) = 0

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$ assuming N(s) = 0

$$C(s) = C_N(s) + C_R(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + N(s)\right]$$

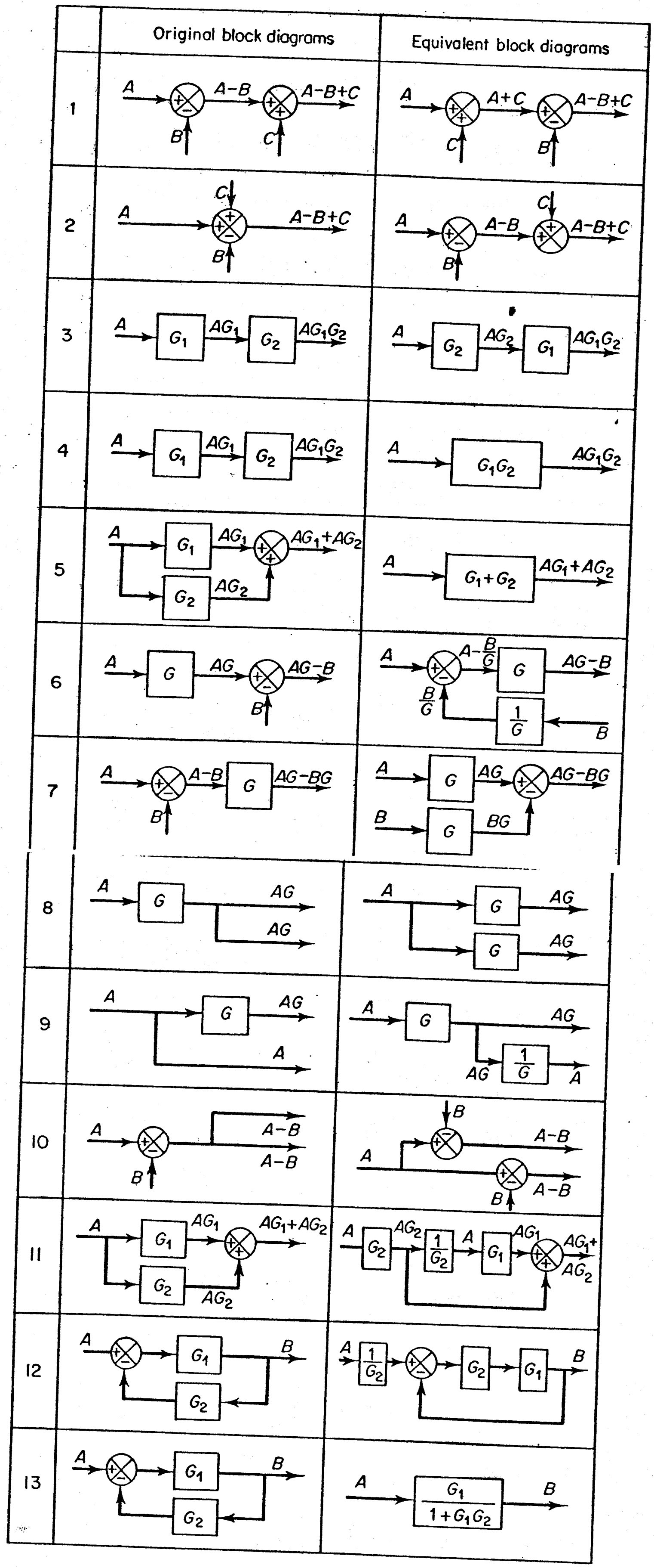
$$C(s) = \left[\frac{G_1(s)\cdot G_2(s)}{1+G_1(s)G_2(s)H(s)}, \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}\right]\begin{bmatrix} R(s) \\ N(s) \end{bmatrix}$$

Block Diagram Reduction



(a) Multiple-loop system;
(b) -(e) successive reduction of the block diagram shown in (a)

• Rules of Block Diagram Algebra

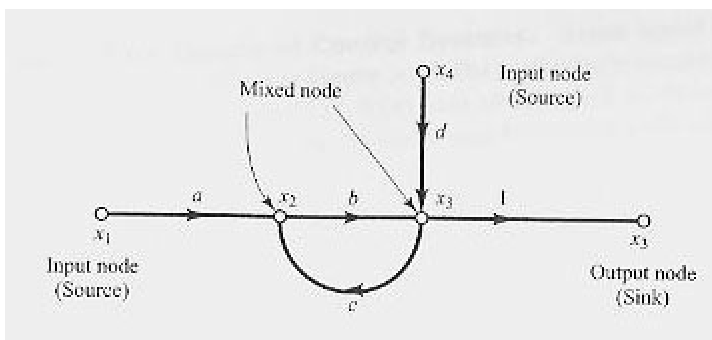


Signal flow Graphs (SFG)

A *Signal Flow Graph (SFG)* is a diagram which represents a set of simultaneous linear equations.

It consists of a network in which nodes are connected by directed branches.

Example:

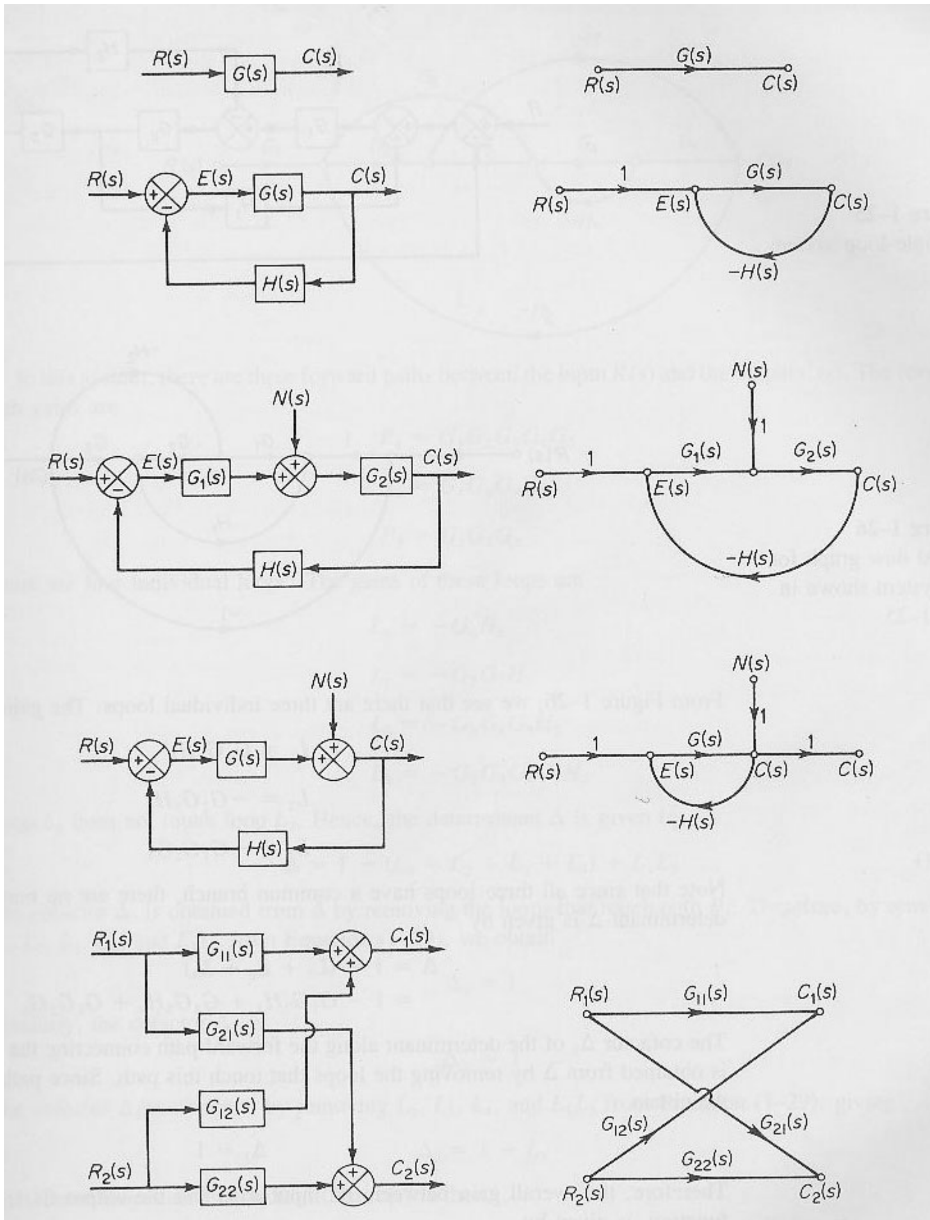


$$x_3 = abx_1 + dx_4 + bcx_3$$

Note that summations are taken over all possible paths from input to output

<u>*Definitions:*</u>

Node:	Point representing a variable or a signal.
Branch:	Gain between two nodes
Source:	A node without outgoing branches (input node)
Sink:	A node with incoming branches only (output nodes)
Mixed node:	Both incoming + outgoing branches
Path:	Connected branches in the direction of the branch arrows. If a node is crossed more than once, it is *closed*.
Forward path:	From input node to output node without crossing nodes more than once.
Loop:	Closed path.
Non touching loops:	Loops that do not possess common nodes.



Block diagrams and corresponding signal flow graphs

Properties of Signal Flow Graphs

1. A branch indicates the functional dependence of one signal on another. A signal passes through only in the direction specified by the arrow of the branch.

2. A node adds the signals of all incoming branches and transmits this sum to all outgoing branches.

3. A mixed node, which has both incoming and outgoing branches, may be treated as an output node (sink) by adding an outgoing branch of unity transmittance. However, we can not change a mixed node to a source by this method.

4. For a given system, the signal flow graph is not unique. Many different signal flow graphs can be drawn for a given system by writing the system equations differently.

Mason's Formula **for Signal Flow Graphs**

$$P = \frac{1}{\Delta}\sum_k P_k \Delta_k$$

P: Overall transmittance between input and output node.

Δ*: determinant of path* =

1 – (sum of all different loop gains)
+ (sum of gain products of all combinations of *2 non-touching loops*)
- (sum of gain products of all combinations of *3 non-touching loops*)
+…..

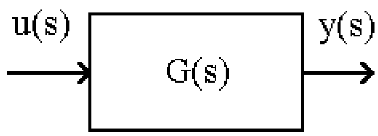
$$\Delta = 1 - \sum_a L_a + \sum_{b,c} L_b L_c - \sum_{d,e,f} L_d L_e L_f + \ldots$$

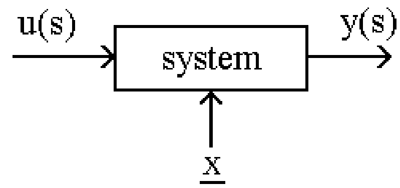
P_k: Gain of k^{th} forward path

Δ_k: Cofactor of the k^{th} forward path.
Δ_k is determined as the determinant Δ but with the loops touching the k^{th} forward path removed.

STATE - SPACE DESCRIPTION

Input - Output	State - Space Description
↕	↕
Transfer Function	State vector $\underline{x} = [x_1, \ldots, x_n]^T$





differential equation of order n	n differential equations of first order

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{c}\underline{x} + du$$

State: A set of variables $\underline{x}(t) = [x_1(t), .., x_n(t)]^T$ such that the knowledge of these variables at $t = t_0$ together with the input u(t) for $t \geq t_0$ determines the behaviour of the system for any time $t \geq t_0$.

Minimal state: A set of variables $\underline{x}$(t) with n minimal.

SISO (Single Input Single Output) system description:

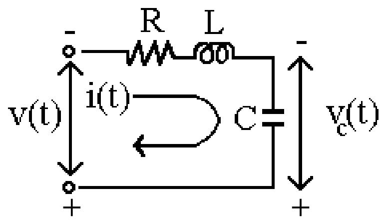
$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$

$\underline{x}$: state vector

y: output
u: input

A: system matrix
$\underline{b}$: input vector
$\underline{c}$: output vector
d: direct input/output transmission

Example:



$$\underline{x} = [i(t), \quad v_c(t)]^T$$
$$y = v_C(t)$$
$$u = v(t)$$

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{cx} + du$$

$$C\frac{dv_c}{dt} = i$$

$$L\frac{di}{dt} + Ri + v_c = v$$

State-space description

$$\dot{x} = \begin{bmatrix} di/dt \\ dv_C/dt \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}}_{A} \begin{bmatrix} i \\ v_c \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\underline{b}} v$$

$$y = v_c = \underbrace{[0 \quad 1]}_{\underline{c}} \begin{bmatrix} i \\ v_C \end{bmatrix} + \underbrace{[0]}_{d} v$$

Input-Output:

$$G(s) = \frac{V_C(s)}{V(s)}$$ from the above equations.

$$G(s) = d + \underline{c}\,(sI - A)^{-1}\,\underline{b}$$

Remark: One could also choose for state variables

$$z_1 = v_C(t) + Ri(t)$$
$$z_2 = v_C(t)$$

Choice of state variables is not unique.

Minimal number of state variables = order of system =
order of differential equation
is unique.

Nonuniqueness of State-Space Realizations:

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$
$$y = \underline{c}\underline{x} + du$$
$$\Downarrow$$
$$\underline{x} = T\underline{z}$$

$\underline{z}$: new state

T : similarity transformation
det T $\neq$ 0

$$\dot{\underline{z}} = T^{-1}AT\underline{z} + T^{-1}\underline{b}u$$
$$y = \underline{c}T\underline{z} + du$$

$$\dot{\underline{z}} = A_1\underline{z} + b_1u$$
$$y = \underline{c}_1\underline{z} + du$$

$$A_1 = T^{-1}AT$$
$$\underline{b}_1 = T^{-1}\underline{b}$$
$$\underline{c}_1 = \underline{c}T$$

Example (from previous page):

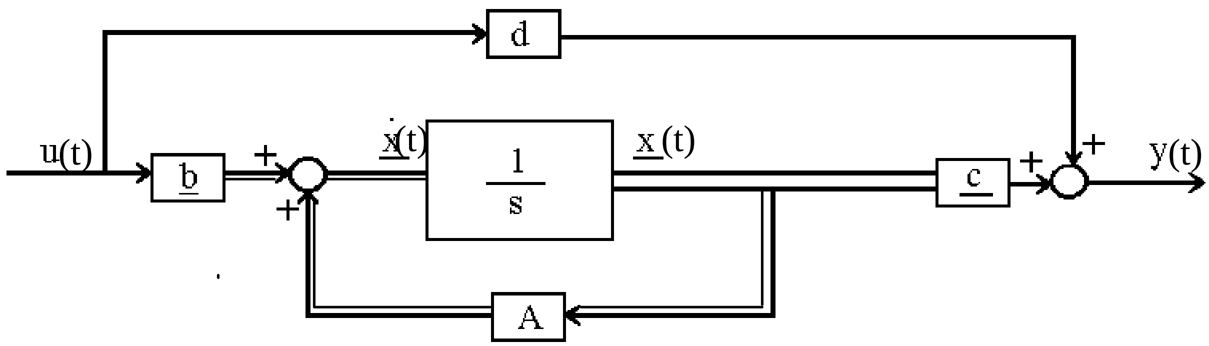
$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} R & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T^{-1}\underline{x}$$

$$T = \frac{1}{R}\begin{bmatrix} 1 & -1 \\ 0 & R \end{bmatrix}$$

CONTINUOUS STATE-SPACE EQUATIONS

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \qquad \underline{x} \in R^n$$

$$y = \underline{c}\underline{x} + du$$



STATE-SPACE REPRESENTATIONS OF CONTINUOUS SYSTEMS

Observable Canonical Form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0 s^n + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} = \frac{b_0 + b_1 s^{-1} + \ldots + b_n s^{-n}}{1 + a_1 s^{-1} + \ldots + a_n s^{-n}}$$

$$Y(s) = b_0 U(s) - s^{-1}[a_1 Y(s) - b_1 U(s)] - s^{-2}[a_2 Y(s) - b_2 U(s)] - \ldots - s^{-n}[a_n Y(s) - b_n U(s)]$$

$$Y(s) = b_0 U(s) + s^{-1}\{b_1 U(s) - a_1 Y(s) + s^{-1}(b_2 U(s) - a_2 Y(s) + s^{-1}(b_3 \ldots))\}$$

$$Y(s) = b_0 U(s) + X_n(s)$$

$$X_n(s) = s^{-1}[b_1 U(s) - a_1 Y(s) + X_{n-1}(s)]$$

$$X_{n-1}(s) = s^{-1}[b_2 U(s) - a_2 Y(s) + X_{n-2}(s)]$$

:

$$X_2(s) = s^{-1}[b_{n-1} U(s) - a_{n-1} Y(s) + X_1(s)]$$

$$X_1(s) = s^{-1}[b_n U(s) - a_n Y(s)]$$

replace Y(s) in the above equations with $Y(s) = b_0 U(s) + X_n(s)$ and transform them in the time domain

$$\dot{x}_n = x_{n-1} - a_1 x_n + (b_1 - a_1 b_0)\, u$$

$$\dot{x}_{n-1} = x_{n-2} - a_2 x_n + (b_2 - a_2 b_0)\, u$$

$$\dot{x}_1 = -a_n x_n + (b_n - a_n b_0)\, u$$

and

$$\begin{bmatrix} \dot{x}_1 \\ : \\ : \\ : \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & \dots & \dots & 0 & -a_n \\ 1 & & & & -a_{n-1} \\ 0 & & & & : \\ 0 & & & & : \\ 0 & \dots & 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ : \\ : \\ : \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0, \quad \dots \quad \dots, \quad 0, \quad 1] \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} + b_0 u$$

Special Case (Observability Canonical Form)

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1}\dot{y} + a_n y = u$$

Define the following state variables:

$$x_1 = y,\quad x_2 = \dot{y},\quad \dots,\quad x_n = y^{(n-1)}$$

Then

$$\dot{x}_1 = x_2,\quad \dot{x}_2 = x_3,\quad \dots,\quad \dot{x}_{n-1} = x_n$$

$$\dot{x}_n = y^{(n)} = -a_n x_1 - \dots - a_1 x_n + u$$

and

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{c}\underline{x} + du$$

where

$$\underline{x} = \begin{bmatrix} x_1 \\ : \\ : \\ : \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & & : \\ : & & & . & : \\ 0 & & & & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ : \\ : \\ 1 \end{bmatrix}$$

$\underline{c} = [1 \quad 0 \quad .. \quad 0]$ and $d = 0$

Observability Canonical Form

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1}\dot{y} + a_n y = b_0 u^{(n)} + b_1 u^{(n-1)} + \ldots + b_{n-1}\dot{u} + b_n u$$

Define states:

$$x_1 = y - \beta_0 u$$

$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u \quad \text{-->} \quad \dot{x}_1 = x_2 + \beta_1 u$$

$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

$$\vdots$$

$$x_n = y^{(n-1)} - \beta_0 u^{(n-1)} - \ldots - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \ldots - a_1 x_n + \beta_n u$$

$$\beta_0 = b_0$$

$$\beta_1 = b_1 - a_1 \beta_0$$

$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$

$$\vdots$$

$$\beta_n = b_n - a_1 \beta_{n-1} - \ldots - a_{n-1} \beta_1 - a_n \beta_0$$

and

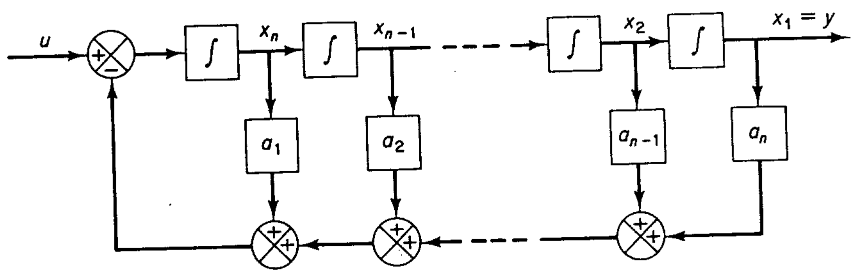
$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{cx} + du$$

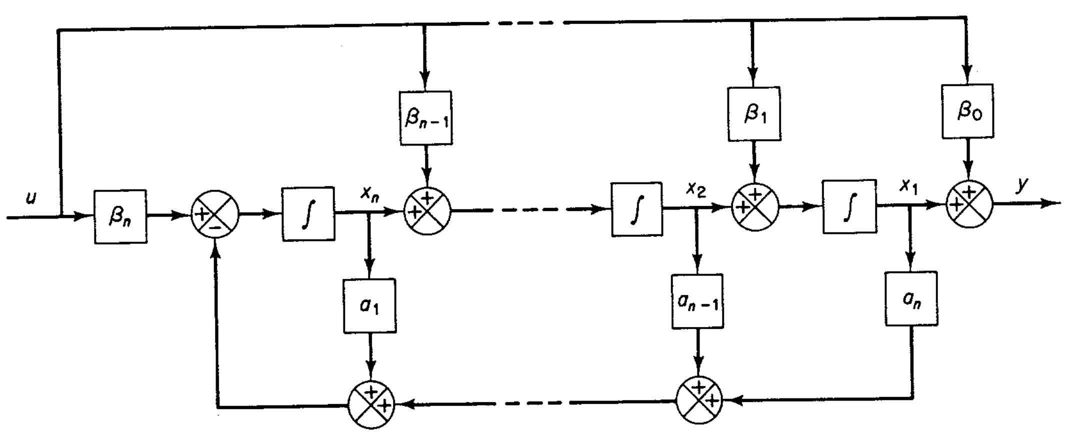
A and $\underline{c}$ as in the previous case,

$$\underline{b} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} \qquad d = [\beta_0]$$

Special Case:



Observability Canonical Form



Comments:

- Choice of state variables is not unique

- Infinite many possibilities for a state-space description of a dynamic system described by a differential equation.

- Forms with special structure (like the above) are called canonical forms.

Input-Output Description from State-Space Description

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u$$

$$y = \underline{c}\underline{x} + du$$

$$sX(s)=AX(s)+\underline{b}U(s)$$

$$(sI-A)X(s)=\underline{b}U(s)$$

$$X(s) = (sI - A)^{-1}\underline{b}U(s)$$

$$\frac{Y(s)}{U(s)} = \underline{c}(sI - A)^{-1}\underline{b} + d$$

$$Y(s) = [\underline{c}(sI - A)^{-1}\underline{b} + d]U(s)$$

Matrix inversion, special case n=2

use $$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

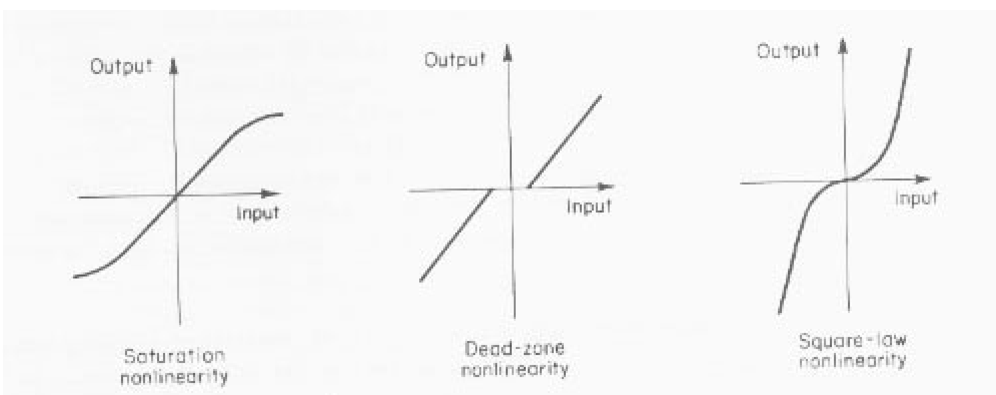
Mathematical Models

Model: The mathematical description of the dynamic characteristics of a system.

- Simplicity versus accuracy
- Time-variant versus time-invariant
- Linear versus nonlinear
  - Linear systems are described by linear differential equations.
  - Nonlinear, such as

$$\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + (x-1)\frac{dx}{dt} = A \sin x$$

are more complicate.



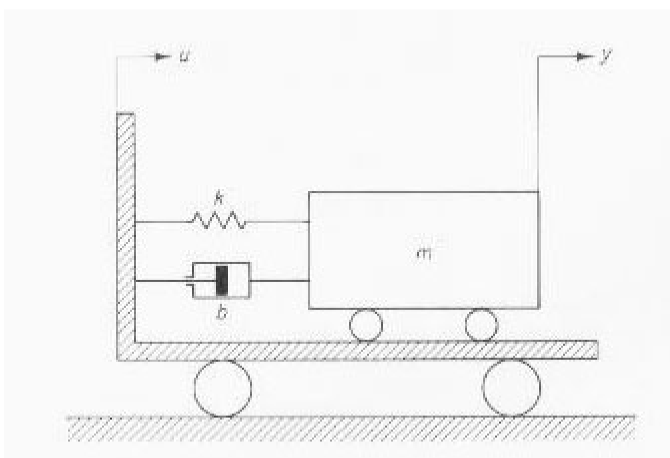
Examples of mathematical modes of nonlinear dynamic systems

Real Systems are nonlinear and time variant. However,
linear time-invariant approximations
around an operating point
are usually good approximations.

Mathematical Models of Simple Systems

Translational Mechanical Systems

Newton's Law: mass x acceleration = $\sum$ Forces



Spring-mass-dashpot system mounted on a cart

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k\left(y - u\right)$$

$$m\frac{d^2y}{dt} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku$$

which leads to the following *transfer function*:

$$\frac{Y(s)}{U(s)} = G(s) = \frac{bs+k}{ms^2+bs+k}$$

or to the following *state-space model*

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

α_1 α_2 b_1 b_2

Using

$$\beta_0 = b_0 = 0$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

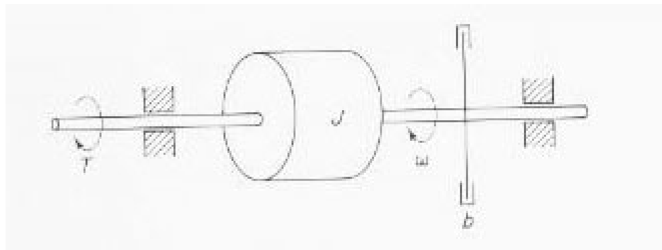
$$x_1 = y - \beta_0 u = y$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u$$

we obtain

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{b}{m}\\ \frac{k}{m} - \left(\frac{b}{m}\right)^2\end{bmatrix}\cdot u \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Mechanical Rotational Systems



Mechanical rotational system

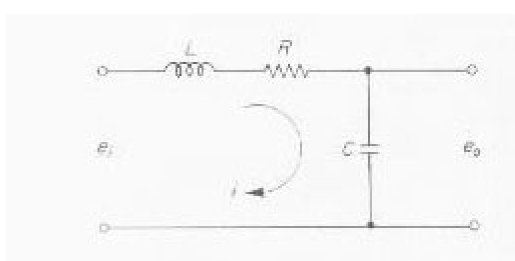
$$Ja = \sum T$$

T → Torque
a → Angular acceleration
J → Inertia

$$J\dot{\omega} + f\omega = T \qquad \frac{\Omega(s)}{T(s)} = \frac{1}{Js + f}$$

Electrical Systems

RLC-Circuit



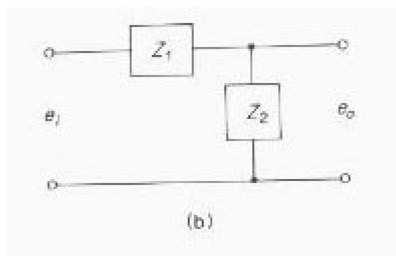
Simple RLC-circuit

The *transfer function* can be obtained from

$$\ddot{e}_0 + \frac{R}{L}\dot{e}_0 + \frac{1}{LC}e_0 = \frac{1}{LC}e_i$$

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

or from



Electrical circuit

$$Z_1 = Ls + R \qquad Z_2 = \frac{1}{LC}$$

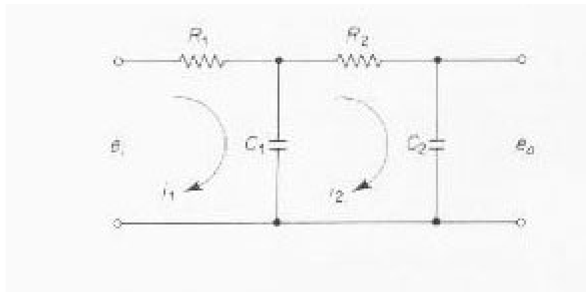
$$\frac{E_0(s)}{E_i(s)} = \frac{Z_2}{Z_1 + Z_i} = \frac{1}{LCs^2 + RCs + 1}$$

A *state-space model* is obtained using

$$x_1 = e_0 \quad x_2 = \dot{e}_0 \quad u = e_i \quad y = e_0 = x_1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{LC}\end{bmatrix}\cdot u \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Loading effect in series connection



RC Circiut

The transfer function can be obtained from

$$\frac{1}{C_1}\int (i_1 - i_2)dt + R_1 i_1 = e_i$$

$$\frac{1}{C_1}\int (i_2 - i_1)dt + R_2 i_2 \cdot \frac{-1}{C_2}\int i_2 dt = -e_0$$

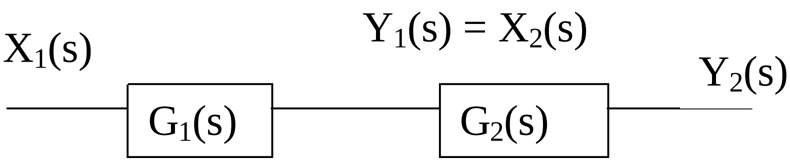
$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s)$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) - \frac{1}{C_2 s} I_2(s) = -E_0(s)$$

and

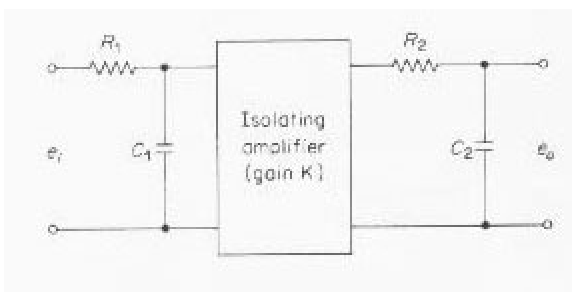
$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_1)s + 1}$$

Transfer functions of cascaded elements



$$\frac{Y_2(s)}{X_1(s)} = G_1(s)G_2(s)$$

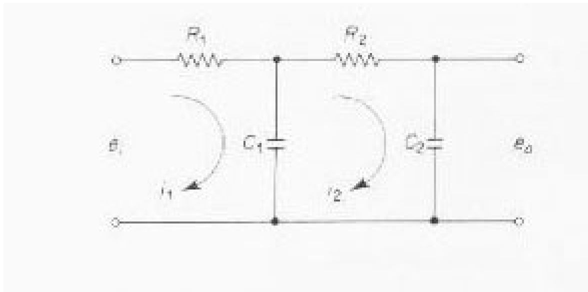
Example:



Cascade of two *non-loading* RC Citcuits

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{(R_1C_1s+1)}K\frac{1}{(R_2C_2s+1)}$$

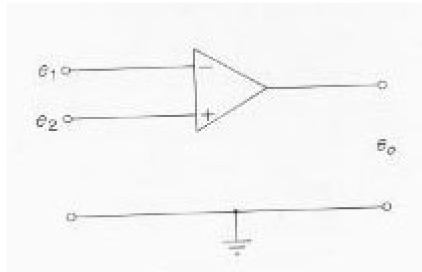
Note the difference in the transfer functions between the last and the previous circuit:



Cascade of two *loading* RC-Circuits

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_1)s + 1}$$

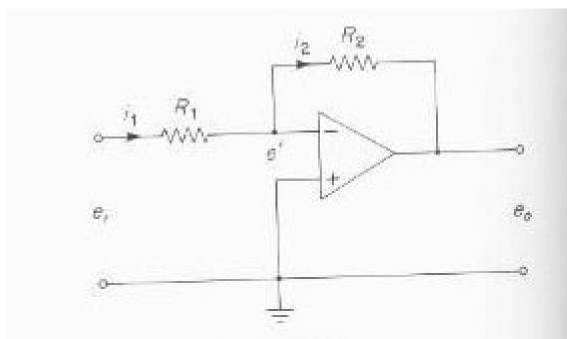
Operational Amplifiers



Operational amplifier

$$e_0 = K(e_2 - e_1) \qquad (K \text{ is} \sim 10^4)$$

Inverting amplifiers



Inverting amplifier

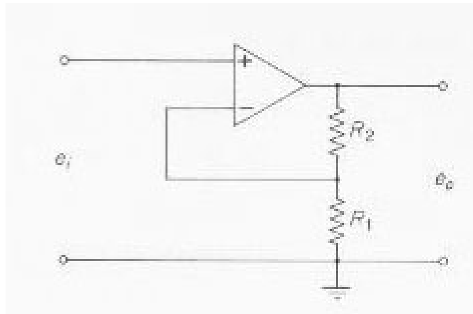
$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = \frac{e' - e_0}{R_2}$$

Since $i_1 = i_2$, $e_0 = -K(0 - e')$ and $K >> 1$,

it follows that $e' = 0$ and

$$e_0 = -\frac{R_2}{R_1} e_i$$

Non-inverting amplifier



Non-inverting operational amplifier

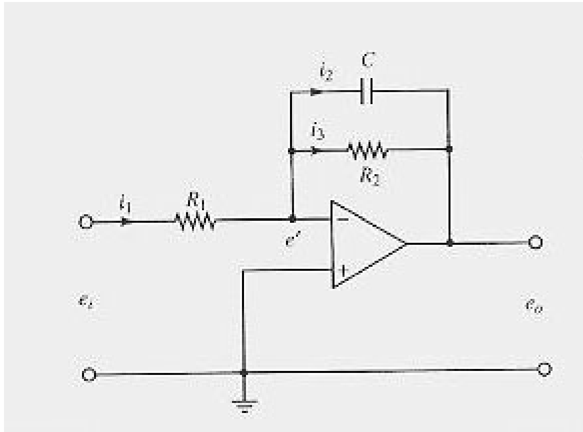
$$e_0 = Ke_i - \frac{KR_1}{R_1 + R_2} e_0$$

$$e_i = \left(\frac{R_1}{R_1 + R_2} + \frac{1}{K} \right) e_0$$

Since K>>1 (K is ~ 10^4), and $\frac{R_1}{R_1 + R_2} >> \frac{1}{K}$

$$e_0 = \left(1 + \frac{R_2}{R_1} \right) e_i$$

First order circuit



First order lag circuit using operational amplifier

Since the current flowing into the amplifier is negligible

$$i_1 = \frac{e_i - e'}{R_1} \qquad i_2 = C\frac{d(e' - e_0)}{dt} \qquad i_3 = \frac{e' - e_0}{R_2}$$

$$i_1 = i_2 + i_3$$

Hence,

$$\frac{e_i - e'}{R_1} = C\frac{d(e' - e_0)}{dt} + \frac{e' - e_0}{R_2}$$

Since $e' = 0$, we have

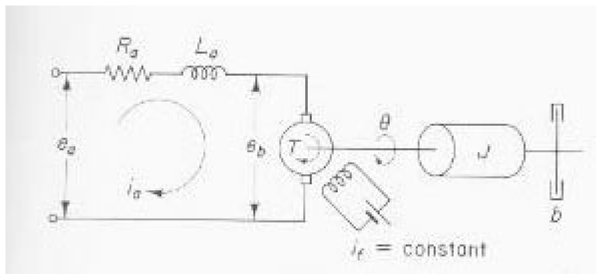
$$\frac{e_i}{R_1} = -C\frac{de_0}{dt} - \frac{e_0}{R_2} \quad \text{and} \quad \frac{E_i(s)}{R_1} = -\frac{R_2Cs+1}{R_2}E_0(s)$$

leading to

$$\frac{E_0(s)}{E_i(s)} = -\frac{R_2}{R_1}\frac{1}{R_2Cs+1}$$

Electromechanical systems

Armature controlled DC Motors



Schematic diagram of an armature-controlled DC motor

Ψ:	Air gap flux (assumed constant)
T:	Torque
e_b:	Voltage induced in the armature due to rotation
i_a, i_f;	Armature and field currents

$\Psi = K_f i_f$ and $T = \Psi K_1 i_a = K_f K_1 i_f i_a = K_t i_a$

$$e_b = K_b \frac{d\theta(t)}{dt} = K_b \omega(t)$$

For the armature circuit, we have:

$$L_a \frac{di_a}{dt} + R_a i_a + e_b = e_a = V_s$$

For the mechanical part we have:

$$J\frac{d^2\theta}{dt} + f\frac{d\theta}{dt} = T = K_t i_a$$

The Laplace Transforms of the above three equations lead to

$$K_b s\Theta(s) = E_b(s)$$

$$(L_a s + R_a)I_a(s) + E_b(s) = E_a(s)$$

$$(Js^2 + fs)\Theta(s) = T(s) = K_t I_a(s)$$

and after eliminating $E_b(s)$ and $I_a(s)$, we have:

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_t}{s\left[L_a Js^2 + (L_a f + R_a J)s + R_a f + K_t K_b\right]}$$

For most small motors $L_a = 0$ → ignore it.

Using $\Omega(s) = s\Theta(s)$ and $E_a(s) = V_s(s)$

$$\frac{\Omega(s)}{V_s(s)} = \frac{K_t}{R_a Js + R_a f + K_t K_b} = \frac{K_m}{1 + s\tau_m}$$

where

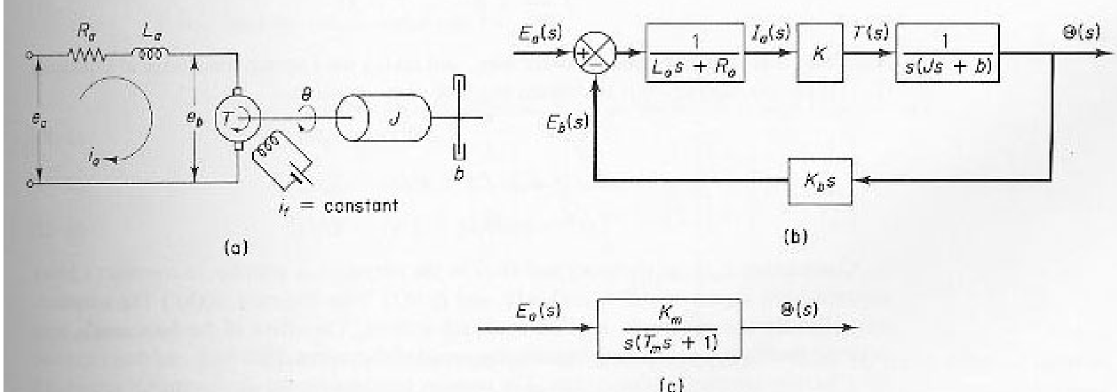
$$K_m = \frac{K}{R_a f + K_t K_b}$$

$$\tau_m = \frac{JR_a}{R_a f + K_t K_b}$$

Armature control of dc servomotors. Consider the armature-controlled dc servomotors shown in Figure 2–20(a), where the field current is held constant. In this system,

R_a = armature resistance, ohm
L_a = armature inductance, henry
i_a = armature current, ampere
i_f = field current, ampere
e_a = applied armature voltage, volt
e_b = back emf, volt
θ = angular displacement of the motor shaft, radian
T = torque developed by the motor, N-m
J = equivalent moment of inertia of the motor and load referred to the motor shaft, kg-m^2
b = equivalent viscous-friction coefficient of the motor and load referred to the motor shaft, N-m/rad/sec

The torque T developed by the motor is proportional to the product of the armature current



(a) Schematic diagram of armature-controlled DC motor
(b) Block diagram obtained from Eqs.
(c) Simplified block diagram.

State-space description of armature controlled DC Motors

Form

$$\frac{\Theta(s)}{E_a(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

we have

$$\ddot{\theta} + \frac{1}{\tau_m}\dot{\theta} = \frac{K_m}{\tau_m} e_a$$

Using the state variables

$$x_1 = \theta \quad \text{and} \quad x_2 = \dot{\theta}$$

and

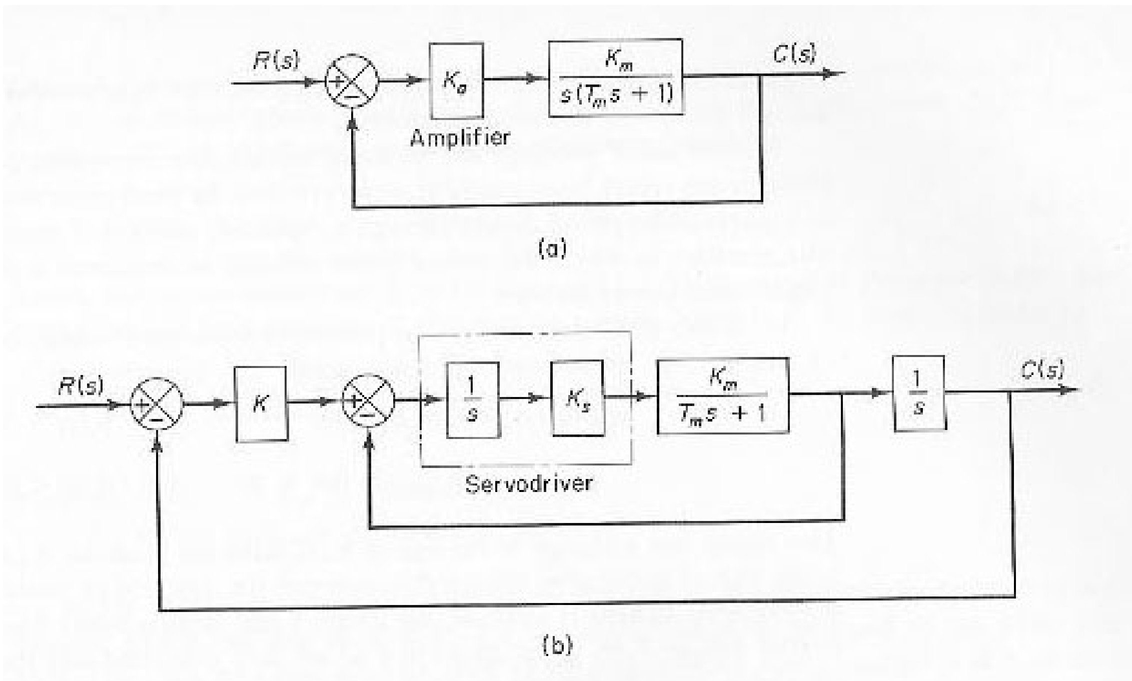
$$u = e_a \quad \text{and} \quad y = \theta = x_1$$

we obtain the following state-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\dfrac{1}{\tau_m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{K_m}{\tau_m} \end{bmatrix} e_a$$

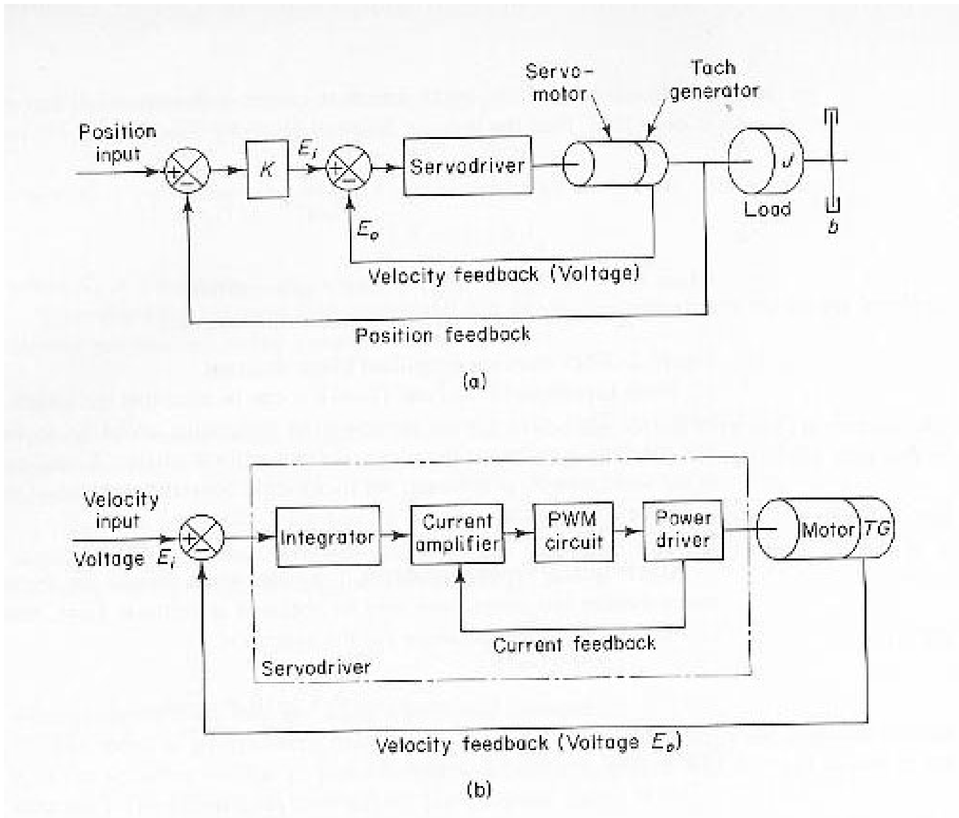
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Positional DC servo system



a) Simple, low-cost positional servo system.
b) High-speed, high-precision positional servo system.

Electronic motion control of DC servomotors



a) High-speed, high precision postional servo system with speed control using a servodriver, servomotor combination.
b) Functional diagram of a servodriver.

Linear approximation of non-linear systems

Consider the system described by:

y = f(x) where

y: output

x: input

$(\bar{x}, \bar{y})$: operating point

The Taylor series expansion of y around $(x - \bar{x})$ gives

$$y = f(\bar{x}) + \left[\frac{df(y)}{dx}\right]_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2!}\left[\frac{d^2 f(y)}{dx^2}\right]_{x=\bar{x}} (x - \bar{x})^2 + \ldots$$

For small $(x - \bar{x})$ higher order derivatives are 0 and

$$y = \bar{y} + k(x - \bar{x})$$

is the linear approximation of the non-linear system

where

$$\bar{y} = f(\bar{x})$$

$$k = \left.\frac{df(y)}{dx}\right|_{x=\bar{x}}$$

Consider the system

$y = f(x_1,x_2)$ where y: output

x_1,x_2: inputs

$((\bar{x}_1,\bar{x}_2)\bar{y})$: operating point

The Taylor series expansion gives

$$y = f(\bar{x}_1,\bar{x}_2) + \left[\frac{\partial f(x_1,x_2)}{\partial x_1}\right]_{\substack{x_1=\bar{x}_1\\ x_2=\bar{x}_2}} (x_1-\bar{x}_1) + \left[\frac{\partial f(x_1,x_2)}{\partial x_2}\right]_{\substack{x_1=\bar{x}_1\\ x_2=\bar{x}_2}} (x_2-\bar{x}_2) + \ldots..$$

Since higher order terms can be considered 0

$$y = \bar{y} + k_1(x_1-\bar{x}_1) + k_2(x_2-\bar{x}_2)$$

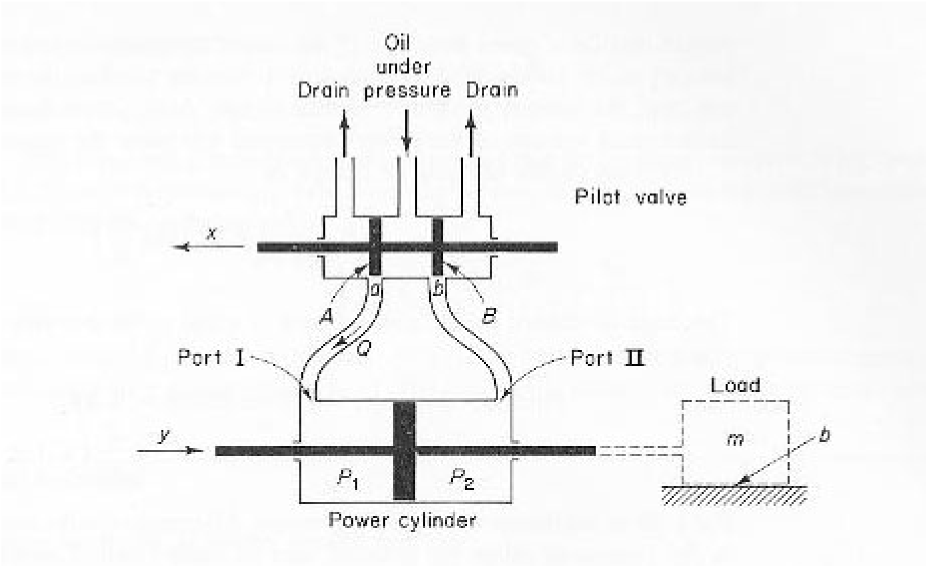
where

$$\bar{y} = f(\bar{x}_1,\bar{x}_2)$$

$$k_1 = \left.\frac{\partial f(x_1,x_2)}{\partial x_1}\right|_{\substack{x_1=\bar{x}_1\\ x_2=\bar{x}_2}}$$

$$k_2 = \left.\frac{\partial y(x_1,x_2)}{\partial x_2}\right|_{\substack{x_1=\bar{x}_1\\ x_2=\bar{x}_2}}$$

Example:



Schematic diagram of a hydraulic servomotor

Q :	Rate of flow to the power cylinder
$\Delta p = p_2 - p_1$:	Pressure difference in the two power cylinders
x :	Displacement of pilot valve

The relationship among the above variables is given by the nonlinear equation:

$$Q = f(x, \Delta p)$$

The linearized equation at $(\overline{Q}, \overline{x}, \Delta\overline{p}) = (0,0,0)$ is given by

$$Q - \overline{Q} = q = k_1(x - \overline{x}) - k_2(\Delta p - \Delta\overline{p})$$

where

$$k_1 = \left.\frac{\partial Q(x, \Delta p)}{\partial x}\right|_{\substack{x=\bar{x} \\ \Delta p=\Delta\bar{p}}} \qquad k_2 = \left.\frac{\partial Q(x, \Delta p)}{\partial \Delta p}\right|_{\substack{x=\bar{x} \\ \Delta p=\Delta\bar{p}}}$$

This gives

$$Q = k_1 x - k_2 \Delta p \qquad k_1, k_2 > 0$$

Using $Q\,dt = A\,p\,dy$ where

A:	Piston area
p:	Oil density
dy:	Displacement of mass

follows

$$\Delta p = \frac{1}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

The force developed by the power piston is given by:

$$A\Delta p = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

This force is applied to the mass *m* and including friction gives:

$$m\ddot{y} + f\,\dot{y} = \frac{A}{k_2}\left(k_1 x - Ap\frac{dy}{dt}\right)$$

where

f: friction coefficient

This gives:

$$m\ddot{y} + \left(f + \frac{A^2 p}{k_2} \right) \dot{y} = \frac{Ak_1}{k_2} x$$

Using

$$x \circ - - \bullet X(s)$$
$$y \circ - - \bullet Y(s)$$

where *X(s)* is the input
Y(s) is the output

$$\frac{Y(s)}{X(s)} = \frac{\frac{Ak_1}{k_2}}{s^2 m + \left(\frac{k_2 f + A^2 p}{k_2} \right) s} = \frac{k}{s(Ts+1)}$$

with

$$k = \frac{Ak_1}{k_2 f + A^2 p} \qquad T = \frac{mk_2}{k_2 f + A^2 p}$$

TRANSIENT RESPONSE ANALYSIS

Test signals:

- Impulse
- Step
- Ramp
- Sin and/or cos

Transient Response: for t between 0 and T

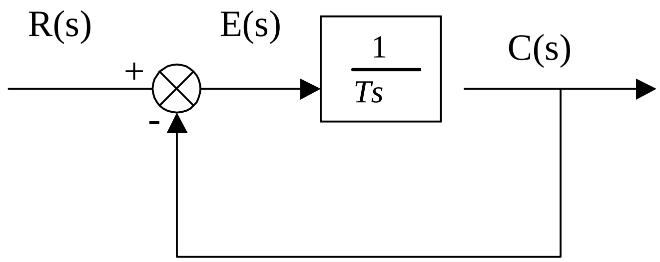
Steady-state Response: for $t \rightarrow \infty$

System Characteristics:

- Stability → transient
- Relative stability → transient
- Steady-state error → steady-state

First order systems

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$



Unit step response:

$$C(s) = \frac{1}{Ts+1}\cdot\frac{1}{s} = \frac{1}{s} - \frac{T}{sT+1}$$

$$c(t) = 1 - e^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = e^{-t/T} \qquad e(\infty) = 0$$

$$c(T) = 1 - e^{-1} = 0.632$$

$$\frac{dc(t)}{dt}\Big|_{t=0} = \frac{1}{T} e^{-t/T}\Big|_{t=0} = \frac{1}{T}$$

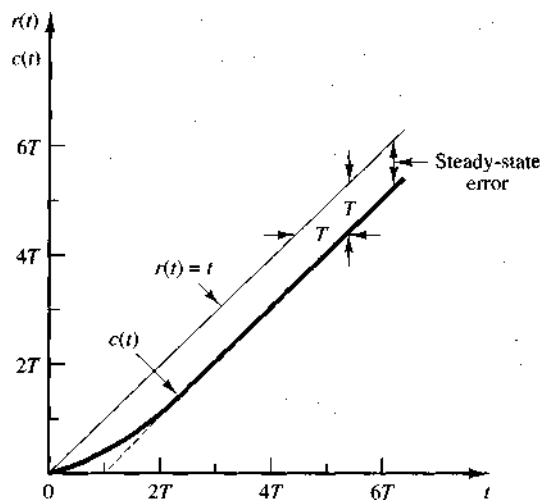
Unit ramp response

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s^2} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

$$c(t) = t - T + Te^{-t/T} \qquad t \geq 0$$

$$e(t) = r(t) - c(t) = T\left(1 - e^{-t/T}\right) \qquad t \geq 0$$

$$e(\infty) = T$$



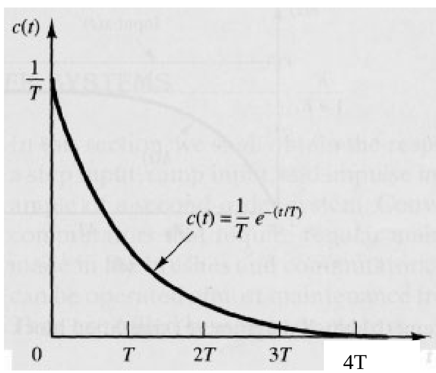
Unit-ramp response of the system

Impulse response:

$R(s) = 1$ $\qquad$ $r(t) = \delta(t)$

$$C(s) = \frac{1}{sT + 1}$$

$$c(t) = \frac{e^{-t/T}}{T} \qquad t \geq 0$$



Unit-impulse response of the system

Input			Output	
Ramp	r(t) = t	$t \geq 0$	$c(t) = t - T + Te^{-t/T}$	$t \geq 0$
Step	r(t) = 1	$t \geq 0$	$c(t) = 1 - e^{-t/T}$	$t \geq 0$
Impulse	r(t) = δ(t)		$c(t) = \frac{e^{-t/T}}{T}$	$t \geq 0$

Observation:

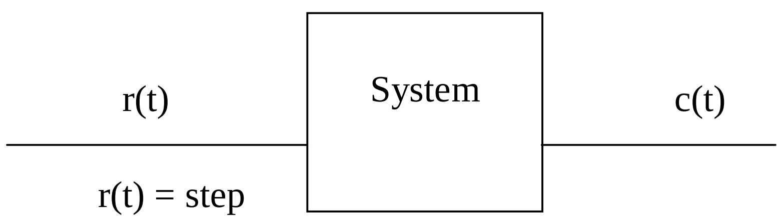
Response to the derivative of an input equals to derivative of the response to the original signal.

$Y(s) = G(s)\,U(s)$ U(s): input

$U_1(s) = s\,U(s) \quad Y_1(s) = s\,Y(s)$ Y(s): output

$$G(s)\,U_1(s) = G(s)\,s\,U(s) = s\,Y(s) = Y_1(s)$$

How can we recognize if a system is 1st order ?



Plot $\log|c(t) - c(\infty)|$

If the plot is linear, then the system is 1st order

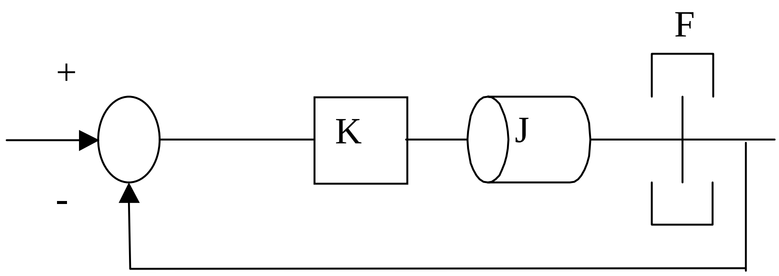
Explanation:

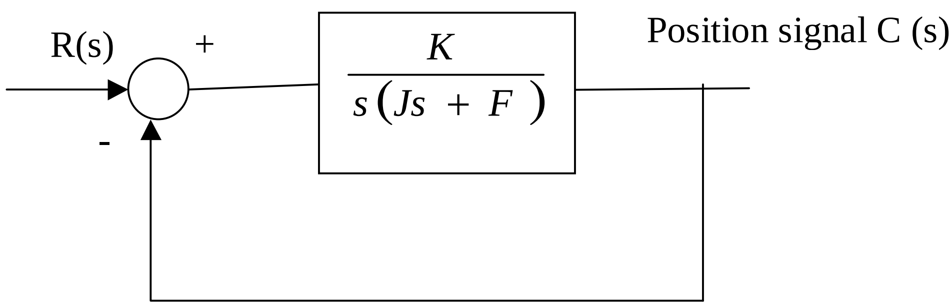
$$c(t) = 1 - e^{-t/T} \qquad c(\infty) = 1$$

$$\log|c(t) - c(\infty)| = \log|e^{-t/T}| = \frac{t}{T}$$

Second Order Systems

Block Diagram





Transfer function:

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + Fs + K}$$

$$= \frac{\frac{K}{J}}{\left[s + \frac{F}{2J} + \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{F}{2J} - \sqrt{\left(\frac{F}{2J}\right)^2 - \frac{K}{J}}\right]}$$

Substitute in the transfer function:

$$\frac{K}{J} = \omega_n^2$$

$$\frac{F}{J} = 2\,\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{F}{2\sqrt{JK}}$$

-

ζ: damping ratio
ω_n: undamped natural frequency
σ: stability ratio

to obtain

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ***Underdamped*** case: $0 < \zeta < 1$

  $F^2 - 4\,J\,K < 0$ two *complex conjugate* poles

- ***Critically damped*** case: $\zeta = 1$

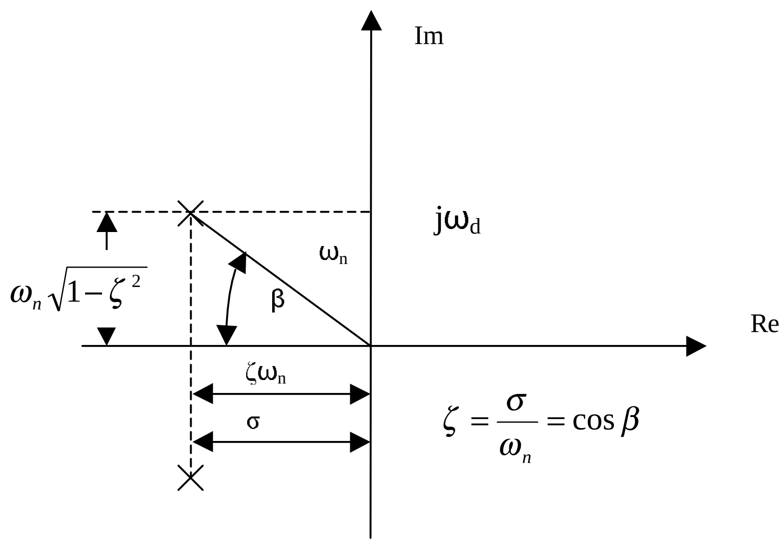
  $F^2 - 4\,J\,K = 0$ two *equal real* poles

- ***Overdamped*** case: $\zeta > 1$

  $F^2 - 4\,J\,K > 0$ two *real* poles

Under damped case $(0 < \zeta < 1)$:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$



$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

ω_n : undamped natural frequency

ω_d : damped natural frequency

ζ : damping ratio

<u>Unit step response:</u>

R(s) = 1/s

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\,2}}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^{\,2}}$$

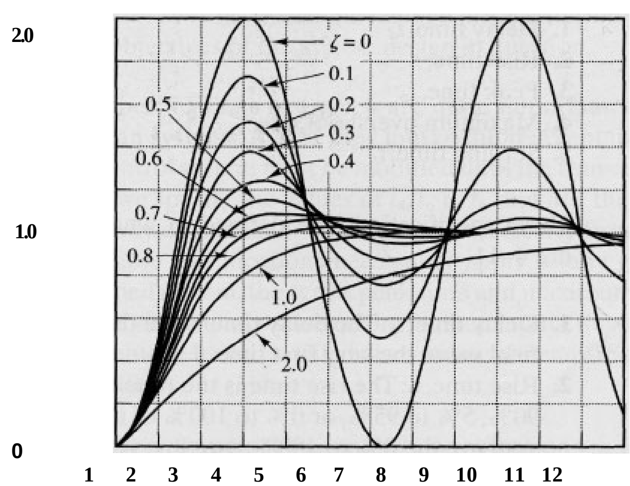
$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\qquad t\geq 0$$

or

$$c(t)=1-\frac{1}{\beta}e^{-\zeta\omega_n t}\sin(\omega_n\beta t+\theta)\qquad t\geq 0$$

$$\beta=\sqrt{1-\zeta^2}\qquad\qquad \theta=\tan^{-1}\frac{\beta}{\zeta}$$

$$\mathrm{e(t)=r(t)\text{ - }c(t)}=e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\qquad t\geq 0$$



Unit step response curves of a second order system

Undamped case (ζ = 0):

Unit step response:

$$c(t) = 1 - \cos \omega_n t \qquad t \geq 0$$

Critically damped case (ζ = 1):

Unit step Response: R(s) = 1/s

$$\frac{C(s)}{R(s)} = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n + \omega_n{}^2} = \frac{\omega_n{}^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \qquad t \geq 0$$

Overdamped case ($\zeta > 1$):

Unit step Response: R(s) = 1/s

$$C(s) = \frac{\omega_n^2}{\left(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}\right)\left(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}\right)} \cdot \frac{1}{s}$$

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad t \geq 0$$

with
$$s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$
$$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

if $|s_2| << |s_1|$, the transfer function can be approximated by

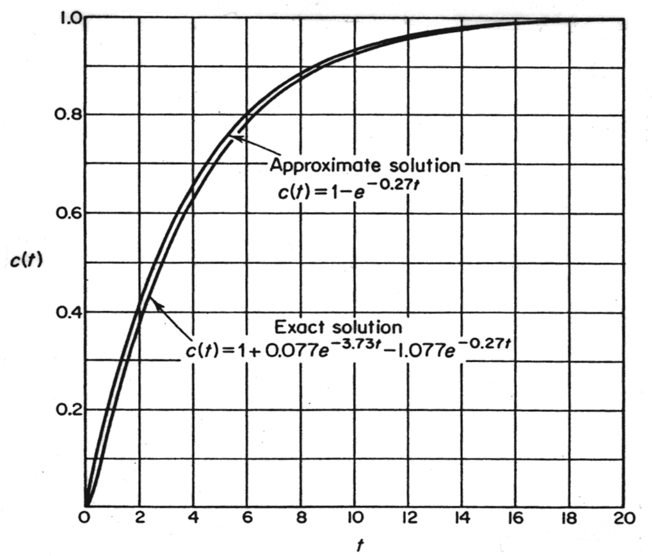
$$\frac{C(s)}{R(s)} = \frac{s_2}{s + s_2}$$

and for R(s) = 1/s

$$c(t) = 1 - e^{-s_2 t} \qquad t \geq 0$$

with

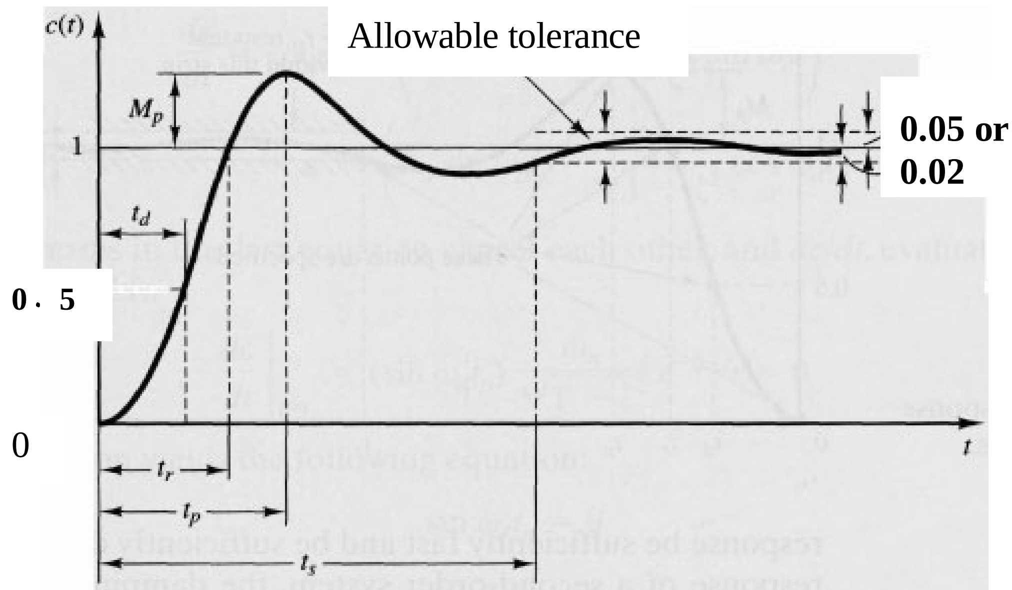
$$s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$



Unit step response curves of a critically damped system.

Transient Response Specifications

Unit step response of a 2nd order underdamped system:



t_d *delay time*: time to reach 50% of c(∞) for the first time.

t_r *rise time* : time to rise from 0 to 100% of c(∞).

t_p *peak time* : time required to reach the first peak.

M_p *maximum overshoot* : $\frac{c(t_p) - c(\infty)}{c(\infty)} \cdot 100\%$

t_s *settling time* : time to reach and stay within a 2% (or 5%) tolerance of the final value c(∞).

$0.4 < \zeta < 0.8$

Gives a good step response for an underdamped system

Rise time t_r time from 0 to 100% of $c(\infty)$

$$c(t_r)=1 \Rightarrow 1-e^{-\zeta\omega_d t_r}(\cos\omega_d t_r+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r)=1$$

$$\cos\omega_d t_r+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r=0$$

$$\tan\omega_d t_r=-\frac{\sqrt{1-\zeta^2}}{\zeta}=-\frac{\omega_d}{\sigma}$$

$$t_r=\frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{\sigma}\right)$$

Peak time t_p: time to reach the first peak of c(t)

$$\left.\frac{dc(t)}{dt}\right|_{t=t_p}=0 \Rightarrow (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}=0$$

$$\sin\omega_d t_p=0$$

$$t_p=\frac{\pi}{\omega_d}$$

Maximum overshoot M_p:

$$t = t_p = \frac{\pi}{\omega_d}$$

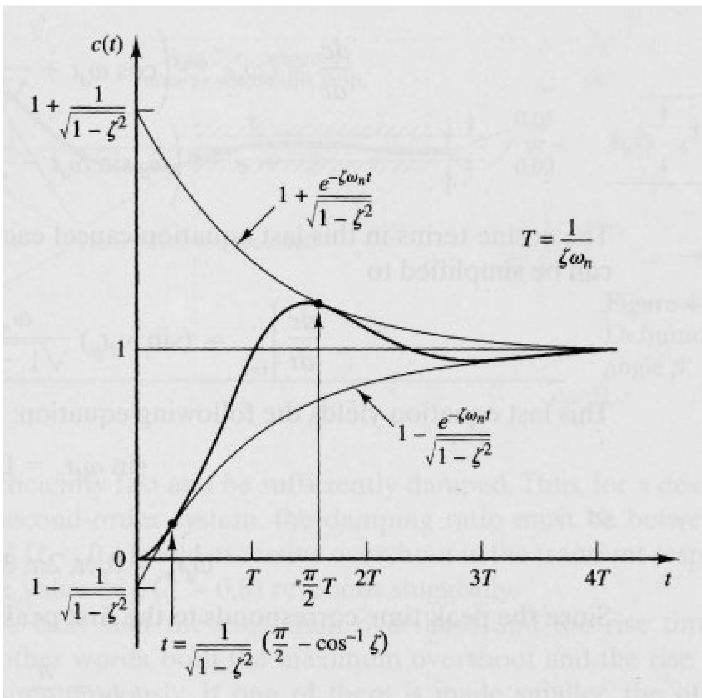
$$M_p = c(t_p) = 1 - e^{-\zeta\omega_n(\pi/\omega_d)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= e^{-\frac{\zeta\omega_n\pi}{\omega_d}} = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-\sigma\pi}{\omega_d}}$$

Settling time t_s:

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

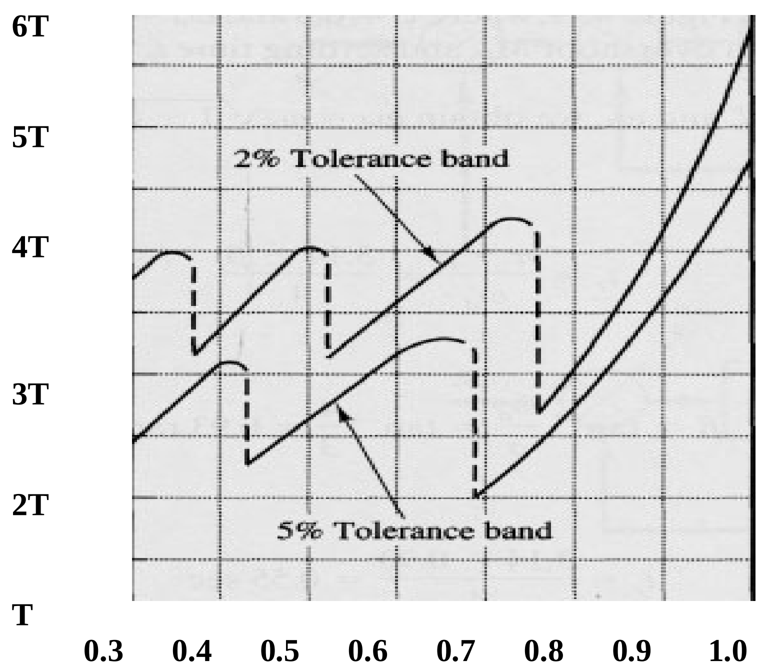
approximate t_s using envelope curves: $env(t) = 1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

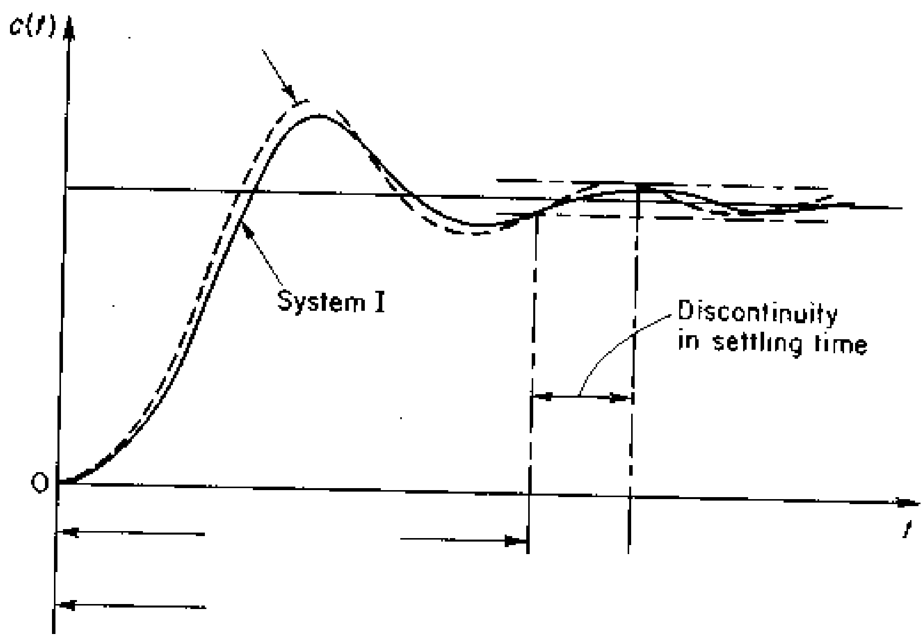


Pair of envelope curves for the unit-step response curve

2% band: $t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$ 5% band $t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$

Settling time t_s versus ζ curves $\{T = 1/(\zeta\omega_n)\}$





Impulse response of second-order systems

$$C(s) = \frac{\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} \qquad\qquad R(s) = 1$$

underdamped case (0< ζ < 1):

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t \qquad\qquad t \geq 0$$

the first peak occurs at $t = t_0$

$$t_0 = \frac{\tan^{-1} \dfrac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n \sqrt{1-\zeta^2}}$$

and the maximum peak is

$$c(t_0) = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} \right)$$

critically damped case (ζ = 1):

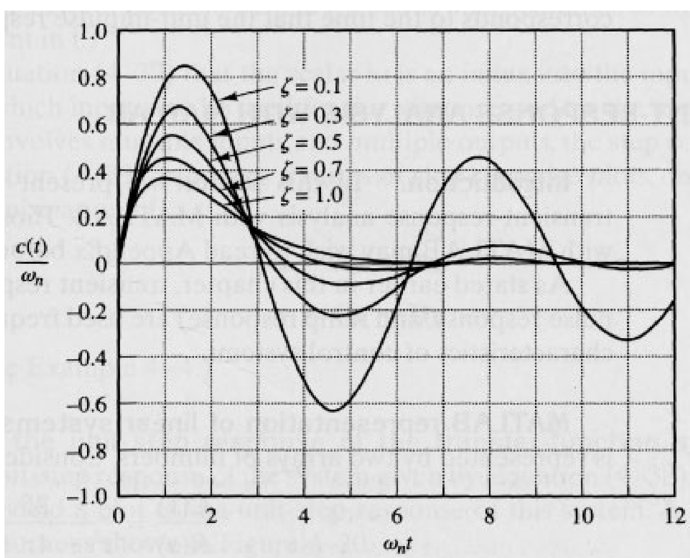
$$c(t) = \omega_n{}^2 t e^{-\omega_n t} \qquad t \geq 0$$

overdamped case (ζ >1):

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-s_2 t} \qquad t \geq 0$$

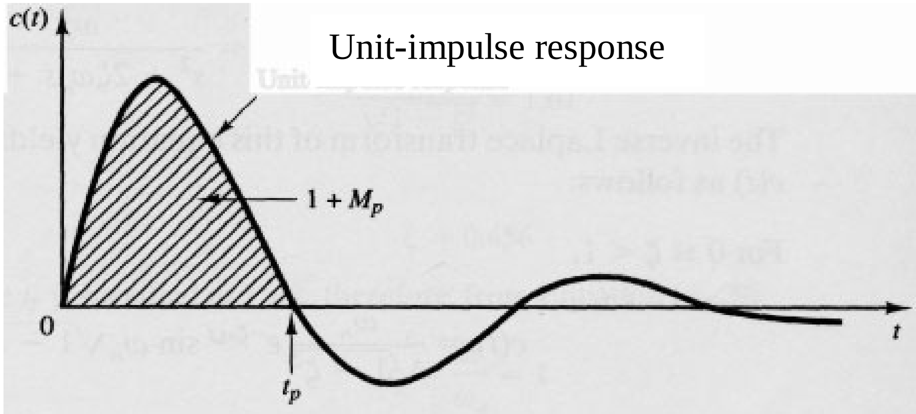
where

$$s_1 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$
$$s_2 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$



Unit-impulse response for 2nd order systems

Remark: Impulse Response = d/dt (Step Response)



Relationship between t_p, M_p and the unit-impulse response curve of a system

Unit ramp response of a second order system

$$C(s) = \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n + \omega_n^{\;2}} \cdot \frac{1}{s^2}$$

$R(s) = 1/ s^2$

for an underdamped system ($0 < \zeta < 1$)

$$c(t) = t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t + \frac{2\zeta^2 - 1}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right) \qquad t \geq 0$$

and the error:

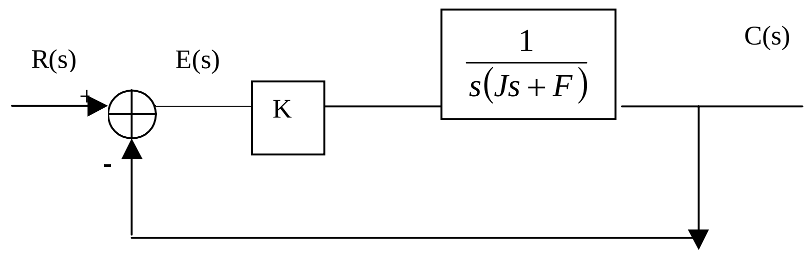
$$e(t) = r(t) - c(t) = t - c(t)$$

at steady-state:

$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n}$$

Examples:

a. Proportional Control



$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n + \omega_n^{\,2}}$$

with

$$\frac{K}{J} = \omega_n^2$$

$$\frac{F}{J} = 2\,\zeta\omega_n = 2\sigma$$

$$\zeta = \frac{F}{2\sqrt{JK}}$$

Choose K to obtain ‘good’ performance for the closed-loop system

For good *transient response*:

$0.4 < \zeta < 0.8$ → acceptable overshoot

ω_n sufficiently large → good settling time

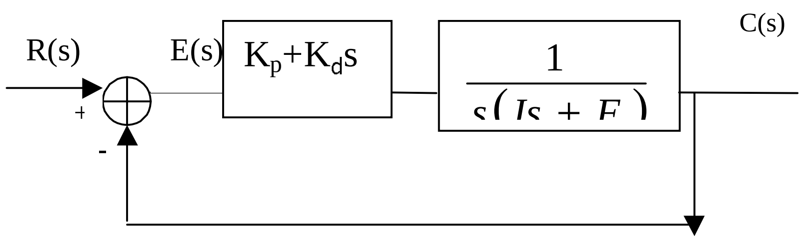
For small *stead- state error in ramp response*:

$$e(\infty) = \lim_{t\to\infty} e(t) = \frac{2\zeta}{\omega_n} = \frac{2F}{2\sqrt{K\zeta}} \cdot \sqrt{\frac{\zeta}{K}} = \frac{F}{K}$$ → large K

Large K reduces $e(\infty)$ but also leads to small ζ and large M_p

→compromise necessary

b. Proportional plus derivative control:



$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (F + K_d)s + K_p}$$

with

$$\varsigma = \frac{F + K_d}{2\sqrt{K_p J}} \qquad \omega_n = \sqrt{\frac{K_p}{J}}$$

The error for a ramp response is:

$$E(s) = \frac{s^2 J + sF}{s^2 J + s(F + K_d) + K_p} \cdot R(s)$$

and at steady-state:

$$e(\infty) = \lim_{s \to 0} sE(s) = \frac{F}{K_p}$$

using $z = \frac{K_p}{K_d}$

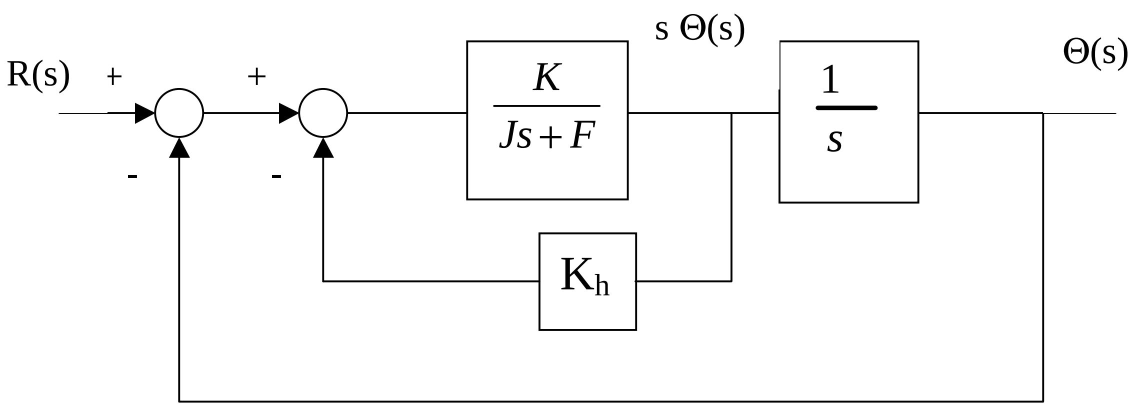
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{z} \cdot \frac{s + z}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Choose K_p, K_d to obtain ‘good’ performance of the closed-loop system

For small *steady-state error in ramp response* → K_p large

For good *transient response* → K_d so that $0.4 < \zeta < 0.8$

c. Servo mechanism with velocity feedback



Transfer function

$$\frac{\Theta(s)}{R(s)} = \frac{K}{Js^2 + (F + KK_h)s + K}$$

where

$$\varsigma = \frac{F + KK_h}{2\sqrt{KJ}}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$ (not affected by velocity feedback)

$$e(\infty) = \frac{F}{K}$$ for a ramp

Choose K, K_h to obtain 'good' performance for the closed-loop system

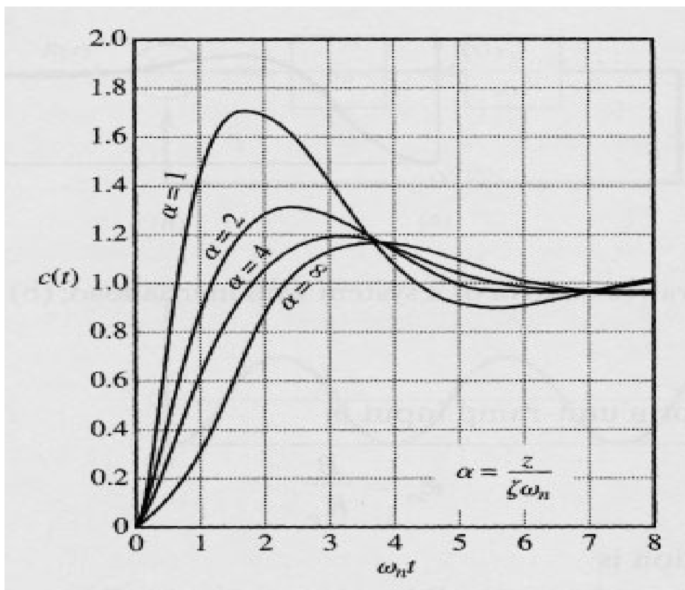
For small *steady-state error in ramp response* → K large

For good *transient response* → K_h so that $0.4 < \zeta < 0.8$

<u>*Remark:*</u> The damping ratio ζ can be increased without affecting the natural frequency ω_n in this case.

Effect of a zero in the step response of a 2nd order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^{\ 2}}{z} \cdot \frac{s+z}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} \qquad \zeta = 0.5$$



Unit-step response curves of 2nd order systems

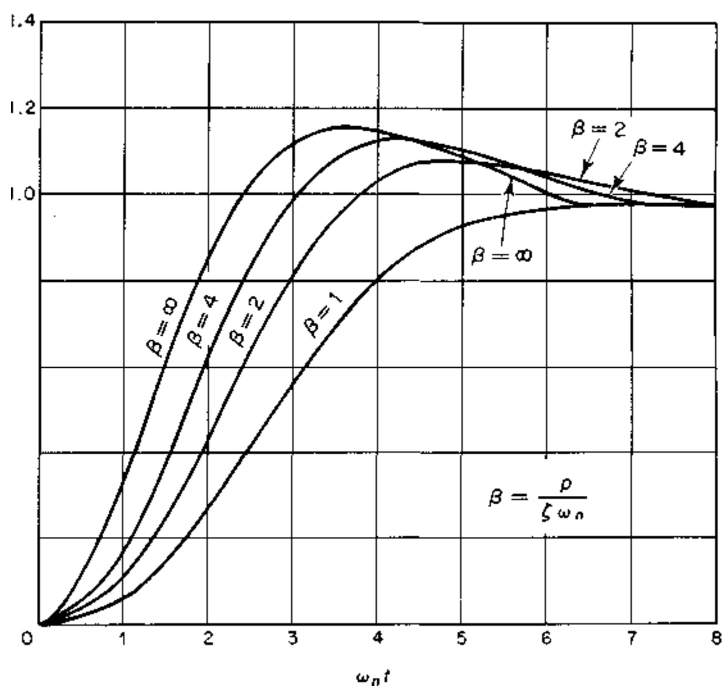
Unit step Response of 3rd order systems

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 p}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)(s+p)} \qquad 0 < \zeta < 1 \qquad R(s)=1/s$$

$$c(t) = 1 - \frac{e^{-pt}}{\beta\zeta^2(\beta-2)+1} - \frac{e^{-\xi\omega_n t}}{\beta\zeta^2(\beta-2)+1} \bullet$$

$$\left\{\beta\zeta^2(\beta-2)\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\beta\zeta\left[\zeta^2(\beta-2)+1\right]}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

where $$\beta = \frac{p}{\zeta\omega_n}$$



Unit-step response curves of the third-order system, $\zeta = 0.5$

The effect of the pole at $s = -p$ is:

- Reducing the maximum overshoot
- Increasing settling time

Transient response of higher-order systems

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + ... + b_{m-1}s + b_m}{s^n + ... + d_{n-1}s + a_n} = \frac{K(s+z_1)...(s+z_m)}{(s+p_1)...(s+p_n)} \qquad n > m$$

Unit step response

$$C(s) = \frac{K\sum_{i=1}^{m}(s+z_i)}{\sum_{j=1}^{q}(s+p_j)\sum_{k=1}^{r}(s^2+2\zeta_k\omega_k s+\omega_k{}^2)} \cdot \frac{1}{s}$$

$0 < \zeta_k < 1$ k=1,…,r and q + 2r = n

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k)+c_k\omega_k\sqrt{1-\zeta_k{}^2}}{s^2+2\zeta_k\omega_k+\omega_k{}^2}$$

$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_n\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right)$$

$$+ \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t}\sin\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right) \qquad t \geq 0$$

Dominant poles: the poles closest to the imaginary axis.

STABILITY ANALYSIS

$$G(s) = \frac{B(s)}{A(s)} = \frac{\sum_{i=0}^{m} b_i s^{m-i}}{\sum_{i=0}^{n} a_i s^{n-i}}$$

<u>Conditions for Stability:</u>

A. ***Necessary*** condition for stability:

All coefficients of A(s) have the same sign.

B. ***Necessary and sufficient*** condition for stability:

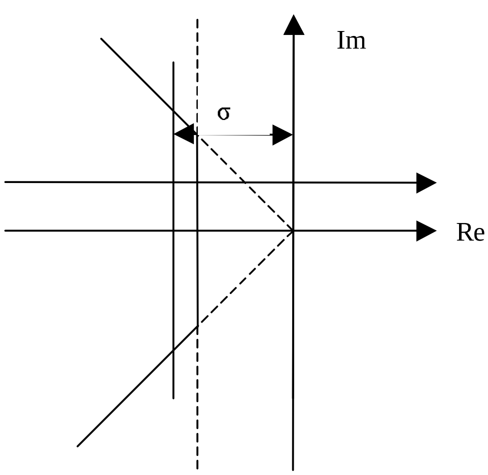
$A(s) \neq 0$ for $\mathrm{Re}[s] \geq 0$

or, equivalently

All poles of G(s) in the left-half-plane (LHP)

<u>Relative stability:</u>

The system is stable and further, all the poles of the system are located in a sub-area of the left-half-plane (LHP).



Necessary condition for stability:

$$
\begin{aligned}
A(s) &= a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \\
&= a_0 (s + p_1)(s + p_2)\ldots(s + p_n) \\
&= a_0 s^n + a_0 (p_1 + p_2 + \ldots p_n) s^{n-1} \\
&\quad + a_0 (p_1 p_2 + \ldots + p_{n-1} p_n) s^{n-2} \\
&\quad \vdots \\
&\quad + a_0 (p_1 p_2 \ldots p_n)
\end{aligned}
$$

$-p_1$ to $-p_n$ are the poles of the system.

If the system is stable → all poles have negative real parts

→ the coefficients of a stable polynomial have the same sign.

Examples:

$A(s) = s^3 + s^2 + s + 1$ can be stable or unstable

$A(s) = s^3 - s^2 + s + 1$ is unstable

Stability testing

Test whether all poles of G(s) (roots of A(s)) have *negative real parts*.

Find all roots of A(s) → too many computations

Easier Stability test?

Routh-Hurwitz Stability Test

$A(s) = \alpha_0 s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n$

$$
\begin{array}{llll}
s^n & \alpha_0 & \alpha_2 & \alpha_4 \ \ldots \\
s^{n-1} & \alpha_1 & \alpha_3 & \alpha_5 \ \ldots \\
s^{n-1} & b_1 & b_2 & b_3 \\
 & c_1 & c_2 & \\
 \ldots\ldots & & & \\
s^2 & e_1 & e_2 & \\
s^1 & f_1 & & \\
s^0 & g_1 & &
\end{array}
$$

$$b_1 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{1}{-a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$c_1 = \frac{1}{-b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = \frac{a_3 b_1 - a_1 b_2}{b_1} \qquad \text{etc}$$

.

Properties of the Ruth-Hurwitz table:

1. Polynomial A(s) is stable (i.e. all roots of A(s) have negative real parts) if there is *no sign change in the first column*.

2. The *number of sign changes in the first column* is equal to the number of roots of A(s) with positive real parts.

Examples:

A(s) = $a_0 s^2 + \alpha_1 s + \alpha_2$

$$\begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & \\ s^0 & a_2 & \end{array}$$

$\alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0$ or
$\alpha_0 < 0, \quad \alpha_1 < 0, \quad \alpha_2 < 0$

For 2nd order systems, the condition that all coefficients of A(s) have the same sign is *necessary and sufficient for stability*.

A(s) = $\alpha_0 s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

$\alpha_0 > 0, \quad \alpha_1 > 0, \quad \alpha_3 > 0, \qquad \alpha_1\alpha_2 - \alpha_0\alpha_3 > 0$

(or all first column entries are negative)

Special cases:

1. The properties of the table do not change when all the coefficients of a row are multiplied by the same positive number.

2. If the first-column term becomes zero, replace 0 by ε and continue.
   - If the signs above and below ε are the same, then there is a pair of (complex) imaginary roots.
   - If there is a sign change, then there are roots with positive real parts.

Examples:

$A(s) = s^3 + 2s^2 + s + 2$

$$\begin{array}{lll} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \rightarrow \varepsilon & \\ s^0 & 2 & \end{array}$$

→ pair of imaginary roots (s= ±j)

$A(s) = s^3 - 3s + 2 = (s-1)^2(s+2)$

$$\begin{array}{lll} s^3 & 1 & -3 \\ s^2 & 0 \approx \varepsilon & 2 \\ s^1 & -3-\frac{2}{\varepsilon} & \\ s^0 & 2 & \end{array}$$

→ two roots with positive real parts

3. If all coefficients in a line become 0, then A(s) has roots of equal magnitude radially opposed on the real or imaginary axis. Such roots can be obtained from the roots of the auxiliary polynomial.

Example:

$A(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

→ auxiliary polynomial p(s)

$$p(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dp(s)}{ds} = 8s^3 + 96s$$

s^3	8	96
s^2	24	-50
s^1	112.7	0
s^0	-50	

- A(s) has two radially opposed root pairs (+1,-1) and (+5j,-5j) which can be obtained from the roots of p(s).
- One sign change indicates A(s) has one root with positive real part.

Note:

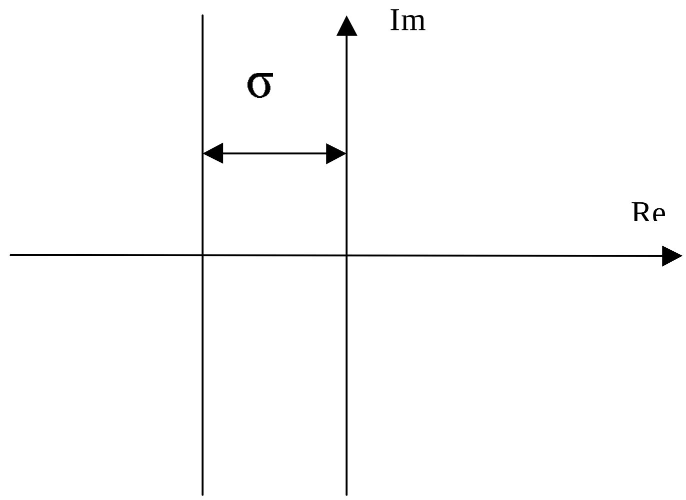
$A(s) = (s+1)(s-1)(s+5j)(s-5j)(s+2)$

$p(s) = 2(s^2-1)(s^2+25)$

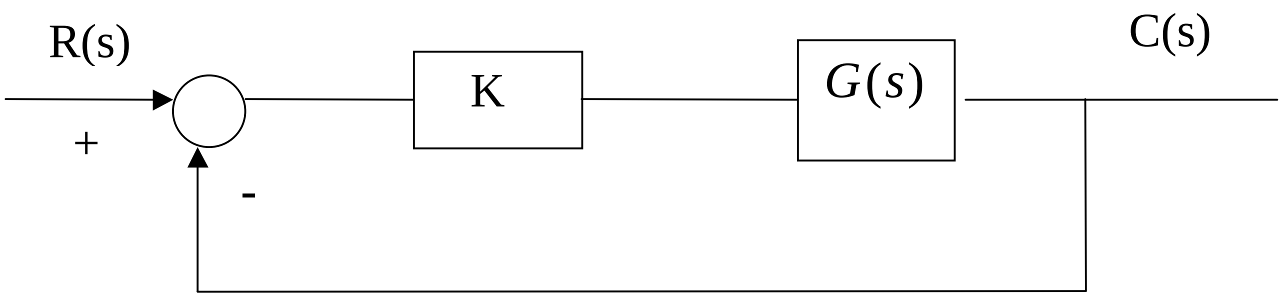
Relative stability

Question: Have all the roots of A(s) a distance of at least σ from the imaginary axis?

Substitute s with s = z - σ in A(s) and apply the Routh-Hurwitz test to A(z)



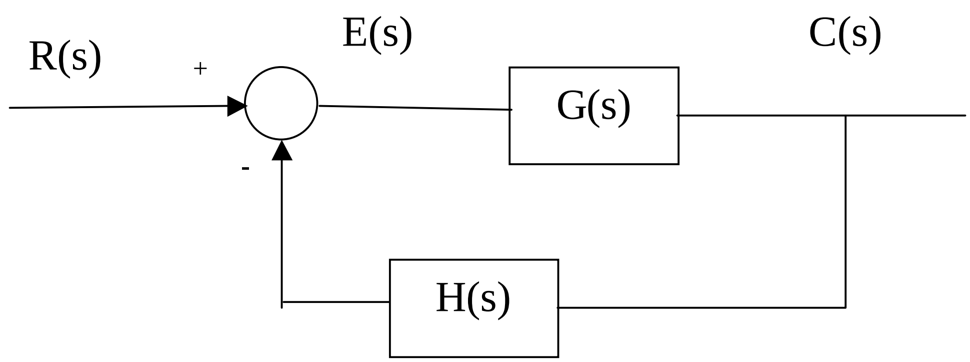
Closed-loop System Stability Analysis



Question: For what value of K is the closed-loop system stable?

Apply the Routh-Hurwitz test to the denominator polynomial of the closed-loop transfer function $\frac{KG(s)}{1 + KG(s)}$.

Steady-State Error Analysis



Evaluate the steady-state performance of the closed-loop system using the steady-state error e_{ss}

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$E(s) = \frac{1}{1 + G(s)H(s)} \cdot R(s)$$

for the following input signals: Unit step input
Unit ramp input
Unit parabolic input

Assumption: the closed-loop system is stable

Question: How can we obtain the steady-state error e_{ss} of the closed-loop system from the open-loop transfer function G(s)H(s) ?

Classification of systems:

For an open-loop transfer function

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots}{s^N (T_1 s+1)(T_2 s+1)\cdots}$$

Type of system: Number of poles at the origin, i.e., N

Static Error Constants: K_p, K_v, K_a

Open-loop transfer function: G(s)H(s)

Closed-loop transfer function: $G_{tot}(s) = \dfrac{G(s)}{1+G(s)H(s)}$

Static Position Error Constant: K_p

Unit step input to the closed-loop system shown in fig, p. B33.

R(s) = 1/s $\qquad e_{ss} = \lim_{s\to 0} sE(s) = \dfrac{1}{1+G(0)H(0)}$

Define: $\qquad K_p = \lim_{s\to 0} G(s)H(s) = G(0)H(0)$

Type 0 system $\qquad K_p = K \qquad e_{ss} = \dfrac{1}{1+K_p}$

Type 1 and higher $\qquad K_p = \infty \qquad e_{ss} = 0$

<u>Static Velocity Error Constant:</u> **K_v**

Unit ramp input to the closed-loop system shown if fig, p. B33.

$R(s) = 1/s^2$ $$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{sG(s)H(s)}$$

<u>Define:</u> $$K_v = \lim_{s\to 0} sG(s)H(s)$$

Type 0 system	$K_v = 0$	$e_{ss} = \infty$
Type 1 system	$K_v = K$	$e_{ss} = 1/K_v$
Type 2 and higher	$K_v = \infty$	$e_{ss} = 0$

<u>Static Acceleration Error Constant:</u> K_a

Unit parabolic input to the closed-loop system shown in fig, p. B33

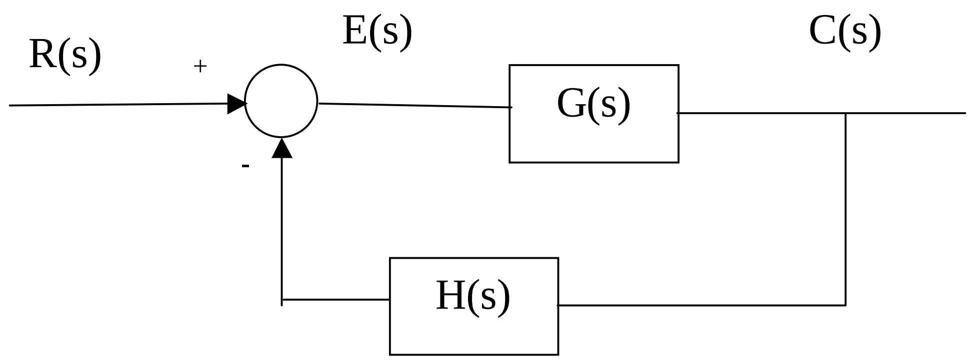
$R(s) = 1/s^3$ $$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \cdot \frac{1}{s^3} = \lim_{s\to 0} \frac{1}{s^2 G(s)H(s)}$$

<u>Define:</u> $$K_a = \lim_{s\to 0} s^2 H(s)G(s)$$

Type 0 system	$K_a = 0$	$e_{ss} = \infty$
Type 1 system	$K_a = 0$	$e_{ss} = \infty$
Type 2 system	$K_a = K$	$e_{ss} = 1/ K_a$
Type 3 and higher	$K_a = \infty$	$e_{ss} = 0$

Summary:

Consider a closed-loop system:



with an open-loop transfer function:

$$G(s)H(s) = \frac{K(T_a s + 1)\cdot(T_b s + 1)\ldots}{s^N (T_1 s + 1)\cdot(T_2 s + 1)\ldots}$$

and static error constants defined as:

$$K_p = \lim_{s \to 0} G(s)H(s) = G(0)H(0)$$

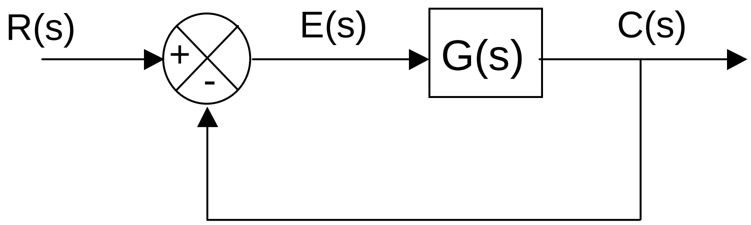
$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_a = \lim_{s \to 0} s^2 H(s)G(s)$$

The steady-state error e_{ss} is given by:

	Unit step r(t) = 1	Unit ramp r(t) = t	Unit parabolic $r(t) = t^2/2$
Type 0	$e_{ss} = \frac{1}{1+K_p} (= \frac{1}{1+K})$	$e_{ss} = \infty$	$e_{ss} = \infty$
Type 1	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_v} (= \frac{1}{K})$	$e_{ss} = \infty$
Type 2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{1}{K_a} (= \frac{1}{K})$

Correlation between the Integral of error in step response and Steady-state error in ramp response



$$E(s) = L[e(t)] = \int_0^\infty e^{-st} e(t)dt$$

$$\lim_{s \to 0} E(s) = \lim_{s \to 0} \int_0^\infty e^{-st} e(t)\,dt = \int_0^\infty e(t)\,dt$$

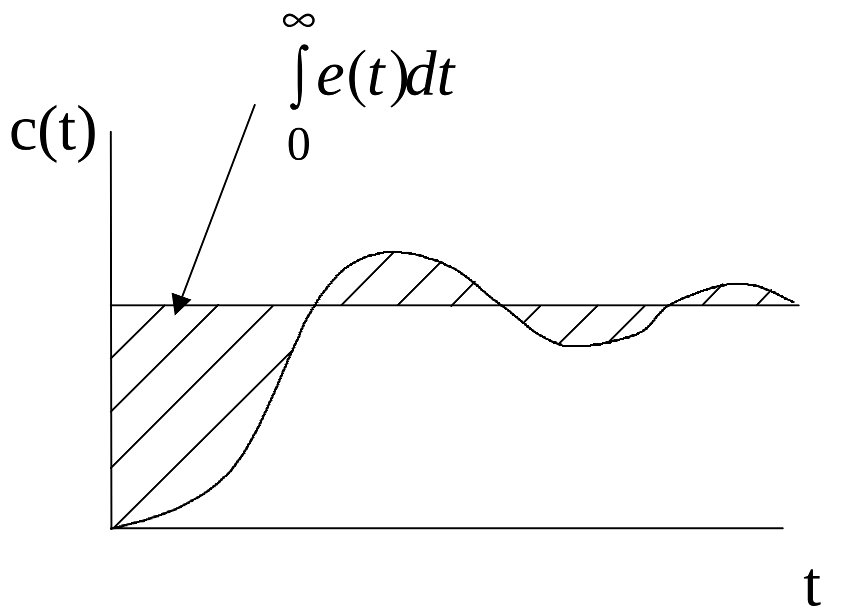
substitute $E(s) = \dfrac{R(s)}{1+G(s)}$ in the above eq.

$$\lim_{s\to 0} \frac{R(s)}{1+G(s)} = \int_0^\infty e(t)\,dt \qquad step: \quad R(s) = \frac{1}{s}$$

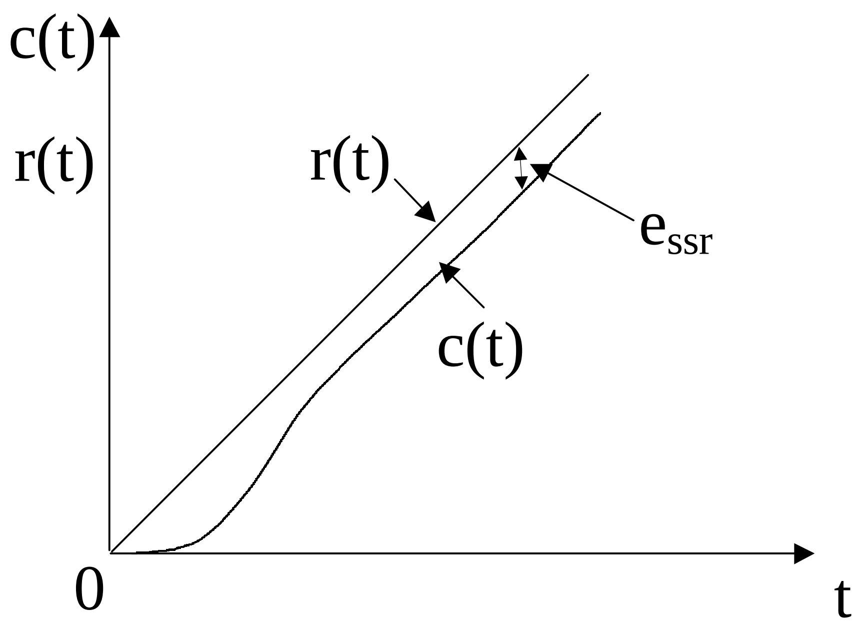
$$\int_0^\infty e(t)dt = \lim_{s\to 0}\left[\frac{1}{1+G(s)} \cdot \frac{1}{s}\right] = \lim_{s\to 0} \frac{1}{s \cdot G(s)} = \frac{1}{K_v}$$

$\dfrac{1}{K_V}$ = Steady-state error in unit-ramp input = e_{ssr}

$$\boxed{e_{ssr} = \int_0^\infty e(t)dt}$$



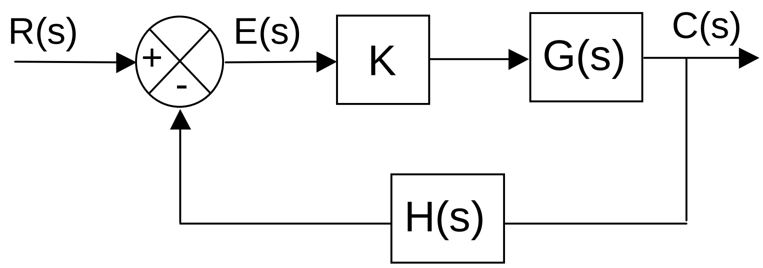
$\int_0^\infty e(t)dt$
c(t)
t



c(t)
r(t)
r(t)
e_{ssr}
c(t)
0
t

ROOT LOCUS

Consider the system



$$\frac{C(s)}{R(s)} = \frac{K \cdot G(s)}{1 + K \cdot G(s) \cdot H(s)}$$

Root locus presents the *poles of the closed-loop system*

when the gain K changes from 0 to ∞

$$1 + K \cdot G(s) \cdot H(s) = 0 \Rightarrow \begin{cases} |K \cdot G(s) \cdot H(s)| = 1 & \textit{Magnitude Condition} \\ \angle G(s) \cdot H(s) = \pm 180^{\circ} \cdot (2k+1) & \textit{Angle Condition} \end{cases}$$

$$k = 0, 1, 2, \ldots$$

Example:

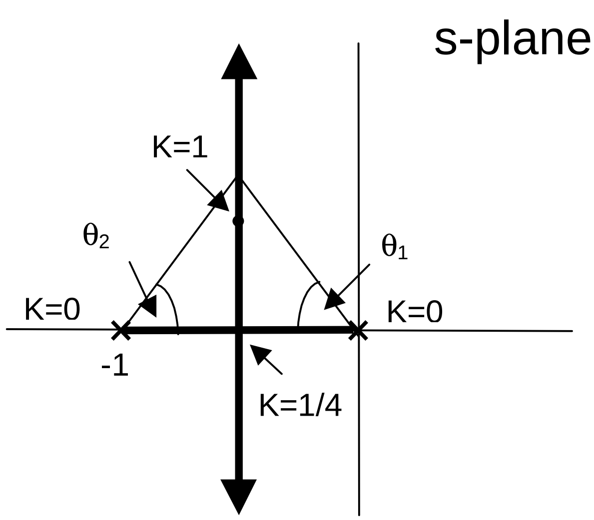
$$K \cdot G(s) \cdot H(s) = \frac{K}{s \cdot (s+1)}$$

$$1 + K \cdot G(s) \cdot H(s) = 0 \Rightarrow s^2 + s + K = 0$$

$$s_{1,2} = -\frac{1}{2} \pm \frac{1}{2} \cdot \sqrt{1 - 4 \cdot K}$$

Angle condition:

$$\angle\left(\frac{K}{s \cdot (s+1)}\right) = -\angle s - \angle s+1 = -(180 - \theta_1) - \theta_2 = \pm 180^\circ$$



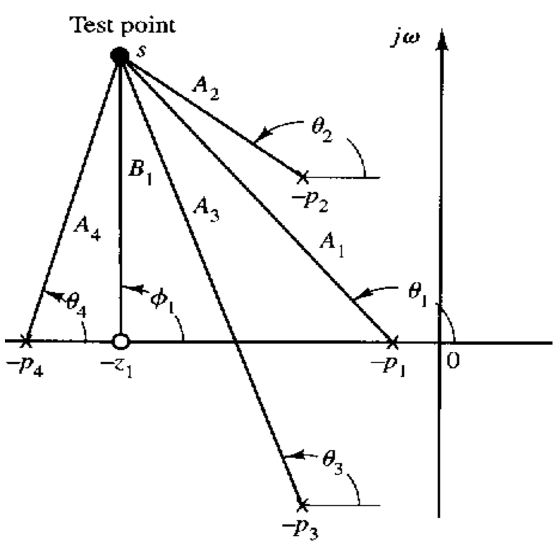
Magnitude and Angle Conditions

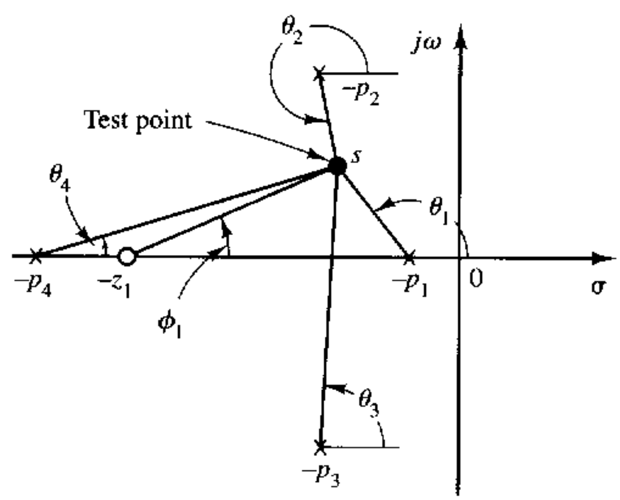
$$K \cdot G(s) \cdot H(s) \;=\; \frac{K \cdot (s+z_1)}{(s+p_1)\cdot(s+p_2)\cdot(s+p_3)\cdot(s+p_4)}$$

$$\left| K \cdot G(s) \cdot H(s) \right| \;=\; \frac{K \cdot B_1}{A_1 \cdot A_2 \cdot A_3 \cdot A_4} \;=\; 1$$

$$\angle G(s) \cdot H(s) \;=\; \varphi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4 \;=\; \pm 180^\circ \cdot (2 \cdot k + 1)$$

for k = 0, 1, 2, …





Construction Rules for Root Locus

Open-loop transfer function:

$$K\,H(s)\cdot G(s) \quad = \quad K\,\frac{B(s)}{A(s)}$$

m: order of open-loop numerator polynomial

n: order of open-loop denominator polynomial

Rule 1: Number of branches

The number of branches is equal to the number of poles of the open-loop transfer function.

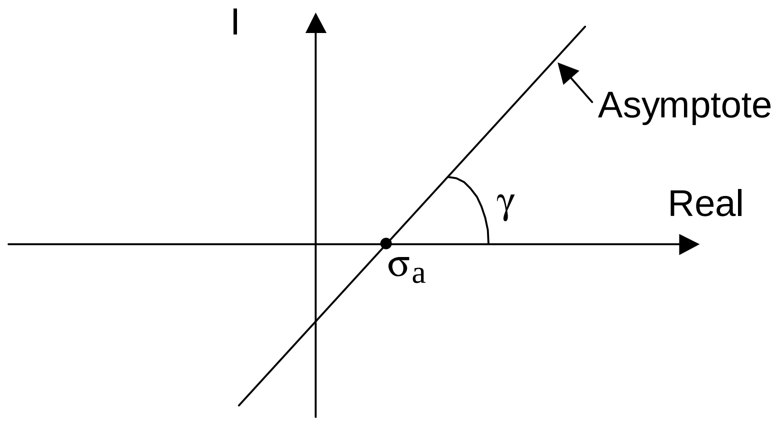
Rule 2: Real-axis root locus

If the total number of poles and zeros of the open-loop system to the right of the s-point on the real axis is odd, then this point lies on the locus.

Rule 3: Root locus end-points

The locus starting point (K=0) are at the open-loop poles and the locus ending points (K=∞) are at the open loop zeros and n-m branches terminate at infinity.

Rule 4: **Slope of asymptotes of root locus as s approaches infinity**



$$\gamma \quad = \quad \frac{\pm 180^\circ \cdot (2k+1)}{n-m}, \quad k \quad = \quad 0, 1, 2, \ldots$$

Rule 5: **Abscissa of the intersection between asymptotes of root locus and real-axis**

$$\sigma_a = \frac{\sum_{i=1}^{n} (-p_i) - \sum_{i=1}^{m} (-z_i)}{n-m}$$

$(-p_i)$ = poles of open-loop transfer function
$(-z_i)$ = zeros of open-loop transfer function

Rule 6: Break-away and break-in points

From the characteristic equation

$$f(s) = A(s) + K \cdot B(s) = 0$$

the break-away and -in points can be found from:

$$\frac{dK}{ds} = -\frac{A'(s) \cdot B(s) - A(s) \cdot B'(s)}{B^2(s)} = 0$$

Rule 7: Angle of departure from complex poles or zeros

Subtract from 180° the sum of all angles from all other zeros and poles of the open-loop system to the complex pole (or zero) with appropriate signs.

Rule 8: Imaginary-axis crossing points

Find these points by solving the characteristic equation for $s=j\omega$ or by using the Routh's table.

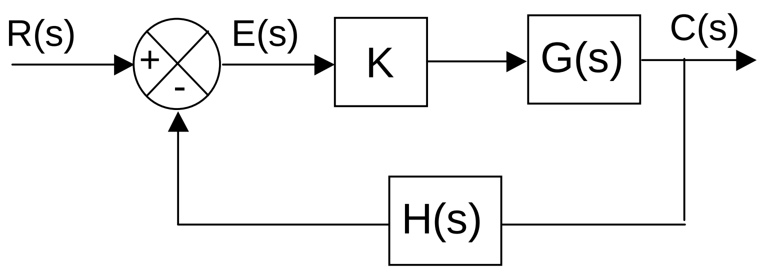
<u>Rule 9:</u> Conservation of the sum of the system roots

If the *order of numerator is lower than the order of denominator by two or more*, then the sum of the roots of the characteristic equation is constant.

Therefore, if some of the roots more towards the left as K is increased, the other roots must more toward the right as K in increased.

Discussion of Root Locus Construction Rules

Consider:



$$K \cdot H(s) \cdot G(s) = K \cdot \frac{B(s)}{A(s)} = K \cdot \frac{\sum_{i=0}^{m} b_i \cdot s^{m-i}}{\sum_{i=0}^{n} \alpha_i \cdot s^{n-i}}$$

m: number of zeros of open-loop *KH(s)G(s)*

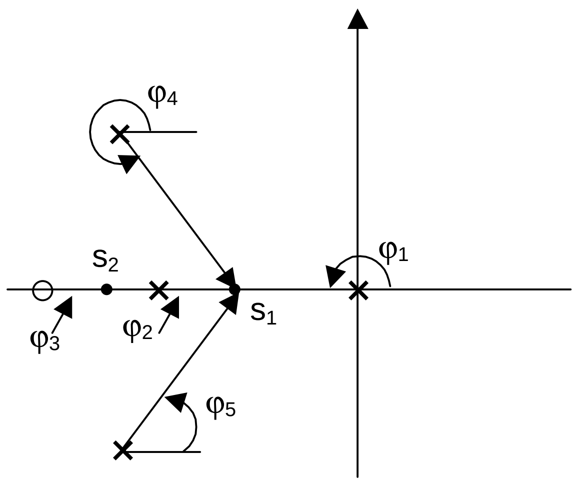
n: number of poles of open-loop *KH(s)G(s)*

Characteristic Equation: $f(s) = A(s) + K \cdot B(s) = 0$

Rule 1: Number of branches

The characteristic equation has n zeros $\Rightarrow$
the root locus has n branches

Rule 2: Real-axis root locus



Consider two points s_1 and s_2 :

$$s_1 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 0 = \varphi_3\,, \quad \varphi_4 + \varphi_5 = 180 \cdot 2 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 3 \cdot 180 \end{cases}$$

$$s_2 \begin{cases} \varphi_1 = 180\,, \quad \varphi_2 = 180, \quad \varphi_3 = 0\,, \quad \varphi_4 + \varphi_5 = 360 \\ \varphi_1 + \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 = 4 \cdot 180 \end{cases}$$

Therefore, s_1 is on the root locus; s_2 is not.

Rule 3: Root locus end-points

Magnitude condition:

$$\left|\frac{B(s)}{A(s)}\right| = \frac{1}{K} = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

K=0 open loop poles

K=∞ m open loop zeros

m-n branches approach infinity

Rule 4: Slope of asymptotes of root locus as s approaches infinity

$$K \cdot \frac{B(s)}{A(s)} = -1$$

$$\lim_{s \to \infty} \frac{K \cdot B(s)}{A(s)} = \lim_{s \to \infty} \frac{K}{s^{n-m}} = -1$$

$$s^{n-m} = -K \ \ for \ s \to \infty$$

Using the angle condition:

$$\angle s^{n-m} = \angle -K = \pm 180^{\circ} \cdot (2 \cdot k + 1), \ k = 1, 2, 3, \ldots$$

or

$$(n-m)\cdot\angle s = \pm 180^\circ \cdot (2\cdot k+1)$$

leading to

$$\angle s = \quad \gamma \quad = \frac{\pm 180^\circ \cdot (2\cdot k+1)}{n-m}$$

Rule 5: Abscissa of the intersection between asymptotes of root-locus and real axis

$$\frac{A(s)}{B(s)} = \frac{s^n + s^{n-1}\cdot\sum_{i=1}^{n} p_i + \ldots + \prod_{i=1}^{m} p_i}{s^m + s^{m-1}\cdot\sum_{i=1}^{m} z_i + \ldots + \prod_{i=1}^{n} z_i} = -K$$

Dividing numerator by denominator yields:

$$s^{n-m} - \left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i\right)\cdot s^{n-m-1} + \ldots = -K$$

For large values of s this can be approximated by:

$$\left(s - \frac{\sum_{i=1}^{m} z_i - \sum_{i=1}^{n} p_i}{n-m}\right)^{n-m} = -K$$

The equation for the asymptote (for $s \to \propto$) was found in Rule 4 as

$$s^{n-m} \;=\; -K$$

this implies $\sigma_a = -\dfrac{-\sum_{i=1}^{m} z_i + \sum_{i=1}^{n} p_i}{n-m} = \dfrac{\sum_{i=1}^{n} -p_i - \sum_{i=1}^{m} -z_i}{n-m}$

<u>Rule 6</u>: Break-away and break-in points

At break-away (and break-in) points the characteristic equation:

$$f(s) \;=\; A(s) + K \cdot B(s) \;=\; 0$$

has multiple roots such that:

$$\frac{df(s)}{ds} \;=\; 0 \;\;\Rightarrow\;\; A'(s) + K \cdot B'(s) \;=\; 0 \qquad \left(A'(s) \;=\; \frac{dA(s)}{ds} \right)$$

$\Rightarrow$ for $K = -\dfrac{A'(s)}{B'(s)}$, *f(s)* has multiple roots

Substituting the above equation into f(s) gives:

$$A(s) \cdot B'(s) - A'(s) \cdot B(s) \;=\; 0$$

Another approach is using:

$$K = -\frac{A(s)}{B(s)} \quad \text{from} \quad f(s) = 0$$

This gives:

$$\frac{dK}{ds} = -\frac{A^{'}(s) \cdot B(s) - A(s) \cdot B^{'}(s)}{B^{2}(s)}$$

and break-away, break-in points are obtained from:

$$\frac{dK}{ds} = 0$$

Extended Rule 6:

Consider

$$f(s) = A(s) + K \cdot B(s) = 0$$

and

$$K = -\frac{A(s)}{B(s)}$$

If the first *(y-1)* derivatives of A(s)/B(s) vanish at a given point on the root locus, then there will be *y* branches approaching and *y* branches leaving this point.

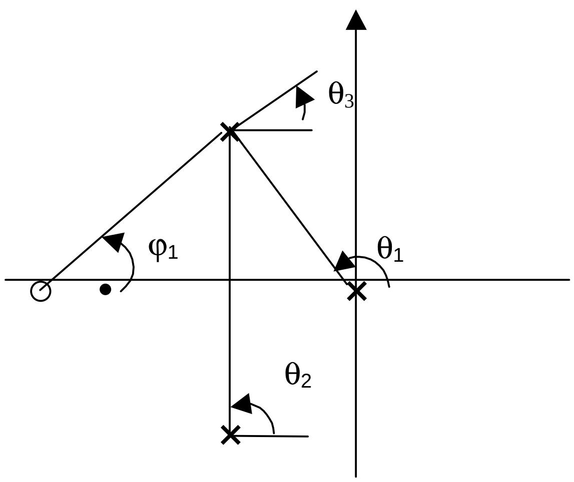
The angle between two adjacent approaching branches is given by:

$$\theta_y = \pm\frac{360^{\circ}}{y}$$

The angle between a leaving branch and an adjacent approaching branch is:

$$\theta_y = \pm \frac{180^\circ}{y}$$

Rule 7: Angle of departure from complex pole or zero



$$\theta_2 = 90^\circ$$
$$\theta_3 = 180^\circ - (\theta_1 + \theta_2 - \varphi_1)$$

Rule 8: Imaginary-axis crossing points

Example: $f(s) = s^3 + b \cdot s^2 + c \cdot s + K \cdot d = 0$

s^3	1	c
s^2	b	Kd
s^1	(bc-Kd)/b	
s^0	Kd	

For crossing points on the Imaginary axis:

$$b \cdot c - K \cdot d = 0 \Rightarrow \quad K = \frac{bc}{d}$$

Further, $b \cdot s^2 + K \cdot d = 0$ leading to

$$s_{1,2} = \pm j \cdot \sqrt{\frac{K \cdot d}{b}} = \pm j\omega$$

The same result is obtained by solving $f(j\omega) = 0$.

<u>Rule 9:</u> Conservation of the sum of the system roots

From

$$A(s) + K \cdot B(s) = \prod_{i=1}^{n} (s + r_i)$$

we have

$$\prod_{i=1}^{n} (s + p_i) + K \cdot \prod_{i=1}^{m} (s + r_i) = \prod_{i=1}^{n} (s + r_i)$$

with

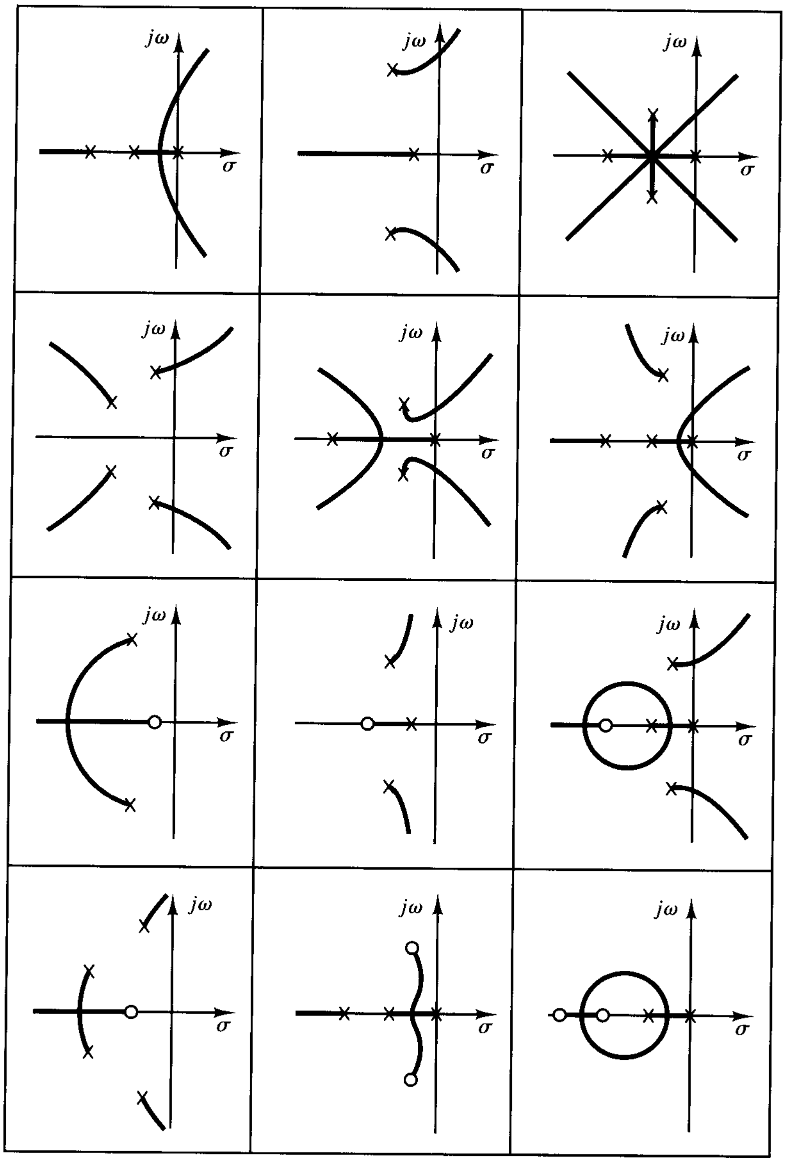
$$A(s) = \prod_{i=1}^{n} (s + p_i) \quad \text{and} \quad B(s) = \prod_{i=1}^{m} (s + z_i)$$

By equating coefficients of s^{n-1} for $n \geq m + 2$, we obtain the following:

Sum of open-loop poles $\quad \sum_{i=1}^{n} -p_i = \sum_{i=1}^{n} -r_i \quad$ Sum of closed-loop poles

i.e. the sum of closed-loop poles is independent of K !

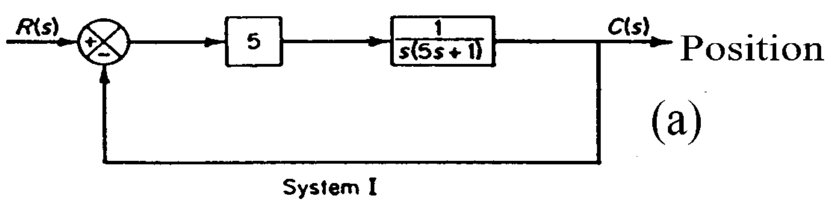
Table 6–1 Open-Loop Pole–Zero Configurations and the Corresponding Root Loci



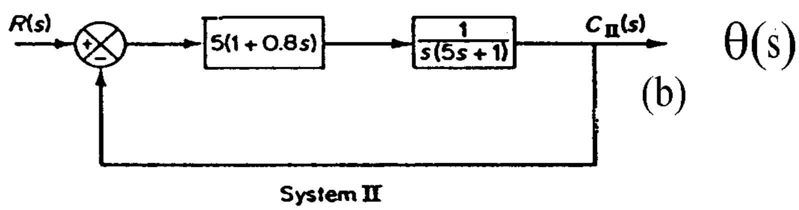
Effect of Derivative Control and Velocity Feedback

Consider the following three systems:

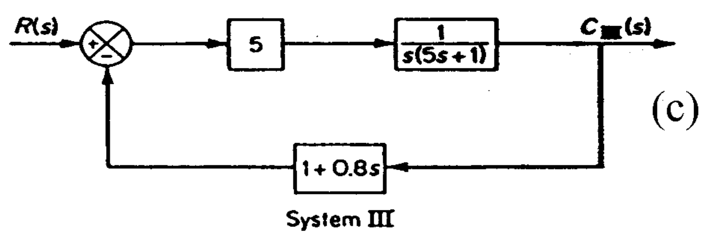
Positional servo.
Closed-loop poles:
$s = -0.1 \pm j \cdot 0.995$



Positional servo with derivative control.
Closed-loop poles:
$s = -0.5 \pm j \cdot 0.866$



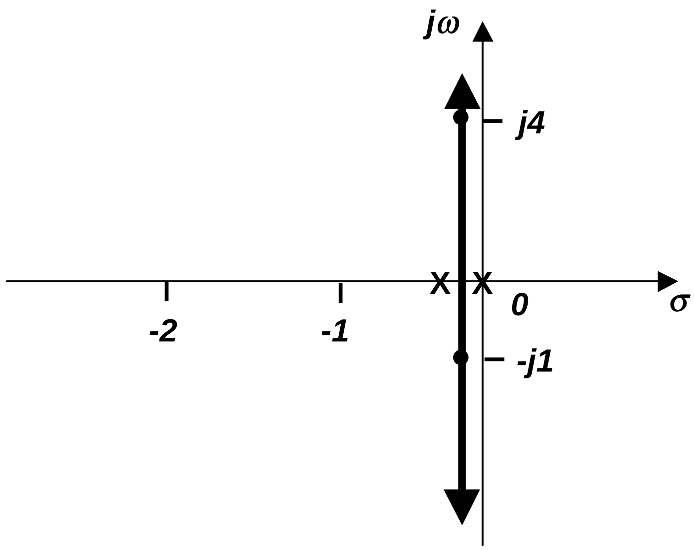
Positional servo with velocity feedback.
Closed-loop poles:
$s = -0.5 \pm j \cdot 0.866$



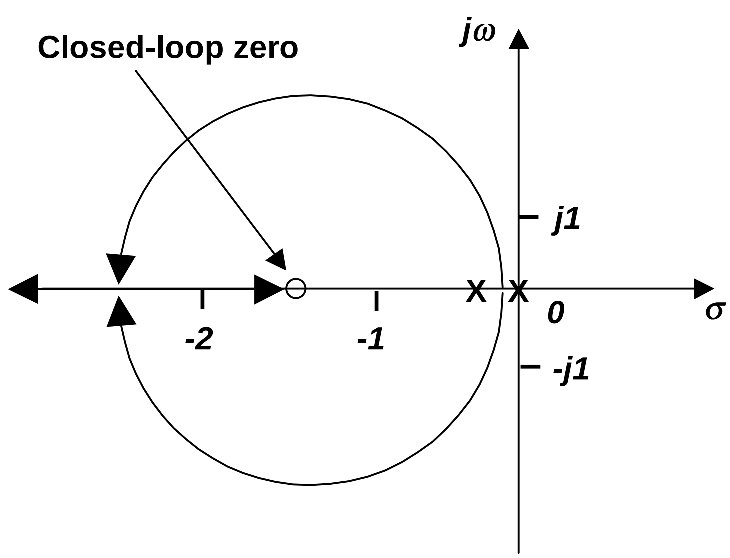
Open-loop of system I: $G_I(s) = \dfrac{5}{s \cdot (5 \cdot s + 1)}$

Open-loop of systems II and III: $G(s) = \dfrac{5 \cdot (1 + 0.8s)}{s \cdot (5 \cdot s + 1)}$

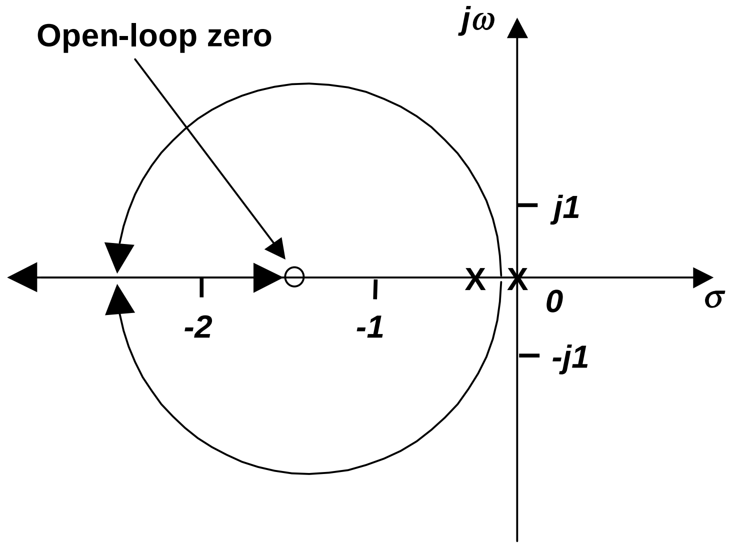
Root locus for the three systems



a) System I



b) System II



c) System III

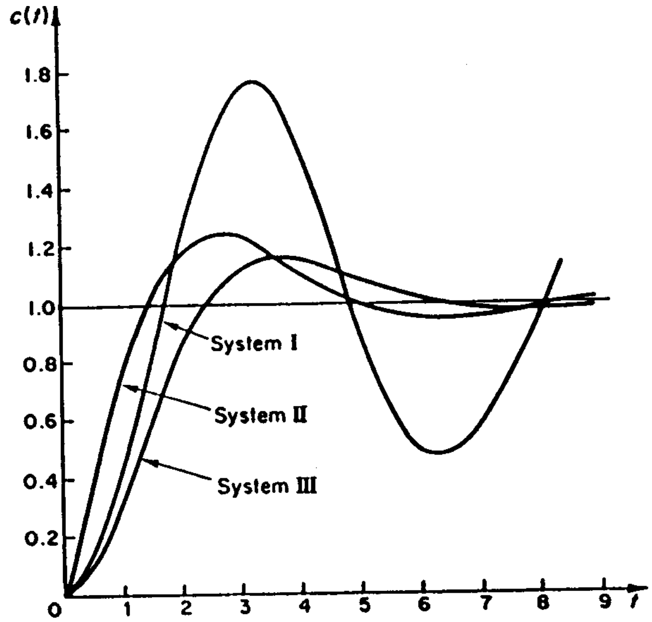
Closed-loop zeros:

System I: none
System II: 1+0.8s=0
System III: none

Observations:

- The root locus presents the *closed loop poles* but gives no information about *closed-loop zeros*.
- Two system with same root locus (same closed-loop poles) may have *different responses due to different closed-loop zeros*.

Unit-step response curves for systems I,II and III :



The closed-loop transfer function of System III is

$$\frac{C_{III}(s)}{R(s)} = \frac{1}{(s + 0.5 + j0.866)(s + 0.5 - j0.866)}$$

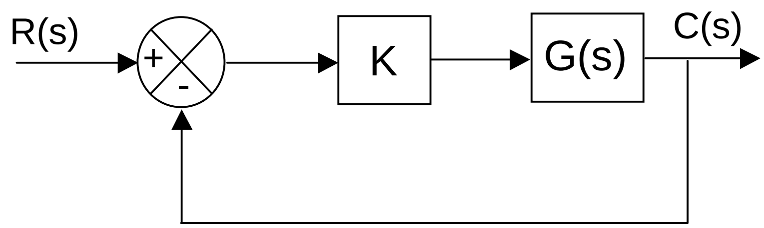
For a unit-impulse input,

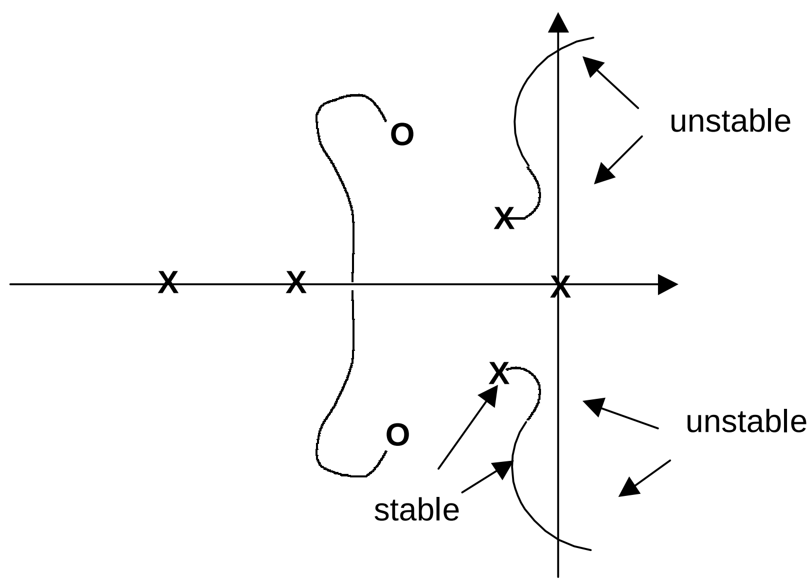
$$C_{III}(s) = \frac{j0.577}{s + 0.5 + j0.866} + \frac{-j0.577}{s + 0.5 - j0.866}$$

- The unit-step response of system II is the fastest of the three.
- This is due to the fact that derivative control responds to the rate of change of the error signal. Thus, it can produce a correction signal before the error becomes large. This leads to a faster response.

Conditionally Stable Systems

System which can be stable or unstable depending on the value of gain K.

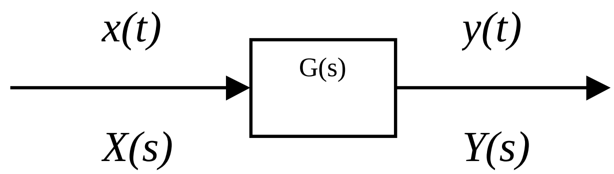




Minimum Phase Systems

All poles and zeros are in the left half plane.

Frequency Response Methods



$$x(t) = X\sin(\omega t) \qquad X(s) = \frac{\omega X}{s^2+\omega^2}$$

$$Y(s) = G(s)\cdot X(s) = G(s)\cdot\frac{\omega X}{s^2+\omega^2} = \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega} + \sum_i \frac{b_i}{s+s_i}$$

$$y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t} + \sum_i b_i e^{-s_i t}$$

stable system $\Leftrightarrow$ $Re(-s_i) < 0$ for all i

for $t \to \infty \quad \Rightarrow \quad y(t) = ae^{-j\omega t} + \bar{a}e^{j\omega t}$

$$a = G(s)\cdot\frac{\omega X}{s^2+\omega^2}\cdot(s+j\omega)\Big|_{(s=-j\omega)} = -\frac{XG(-j\omega)}{2j}$$

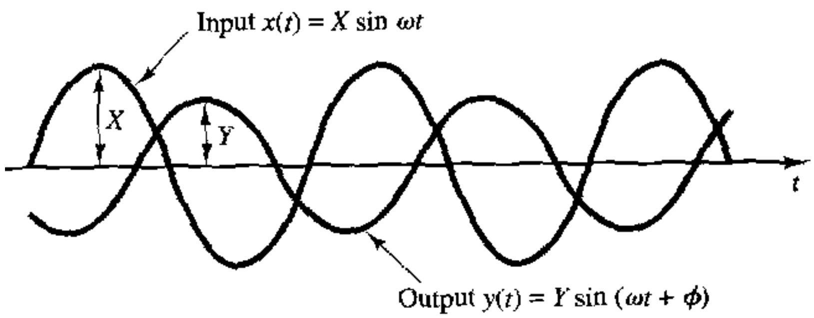
$$\bar{a} = G(s)\cdot\frac{\omega X}{s^2+\omega^2}(s-j\omega)\Big|_{(s=j\omega)} = \frac{XG(j\omega)}{2j}$$

$$G(j\omega) = |G(j\omega)| \cdot e^{j\varphi} \qquad \varphi = \tan^{-1}\left(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))}\right)$$

and

$$G(-j\omega) = |G(j\omega)| \cdot e^{-j\varphi}$$

$$y(t) = X \cdot |G(j\omega)| \cdot \frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j} = Y \cdot \sin(\omega t + \varphi$$



$\varphi > 0$ phase lag

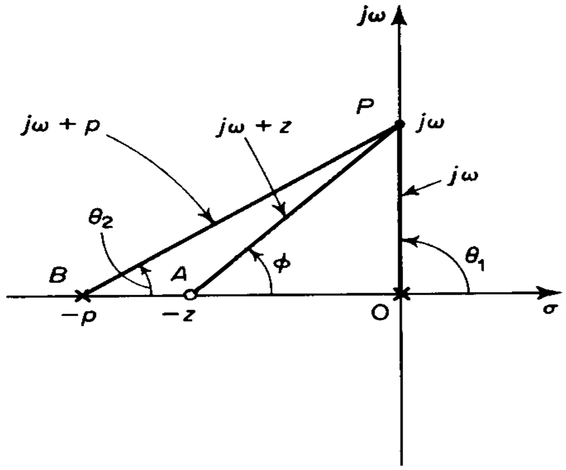
$\varphi < 0$ phase lead

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$|G(j\omega)| = \left|\frac{Y(j\omega)}{X(j\omega)}\right|$$ Magnitude response

$$\varphi = \angle(G(j\omega)) = \angle\left(\frac{Y(j\omega)}{X(j\omega)}\right)$$ Phase response

Connection between pole locations and Frequency Response



Determination of the frequency response in the complex plane.

$$G(s) = \frac{K(s+z)}{s(s+p)}$$

$$|G(j\omega)| = \frac{|K| \cdot |j\omega + z|}{|j\omega| \cdot |j\omega + p|}$$

$$\angle G(j\omega) = \varphi - \theta_1 - \theta_2$$

Frequency Response Plots

- Bode Diagrams
- Polar Plots (Nyquist Plots)
- Log-Magnitude-Versus-Phase Plots (Nichols Plots)

Bode Diagrams

- Magnitude response $|G(j\omega)|$

  $20\log|G(j\omega)|$ in dB

- Phase response $\angle G(j\omega)$ in degrees

Basic factors of G(jω):

- Gain K
- Integral or derivative factors $(j\omega)^{\pm 1}$
- First-order factors $(1+j\omega T)^{\pm 1}$
- Quadratic factors $\left(1+2j\frac{j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2\right)^{\pm 1}$

1. Gain Factor K

Horizontal straight line at magnitude $20\log(K)\,dB$

Phase is zero

2. Integral or derivative factors $(j\omega)^{\pm 1}$

- $(j\omega)^{-1}$

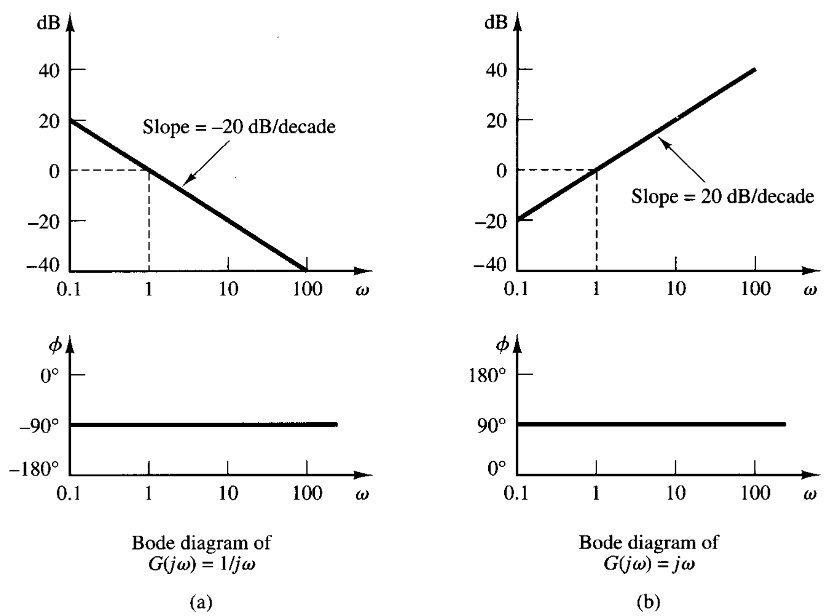
$$20\log\left|\frac{1}{j\omega}\right| = -20\log\omega$$

magnitude: straight line with slope –20 dB/decade
phase: -90^{o}

- $(j\omega)$

$$20\log|j\omega| = 20\log\omega$$

magnitude: straight line with slope 20 dB/decade
phase: $+90^{o}$



(a) Bode diagram of $G(j\omega) = 1/j\omega$; (b) Bode diagram of $G(j\omega) = j\omega$.

3. First order factors $(1+j\omega T)^{\pm 1}$

- $(1+j\omega T)^{-1}$

Magnitude:

$$20\log\left|\frac{1}{1+j\omega T}\right| = -20\log\sqrt{1+\omega^2 T^2}\ dB$$

for $\omega << T^{-1} \Rightarrow 0\ dB$ magnitude

for $\omega >> T^{-1} \Rightarrow -20\log(\omega T)\ dB$ magnitude

Approximation of the magnitude:

for ω between 0 and $\omega = \frac{1}{T}$ 0 dB

for $\omega >> \frac{1}{T}$ straight line with slope –20 dB /decade

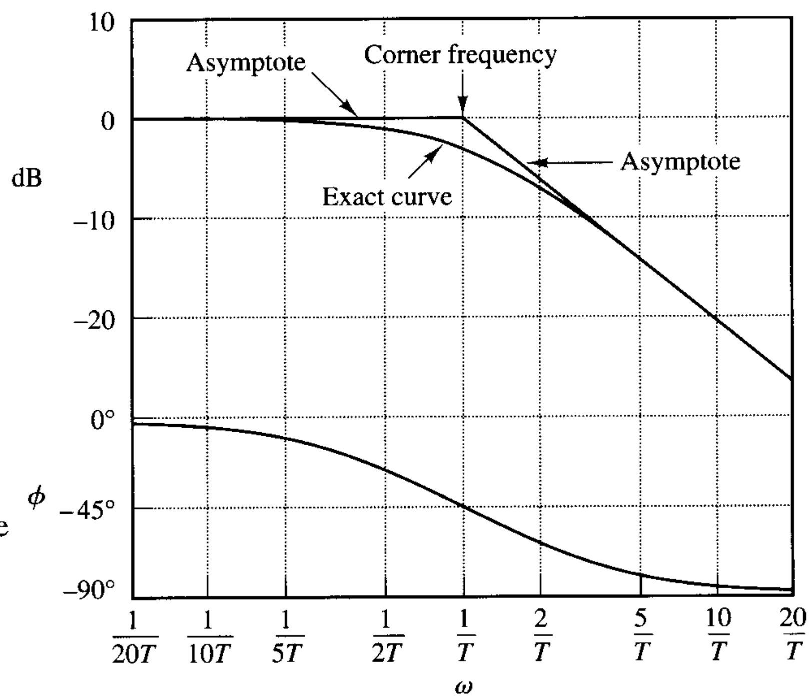
Phase:

$$\angle(1+j\omega T)^{-1} = -tan^{-1}(\omega T)$$

for $\omega = 0$ $\varphi = 0^\circ$

for $\omega = \frac{1}{T} \Rightarrow -\tan^{-1}\left(\frac{T}{T}\right) = 1$ $\varphi = -45^\circ$

for $\omega = \infty$ $\varphi = -90^\circ$

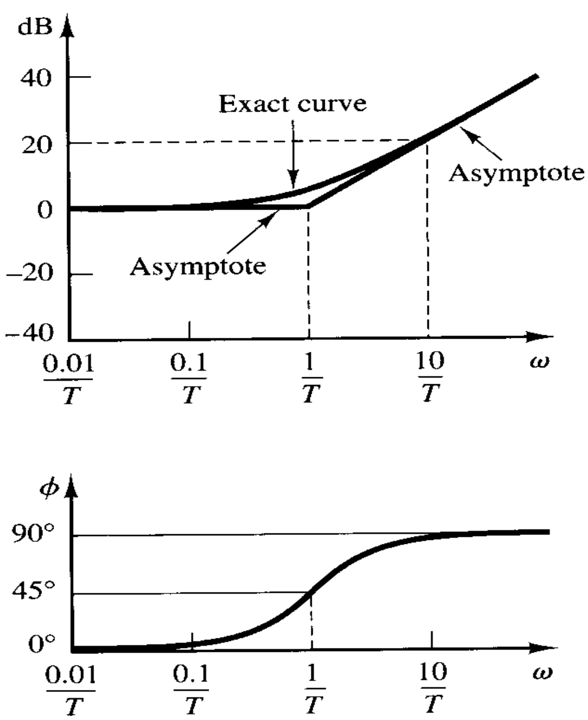


Log-magnitude curve together with the asymptotes and phase angle curve of $1/(1 + j\omega T)$.

- $(1+ j\omega T)^{+1}$

Using $$20\log|1+ j\omega T| = -20\log\left|\frac{1}{1+ j\omega T}\right|$$

$$\angle(1+ j\omega T) = \tan^{-1}(\omega T) = -\angle\left(\frac{1}{1+ j\omega T}\right)$$



Log-magnitude curve together with the asymptotes and phase-angle curve for $1 + j\omega T$.

4. Quadratic Factors

$$G(j\omega)=\frac{1}{1+2\varsigma\, j\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2} \qquad 0<\varsigma<1$$

Magnitude:

$$20\log\left|G(j\omega)\right| = -20\log\sqrt{\left(1-\frac{\omega^2}{\omega^2{}_n}\right)^2-\left(2\,j\left(\frac{\omega}{\omega_n}\right)\right)^2}$$

for

$$\omega \ll \omega_n \quad \Rightarrow \quad 0 \;\; dB$$

for

$$\omega \gg \omega_n \quad \Rightarrow \quad -20\log\left(\frac{\omega^2}{\omega^2{}_n}\right) = -40\log\left(\frac{\omega}{\omega_n}\right) \;\; dB$$

Phase:

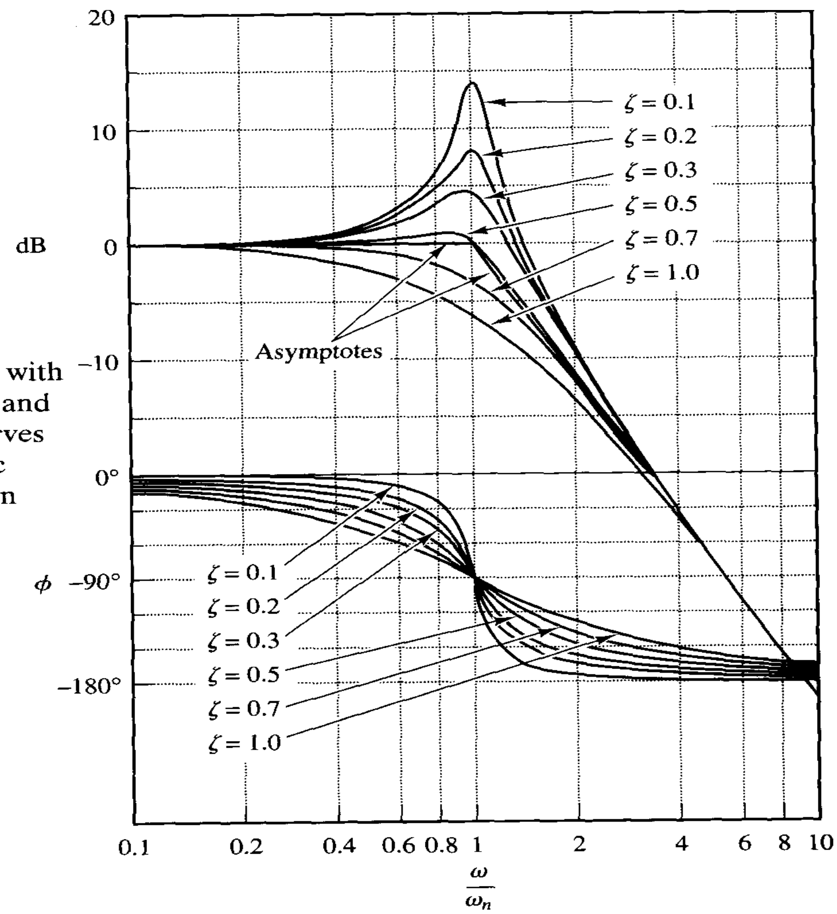
$$\varphi=\tan^{-1}\angle G(j\omega)=-\tan^{-1}\left[\frac{2\varsigma\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]$$

Resonant Frequency:

$$\omega_T=\omega_n\sqrt{1-2\varsigma^2}$$

Resonant Peak Value:

$$M_T=\left|G(j\omega)\right|_{\max}=\frac{1}{2\xi\varsigma\sqrt{1-\varsigma^2}}$$

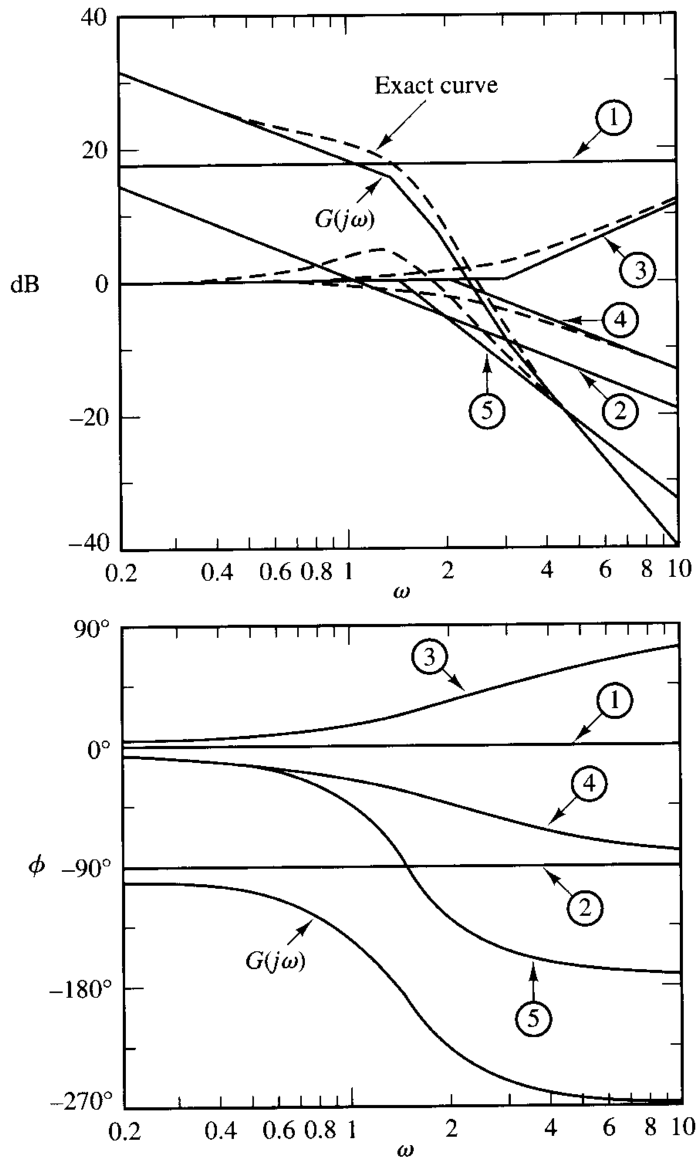


Log-magnitude curves together with the asymptotes and phase-angle curves of the quadratic transfer function

<u>Example:</u>

$$G(j\omega) = \frac{10(j\omega+3)}{(j\omega)\cdot(j\omega+2)\cdot\left((j\omega)^2+j\omega+2\right)}$$

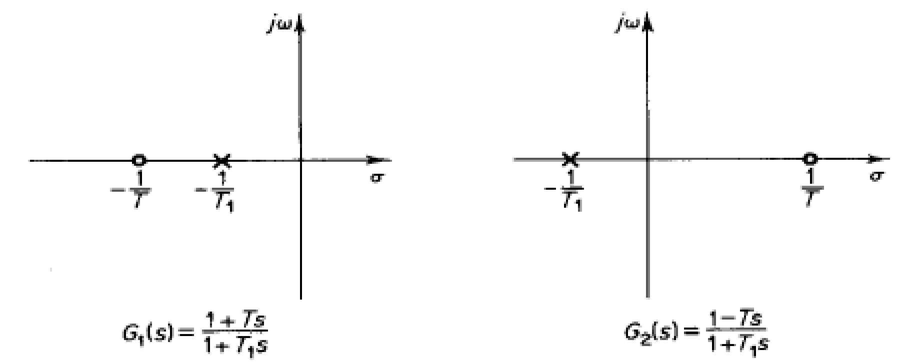
$$G(j\omega) = \frac{7.5\cdot\left(\frac{(j\omega)}{3}+1\right)}{j\omega\left(\frac{j\omega}{2}+1\right)\left(\frac{(j\omega)^2}{2}+\frac{j\omega}{2}+1\right)}$$



Bode diagram of the system considered in Example 8–1.

Frequency Response of non-Minimum Phase systems

Minimum phase systems have all poles and zeros in the left half s-plane and were discussed before



Pole-zero configurations of a minimum phase system $G_1(s)$ and nonminimum phase system $G_2(s)$.

Consider

$$A_1(s) = 1 + Ts \quad A_2(s) = 1 - Ts \quad A_3(s) = Ts - 1$$

then

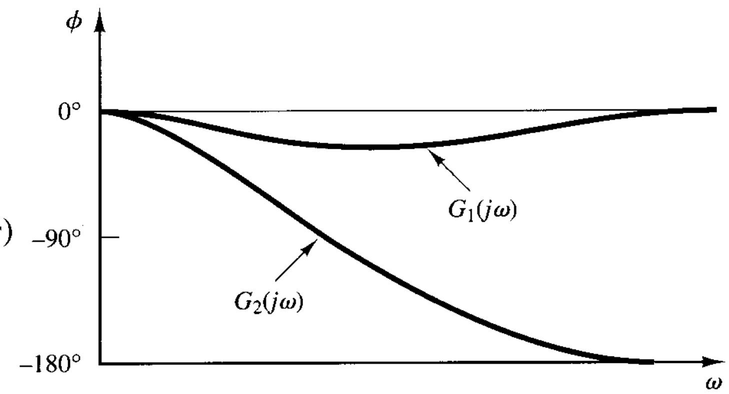
$$|A_1(j\omega)| = |A_2(j\omega)| = |A_3(j\omega)|$$

$$\angle A_2(j\omega) = -\angle A_1(j\omega)$$

$$\angle A_3(j\omega) = 180 - \angle A_1(j\omega)$$

We know that	phase of	$A_1(j\omega)$	from	0^o to $+90^o$
→	phase of	$A_2(j\omega)$	from	0^o to -90^o
	phase of	$A_3(j\omega)$	from	180^o to $+90^o$

Phase-angle characteristics of the systems $G_1(s)$ and $G_2(s)$



Phase-angle characteristic of the two systems $G_1(s)$ and $G_2(s)$ having the same magnitude response but $G_1(s)$ is minimum phase while $G_2(s)$ is not.

Frequency Response of Unstable Systems

Consider

$$G_1(s) = \frac{1}{1+Ts} \qquad G_2(s) = \frac{1}{1-Ts} \qquad G_3(s) = \frac{1}{Ts-1}$$

then

$$|G_1(j\omega)| = |G_2(j\omega)| = |G_3(j\omega)|$$

$$\angle G_2(j\omega) = -\angle G_1(j\omega)$$

$$\angle G_3(j\omega) = -180^o - \angle G_1(j\omega)$$

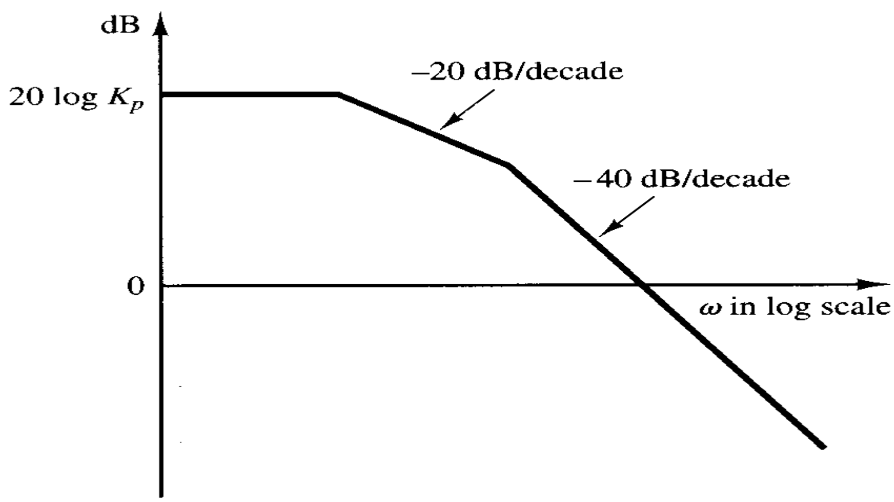
We know that	phase of	$G_1(j\omega)$	from	0^o to -90^o
→	phase of	$G_2(j\omega)$	from	0^o to $+90^o$
	phase of	$G_3(j\omega)$	from	-180^o to -90^o

Relationship between System Type and Log-Magintude curve

Type of system determines:
- the slope of the log-magnitude curve at low frequencies
- for minimum phase, also the phase at low frequencies

Type 0

Position Error Coefficient $K_p \neq 0$



- Slope at low frequencies: 0 db/decade
- Phase at low frequencies (minimum phase): 0^o

$$\lim_{\omega \to 0} G(j\omega) = K_p$$

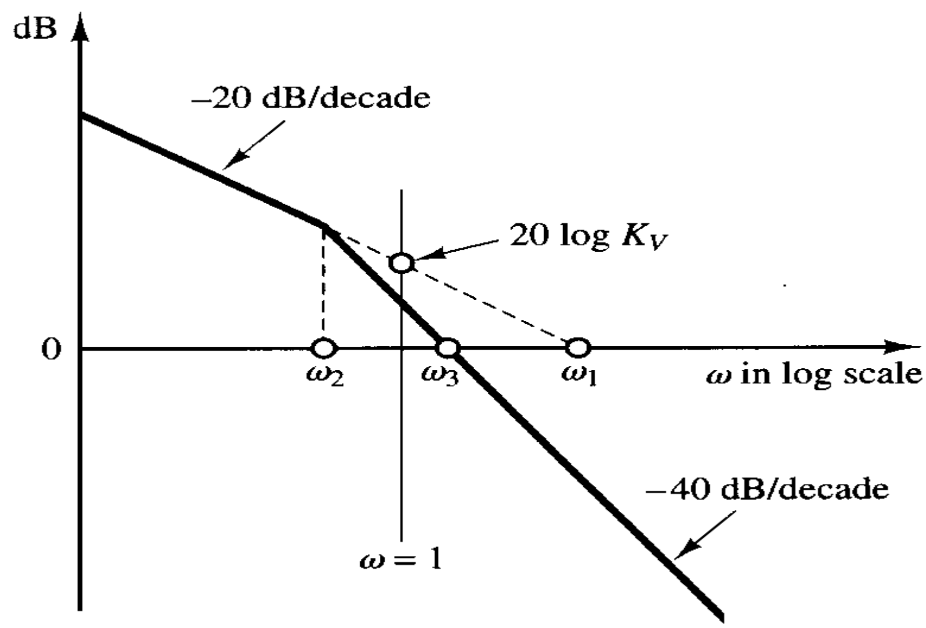
Type 1

Velocity Error Coefficient $K_v \neq 0 \quad (K_p = \infty)$

$$K_v = \lim_{\omega \to 0} j\omega G(j\omega)$$

$$G(j\omega) = \frac{K_v}{j\omega} \quad \text{for} \quad \omega << 1$$

$$20 \log K_v = 20 \log \left| \frac{K_v}{j\omega} \right| \quad \text{for} \quad \omega = 1$$



- Slope at low frequencies: -20 db/decade
- Phase at low frequencies (minimum phase): -90^o

Type 2

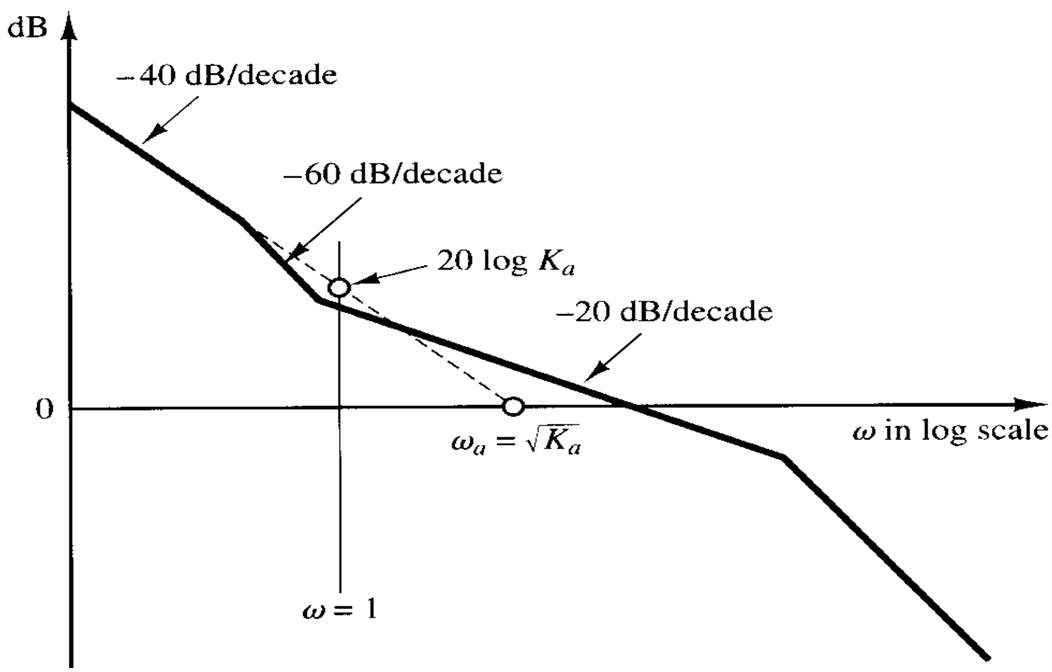
Acceleration Error Coefficient K_a

$$K_a = \lim_{\omega \to 0} (j\omega)^2 G(j\omega) \neq 0$$

$$(K_p = K_v = \infty)$$

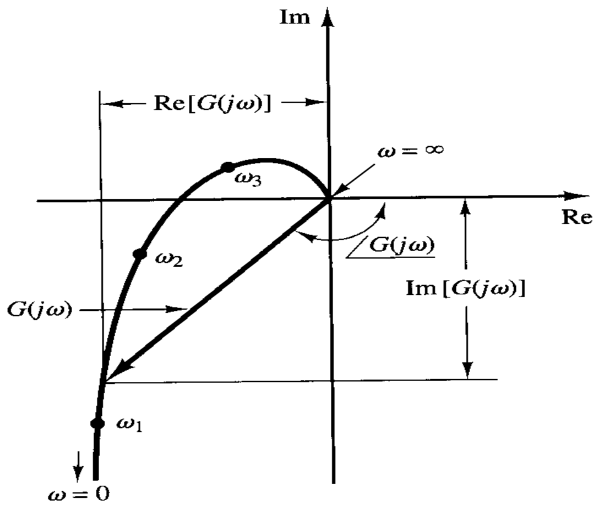
$$G(j\omega) = \frac{K_a}{(j\omega)^2} \quad \text{for} \quad \omega \ll 1$$

$$20 \log K_a = 20 \log \left| \frac{K_a}{(j\omega)^2} \right| \quad \text{for} \quad \omega = 1$$



- Slope at low frequencies: -40 db/decade
- Phase at low frequencies (minimum phase): -180°

Polar Plots (Nyquist Plots)



$$G(j\omega) = |G(j\omega)| * \angle G(j\omega)$$

$$= \mathrm{Re}[G(j\omega)] + \mathrm{Im}[G(j\omega)]$$

Advantage over Bode plots: only one plot

Disadvantage : Polar plot of $G(j\omega)=G_1(j\omega)\cdot G_2(j\omega)$ is more difficult to construct than its Bode plot.

Basic factors of G(jω):

Integral or derivative factors $(j\omega)^{\pm 1}$

$$G(j\omega) = \frac{1}{j\omega} = -j\frac{1}{\omega} = \frac{1}{\omega}\angle -90^\circ$$

$$G(j\omega) = j\omega = \omega \angle 90^\circ$$

First order factors $(1+j\omega T)^{\pm 1}$

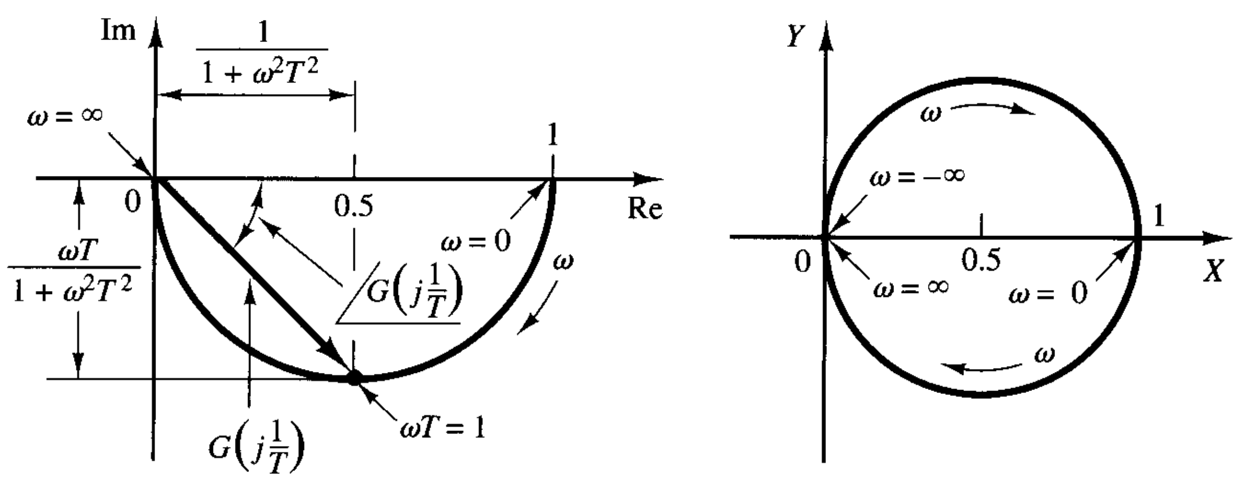
$$G(j\omega) = \frac{1}{1+j\omega T} = X + jY$$

$$X = \frac{1}{1+\omega^2 T^2}, \quad Y = \frac{-\omega T}{1+\omega^2 T^2}$$

It can be show that $(X-0.5)^2 + Y^2 = (0.5)^2$

$\Rightarrow$ Polar plot is a circle with

- Center (1/2, 0) and
- Radius 0.5.



(a) (b)

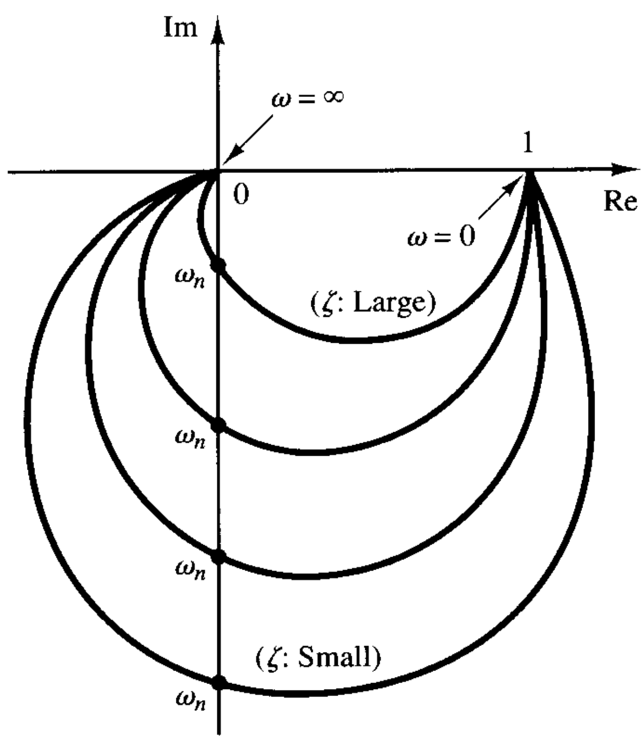
Quadratic Factors

$$G(j\omega)=\frac{1}{1+2\zeta\, j\left(\dfrac{\omega}{\omega_n}\right)+\left(\dfrac{j\omega}{\omega_n}\right)^2} \qquad 1>\zeta>0$$

$$\lim_{\omega\to 0} G(j\omega) = 1\angle 0^\circ$$

$$\lim_{\omega\to\infty} G(j\omega) = 0\angle -180^\circ$$

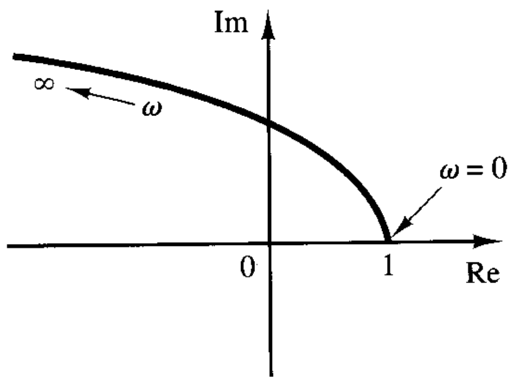
$$G(j\omega_n)=\frac{1}{j2\varsigma}\angle -90^\circ$$



$$\left(1+2\varsigma\left(\frac{j\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right)^{+1}$$

$$\lim_{\omega\to 0} G(j\omega) = 1\angle 0^\circ$$

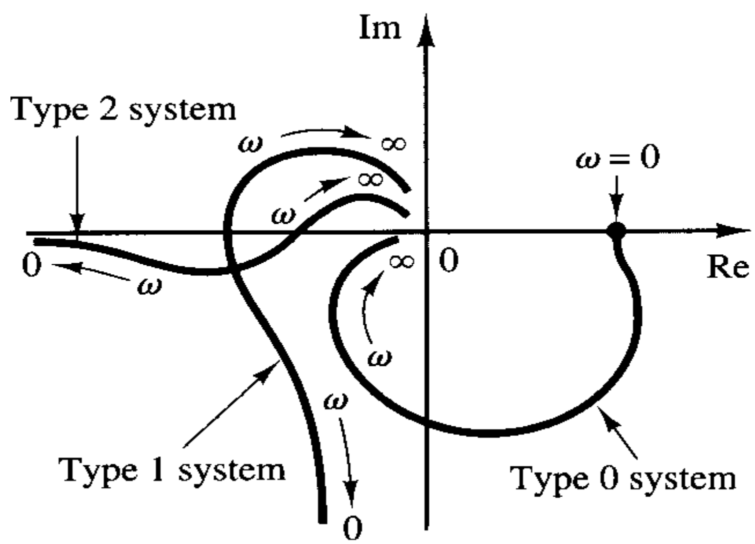
$$\lim_{\omega\to\infty} G(j\omega) = \infty\angle 180^\circ$$



General shapes of polar plots

$$G(j\omega)=\frac{K(1+j\omega\overline{T}_1)\ldots(1+j\omega\overline{T}_m)}{(j\omega)^{\lambda}(1+j\omega T_{\lambda+1})\ldots(1+j\omega T_n)}$$

n = order of the system (denominator)
λ = type of system $\qquad \lambda > 0$
m = order of numerator $\qquad n > m$

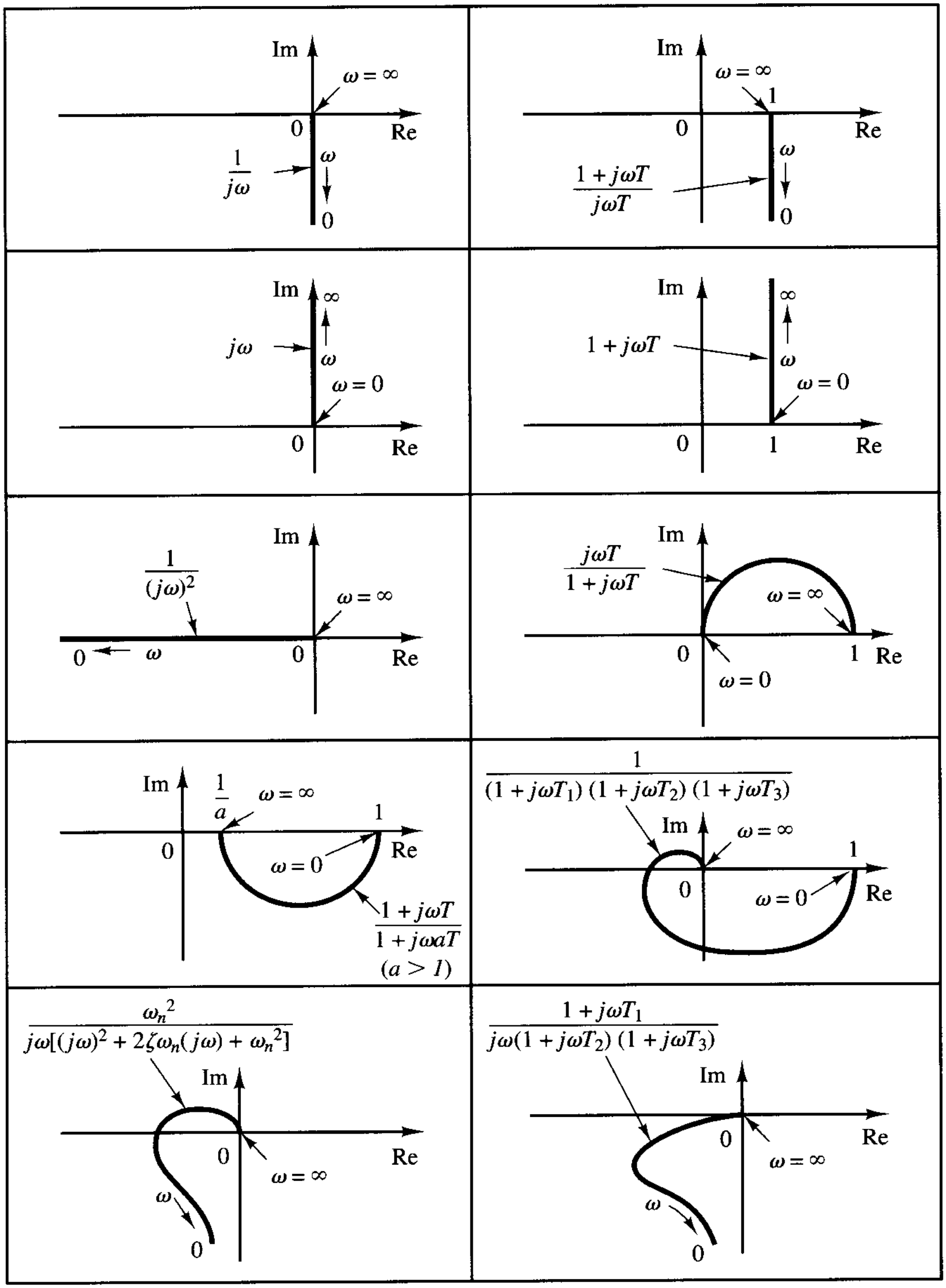


For low frequencies: The phase at $\omega \to 0$ is $\lambda\left(-90^\circ\right)$

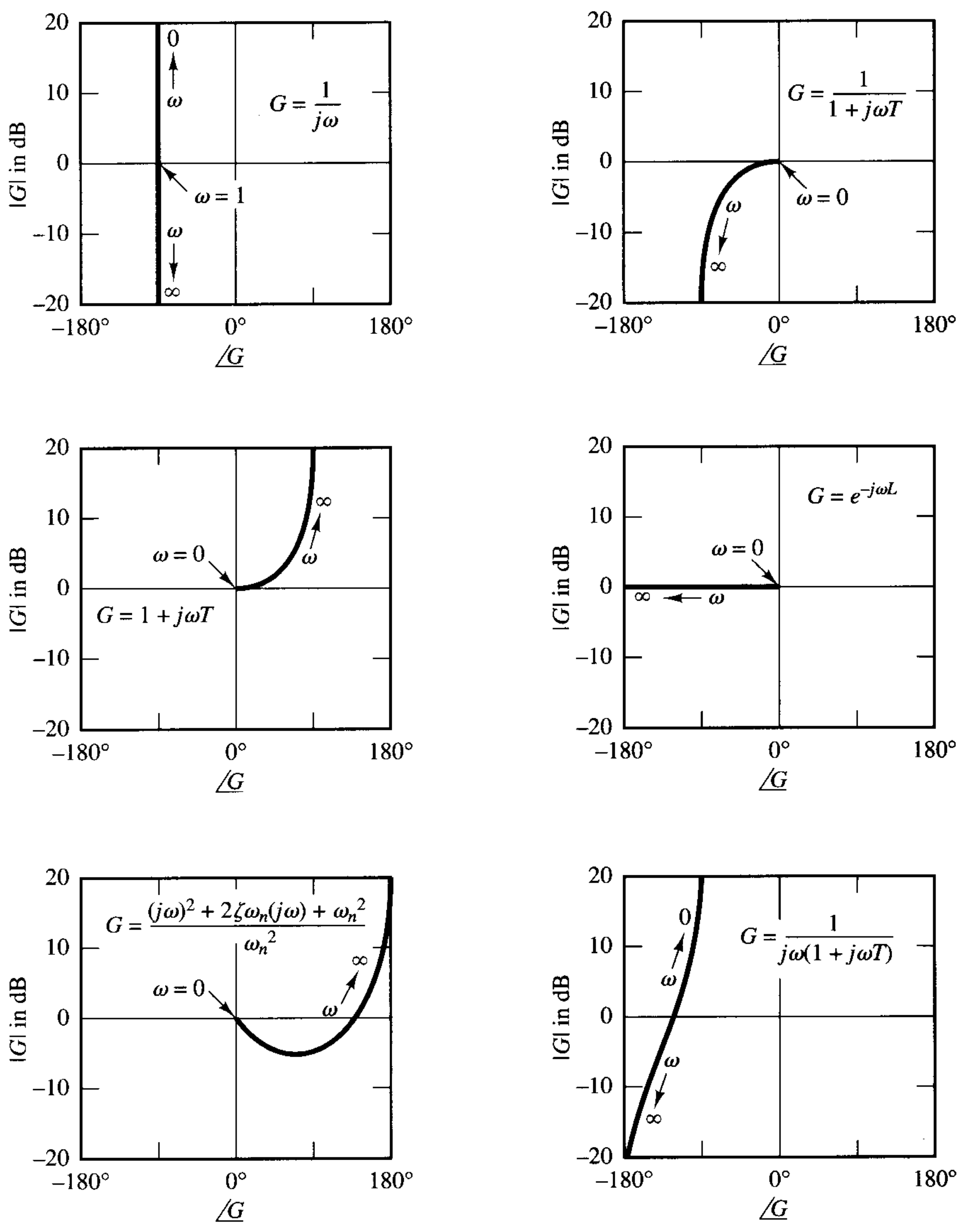
For system type 1, the low frequency asymptote is obtained by taking:

$$\text{Re}\left[G(j\omega)\right] \text{ for } \omega \to 0$$

For high frequencies: The phase is: (n - m) (-90^{o})


Im
Re
0
ω = ∞
1/jω
ω
0
Im
Re
0
1
ω = ∞
(1 + jωT)/jωT
ω
0
Im
Re
0
∞
jω
ω
ω = 0
Im
Re
0
1
∞
1 + jωT
ω
ω = 0
Im
Re
0
1/(jω)²
ω = ∞
0
ω
Im
Re
0
1
jωT/(1 + jωT)
ω = ∞
ω = 0
Im
Re
0
1
1/a
ω = ∞
ω = 0
(1 + jωT)/(1 + jωaT)
(a > 1)
1/((1 + jωT1) (1 + jωT2) (1 + jωT3))
Im
Re
0
1
ω = ∞
ω = 0
ωn²/(jω[(jω)² + 2ζωn(jω) + ωn²])
Im
Re
0
ω = ∞
ω
0
(1 + jωT1)/(jω(1 + jωT2) (1 + jωT3))
Im
Re
0
ω = ∞
ω
0

Log-Magnitude-Versus-Phase Plots (Nichols Plots)

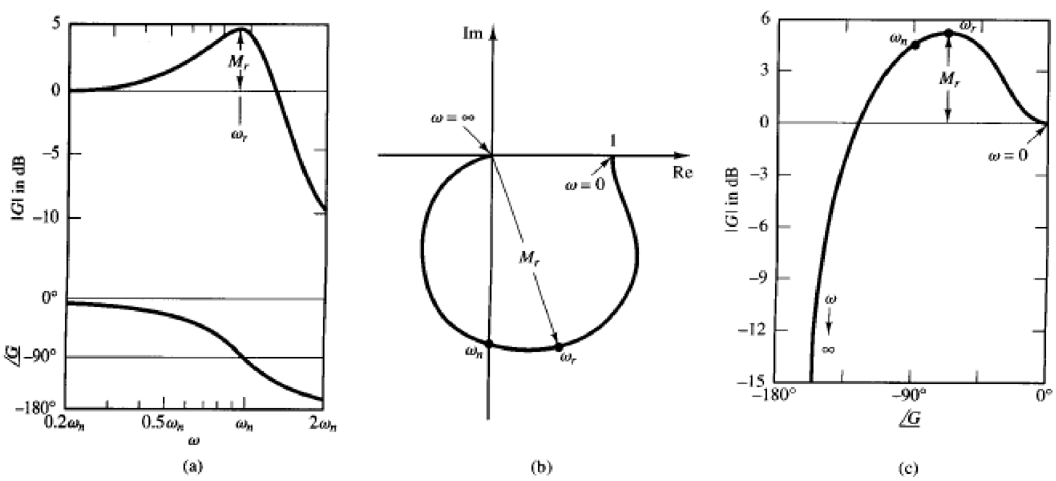


Example:

Frequency Response of a quadratic factor

The same information presented in three different ways:

- Bode Diagram
- Polar Plot
- Log-Magnitude-Versus-Phase Plots

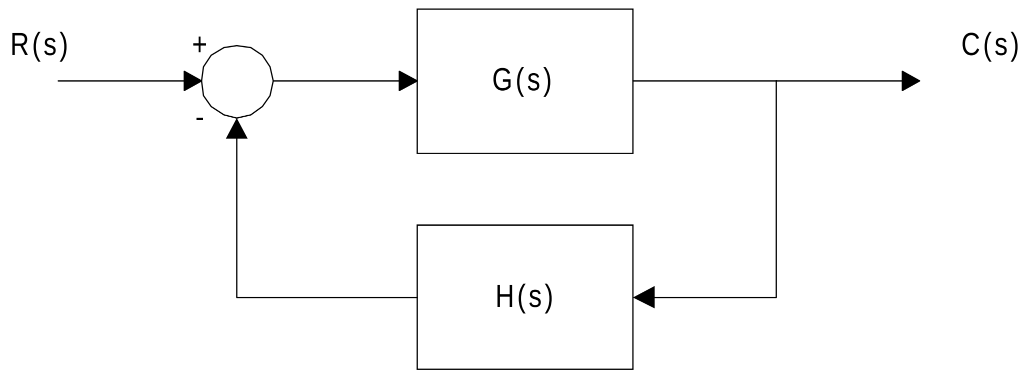


Three representations of the frequency response of $\dfrac{1}{1 + 2\zeta\left(j\dfrac{\omega}{\omega_n}\right) + \left(j\dfrac{\omega}{\omega_n}\right)^2}$, for $\zeta > 0$.

(a) Bode diagram; (b) polar plot; (c) log-magnitude versus phase plot.

Nyquist Stability Criterion

The Nyquist stability criterion relates the stability of the ***closed loop*** system to the frequency response of the ***open-loop*** system.



Open-loop: $G(s) \cdot H(s)$

Closed-loop: $\frac{G(s)}{1 + G(s) \cdot H(s)}$

Advantages of the Nyquist Stability Criterion:

- Simple graphical procedure to determine whether a system is stable or not
- The degree of stability can be easily obtained
- Easy for compensator design
- The response for steady-state sinusoidal inputs can be easily obtained from measurements

Preview

Mathematical Background

- Mapping theorem
- Nyquist path

Nyquist stability criterion

Z=N+P

Z: Number of zeros of (1+H(s)G(s)) in the right half plane = number of unstable poles of the closed-loop system

N: Number of clockwise encirclements of the point –1+j0

P: Number of poles of G(s)H(s) in the right half plane

Application of the Stability Criterion

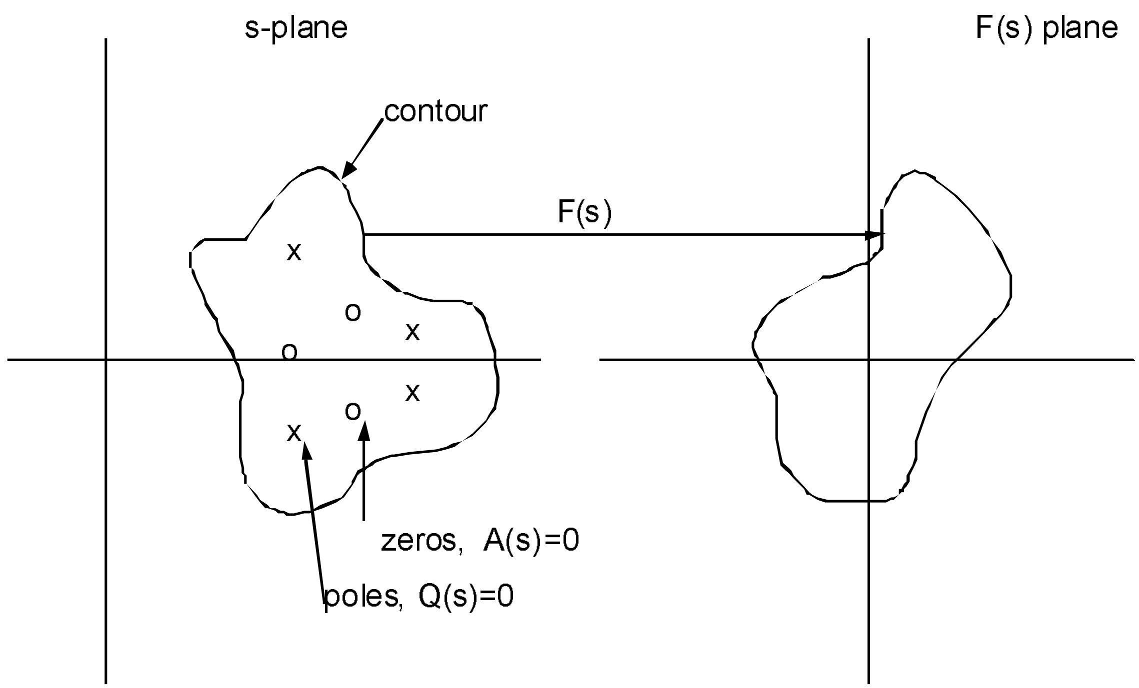
- Sketch the Nyquist plot for $\omega \in (0^+,\infty)$
- Extend to $\omega \in (-\infty,+\infty)$
- Apply the stability criterion (find N and P and compute Z).

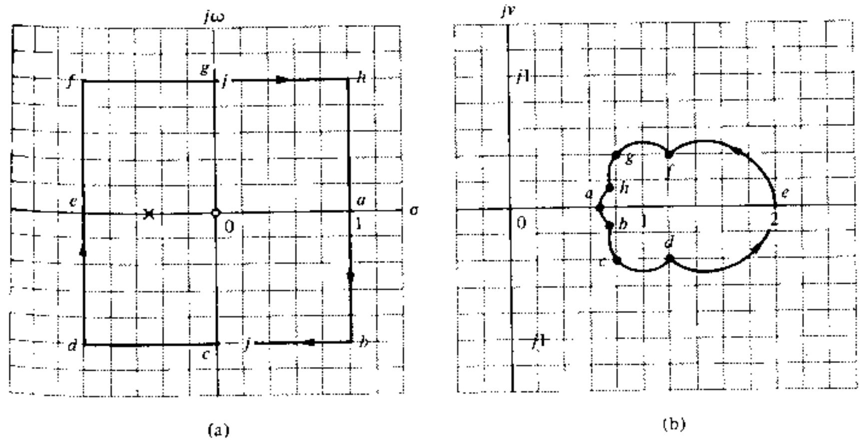
Mapping Theorem

The total number N of clockwise encirclements of the origin of the F(s) plane, as a representative point s traces out the entire contour in the clockwise direction, is equal to Z – P.

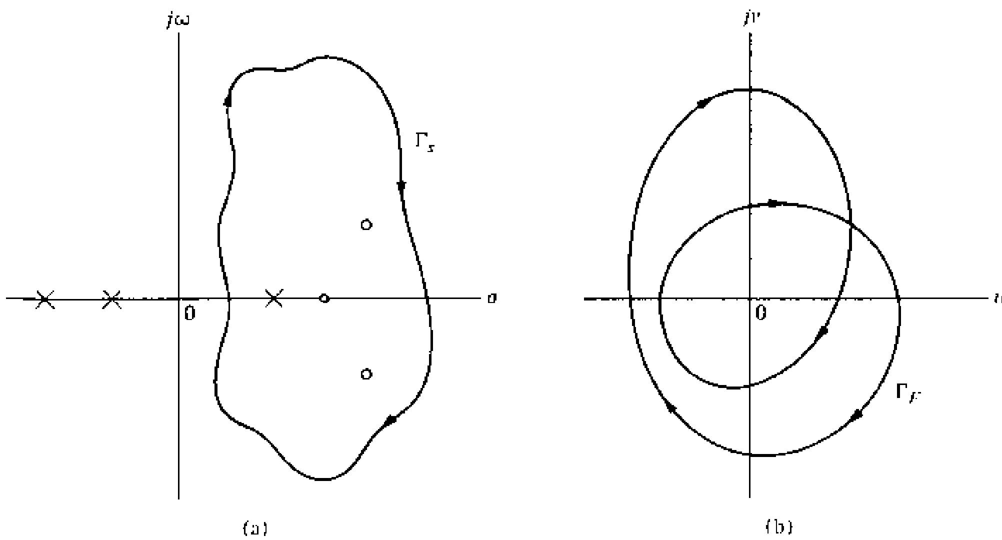
$$F(s) = \frac{A(s)}{Q(s)}$$

P: Number of poles, Q(s) = 0
Z: Number of zeros, A(s) = 0

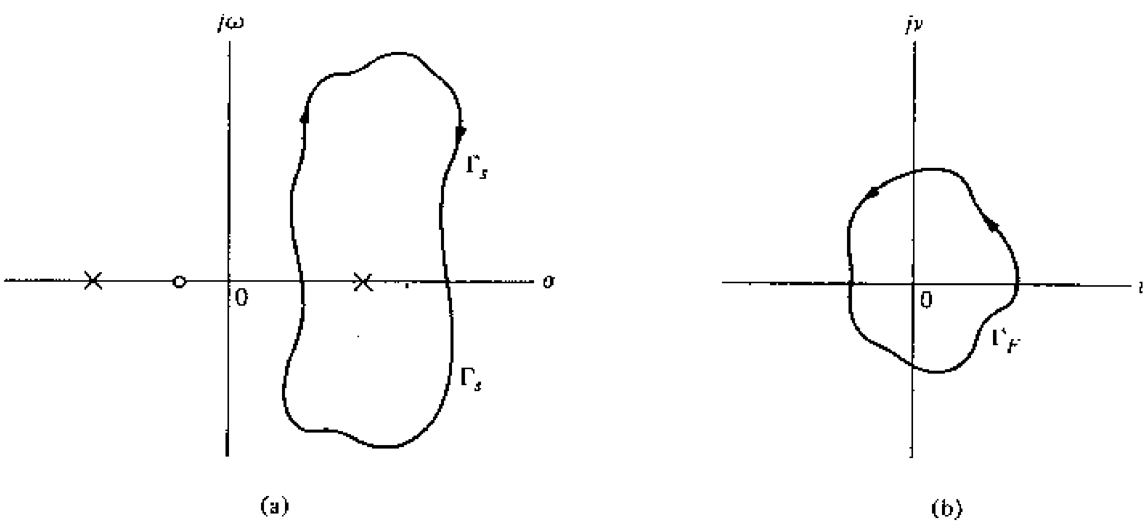




Mapping for F(s) = s/(s+0.5), (Z = P =1)

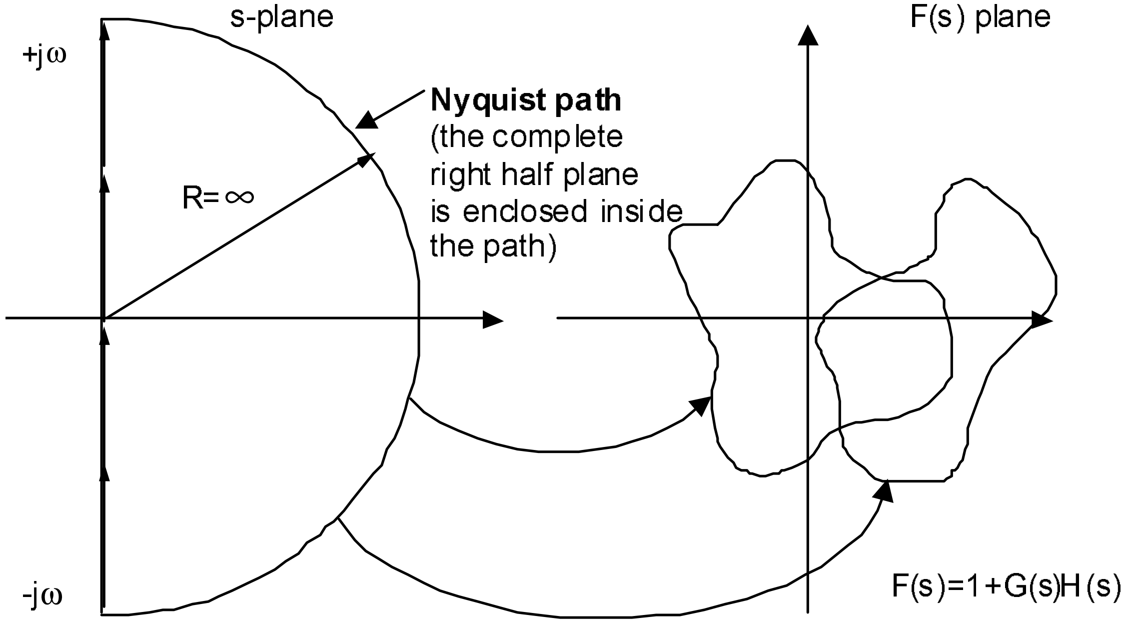


Example of Mapping theorem (Z – P = 2).



Example of Mapping theorem (Z – P = - 1).

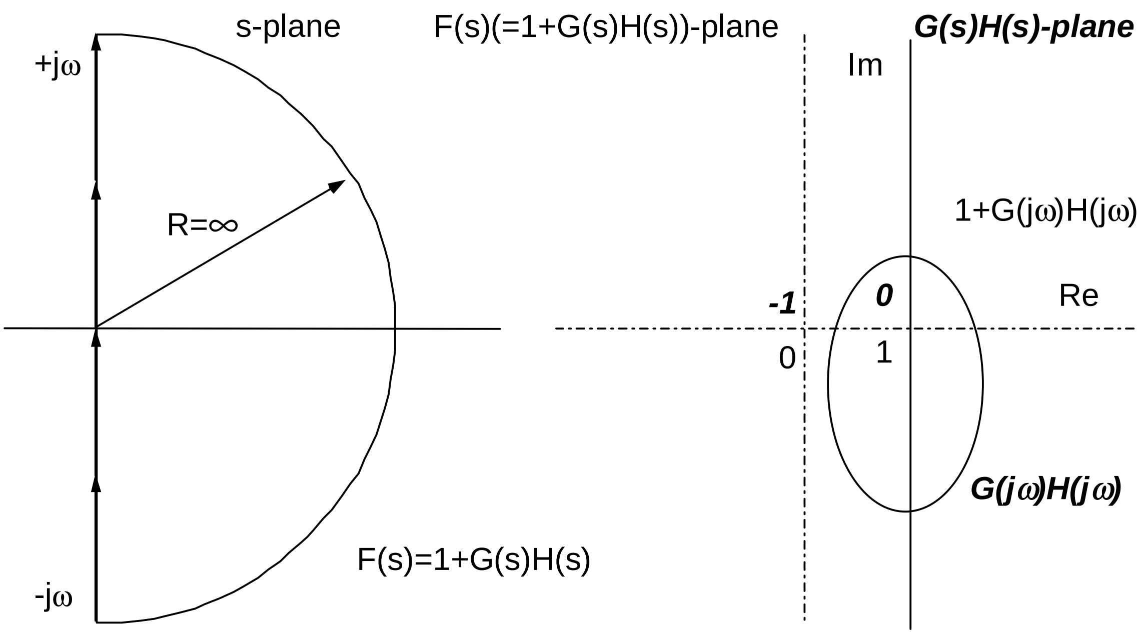
Application of the mapping theorem to stability analysis

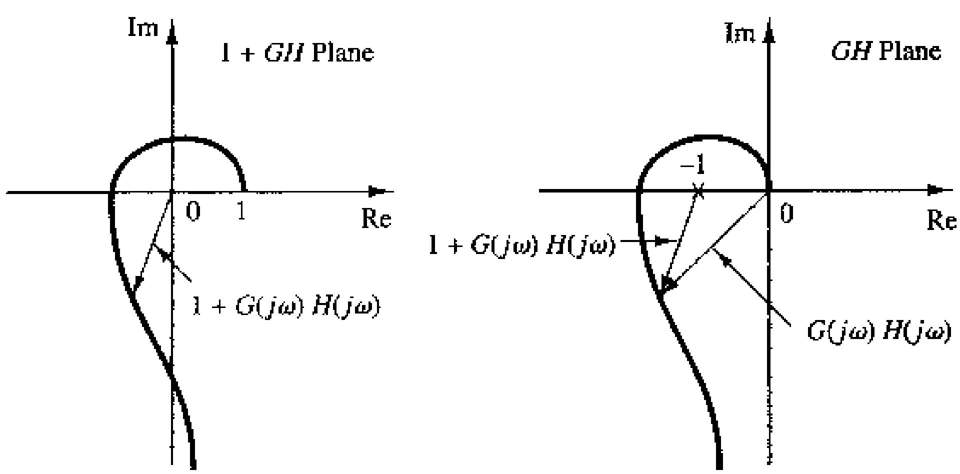


Mapping theorem: The number of clockwise encirclements of the origin is equal to the difference between the zeros and poles of F(s) = 1+ G(s)H(s).

Zeros of F(s) = poles of closed-loop system
Poles of F(s) = poles of open-loop system

Frequency response of open-loop system: $G(j\omega)H(j\omega)$

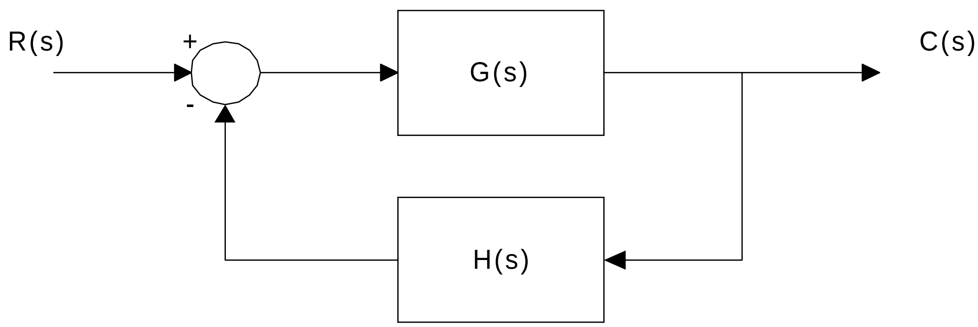




Frequency response of a type 1 system

Nyquist stability criterion

Consider



The Nyquist stability criterion states that:

$$Z = N + P$$

Z: Number of zeros of 1+H(s)G(s) in the right half s-plane = number of poles of closed-loop system in right half s-plane.

N: Number of clockwise encirclements of the point –1+j0 (when tracing from $\omega = -\infty$ to $\omega = +\infty$).

P: Number of poles of G(s)H(s) in the right half s-plane

Thus: if $Z = 0$ $\rightarrow$ closed-loop system is stable

if $Z > 0$ $\rightarrow$ closed-loop system has Z unstable poles

if $Z < 0$ $\rightarrow$ impossible, a mistake has been made

Alternative form for the Nyquist stability criterion:

If the open-loops system G(s)H(s) has k poles in the right half s-plane, then the closed-loop system is stable if and only if the G(s)H(s) locus for a representative point s tracing the modified Nyquist path, encircles the –1+j0 point k times in the counterclockwise direction.

Frequency Response of $G(j\omega)H(j\omega)$ for $\omega : (-\infty,+\infty)$

a) $\omega : (0^+,+\infty)$: using the rules discussed earlier

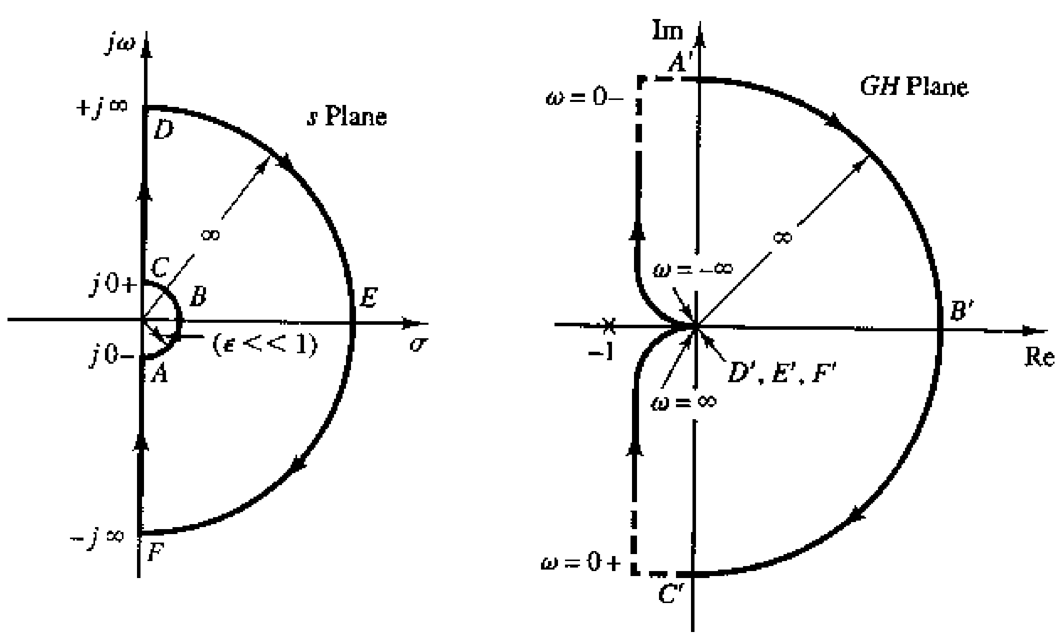
b) $\omega : (0^-,-\infty)$: $G(-j\omega)H(-j\omega)$ is symmetric with $G(j\omega)H(j\omega)$ (real axis is symmetry axis)

c) $\omega : (0^-,0^+)$: next page

Poles at the origin for G(s)H(s):

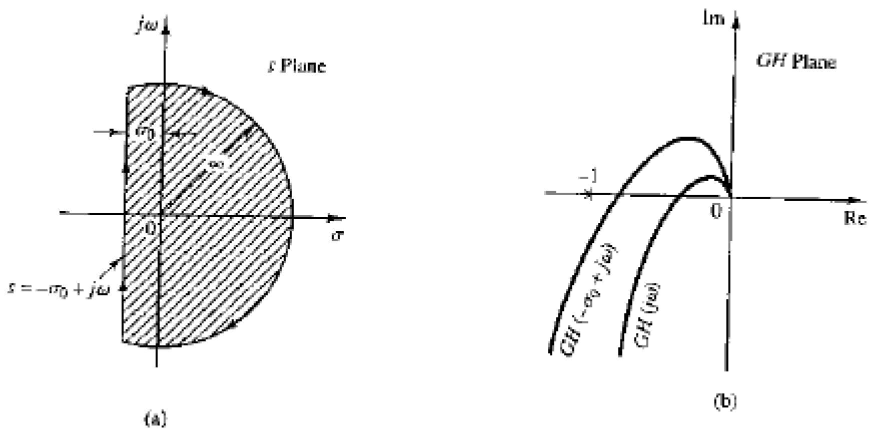
$$G(s)\cdot H(s) = \frac{(\ldots)}{s^{\lambda}(\ldots)}$$

If G(s)H(s) involves a factor $\frac{1}{s^{\lambda}}$, then the plot of G(jω)H(jω), for ω between 0^- and 0^+, <u>has λ clockwise semicircles of infinite radius about the origin</u> in the GH plane. These semicircles correspond to a representative point s moving along the Nyquist path with a semicircle of radius ε around the origin in the s plane.



Relative Stability

Consider a modified Nyquist path which ensures that the closed-loop system has no poles with real part larger than $-\sigma_0$:



Another possible modified Nyquist path:

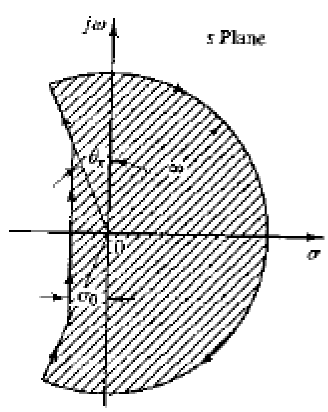


Figure 8–69
Modified Nyquist path.

Phase and Gain Margins

A measure for relative stability of the closed-loop system is how close G(jω), the frequency response of the open-loop system, comes to –1+j0 point. This is represented by phase and gain margins.

Phase margin: The amount of additional phase lag at the Gain Crossover Frequency ω_o required to bring the system to the verge of instability.

Gain Crossover Frequency: ω_o for which $\left|G\left(j\omega_o\right)\right|=1$

Phase margin: $\gamma = 180^\circ + \angle G(j\omega_o) = 180^\circ + \phi$

Gain margin: The reciprocal of the magnitude $\left|G\left(j\omega_1\right)\right|$ at the Phase Crossover Frequency ω_1 required to bring the system to the verge of instability.

Phase Crossover Frequency: ω_1 where $\angle G\left(j\omega_1\right) = -180^\circ$

Gain margin: $$K_g = \frac{1}{\left|G(j\omega_1)\right|}$$

Gain margin in dB: $$K_g \ in \ dB = -20\log\left|G\left(j\omega_1\right)\right|$$

K_g in dB > 0 = stable (for minimum phase systems)
K_g in dB < 0 = unstable (for minimum phase systems)

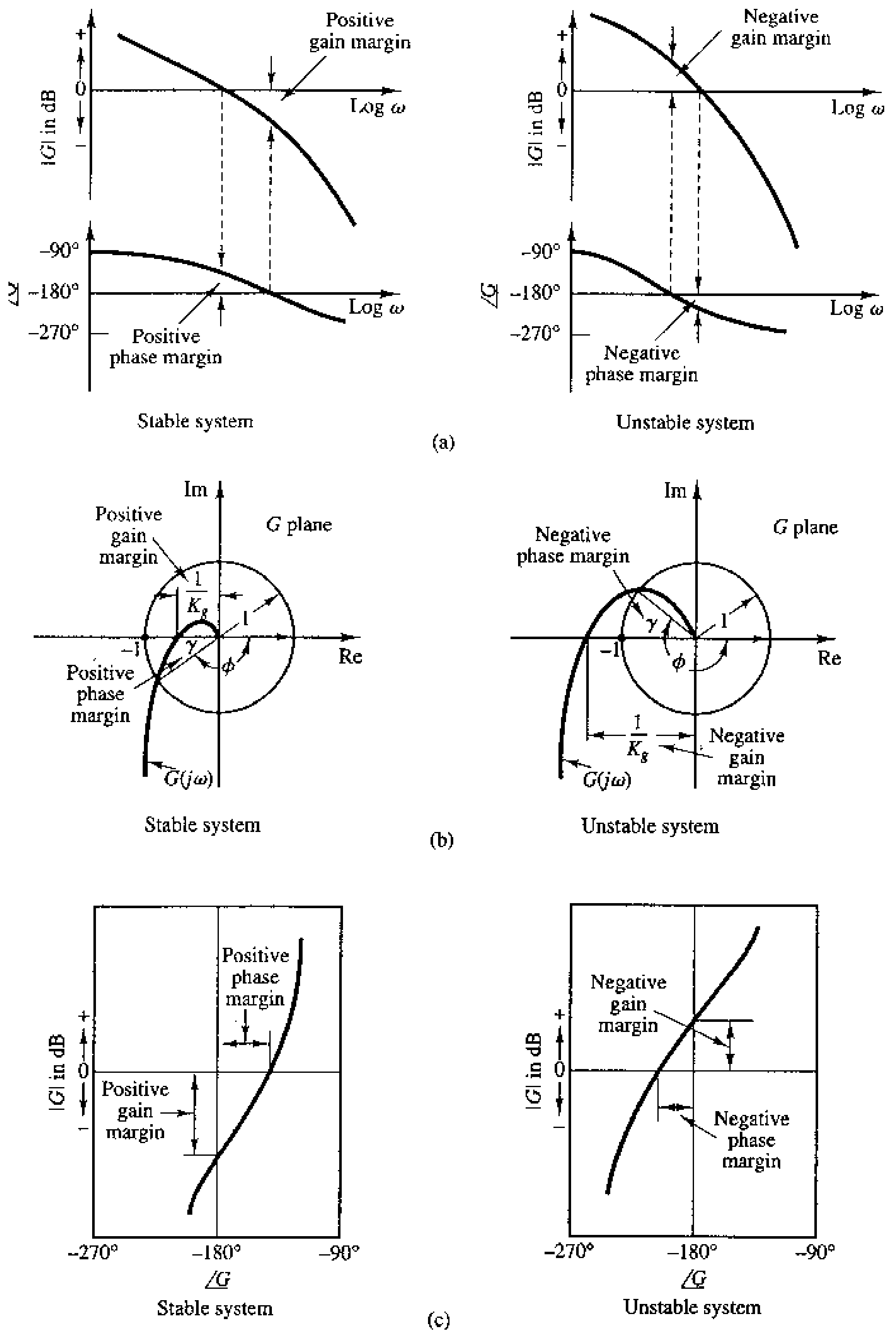


Figure: Phase and gain margins of stable and unstable systems (a) Bode diagrams; (b) Polar plots; (c) Log-magnitude-versus-phase plots.

If the open-loop system is minimum phase and has both phase and gain margins positive,

→ then the closed-loop system is stable.

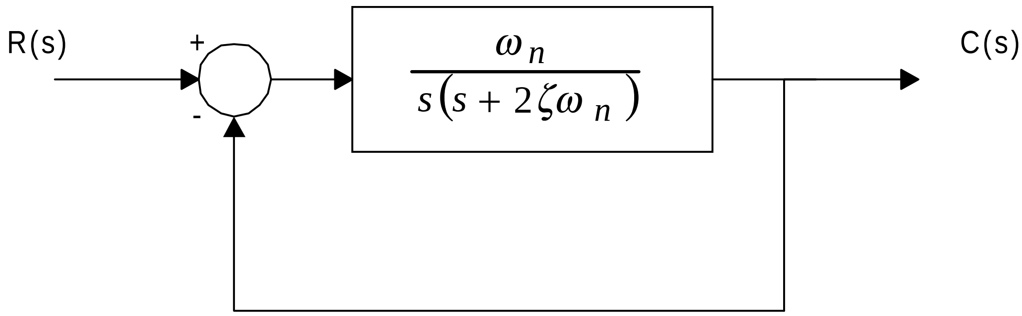
- For good relative stability both margins are required to be positive.

- Good values for minimum phase system:

    - Phase margin : 30° – 60°

    - Gain margin: above 6 dB

Correlation between damping ratio and frequency response for 2nd order systems



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad\qquad \frac{C(j\omega)}{R(j\omega)} = M(\omega)\cdot e^{j\alpha(\omega)}$$

Phase margin:

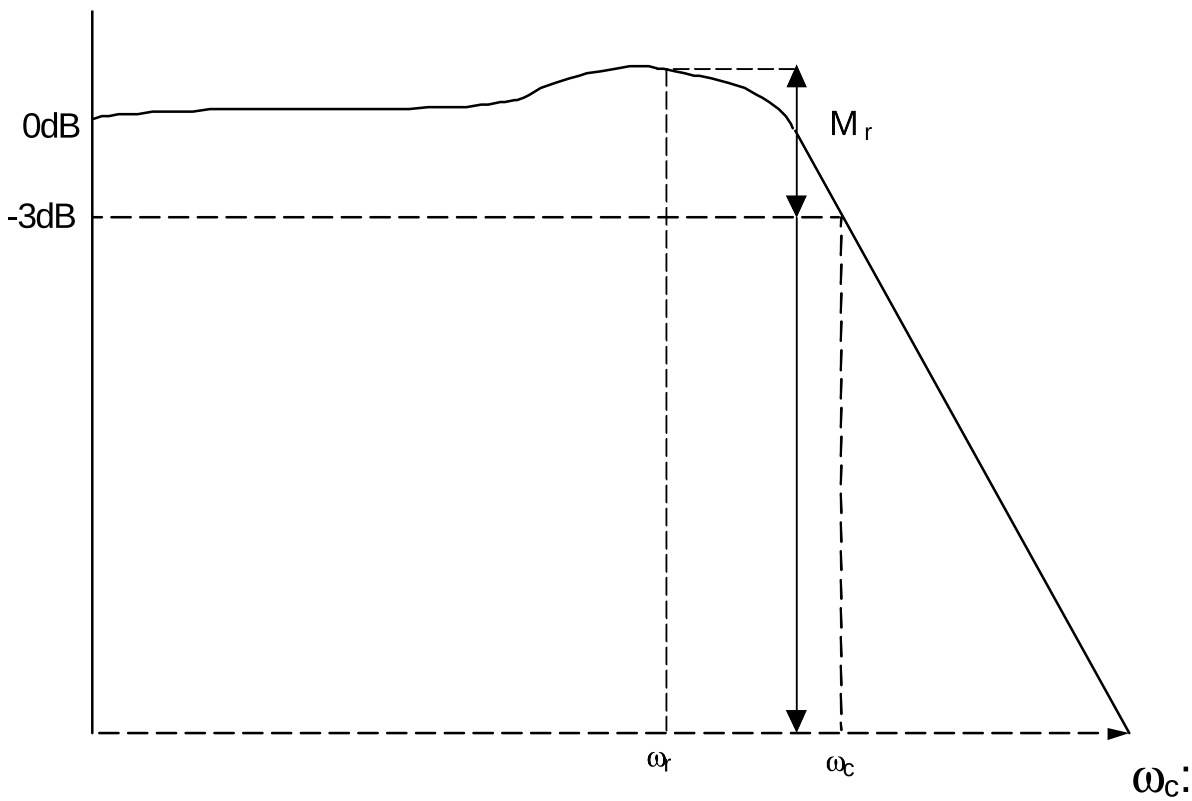
$\gamma = 180° + \angle G(j\omega)$ G(jω): open loop transfer function

$$|G(j\omega)| = \left|\frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}\right|$$ becomes unity for $\omega_1 = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$

and $$\gamma = \tan^{-1}\left[\frac{2\zeta\omega_n}{\omega_1}\right] = \tan^{-1}\left[\frac{2\zeta}{\sqrt{\sqrt{1+\zeta^4} - 2\zeta^2}}\right]$$

→ γ depends only on ζ

Performance specifications in the frequency domain:



ω_c : *Cutoff Frequency*

$0 \le \omega \le \omega_c$: *Bandwidth*

Slope of log-magnitude curve: *Cutoff Rate*

- ability to distinguish between signal and noise

ω_r: *Resonant Frequency*

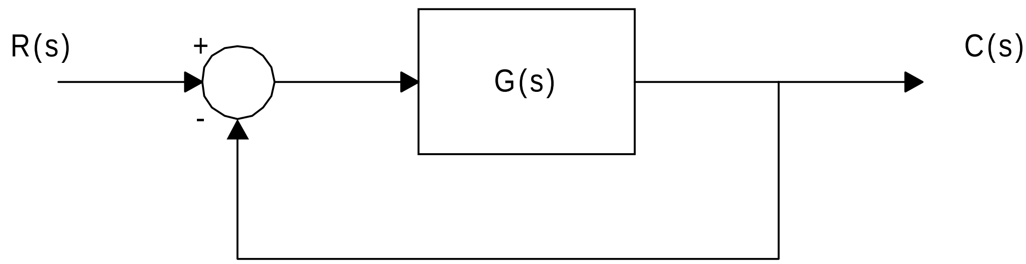
- indicative of transient response speed
- $\omega_r \rightarrow$ increase, transient response faster (dominant complex conjugate poles assumed)

$M_r = \max|G(j\omega)|$: *Resonant Peak*

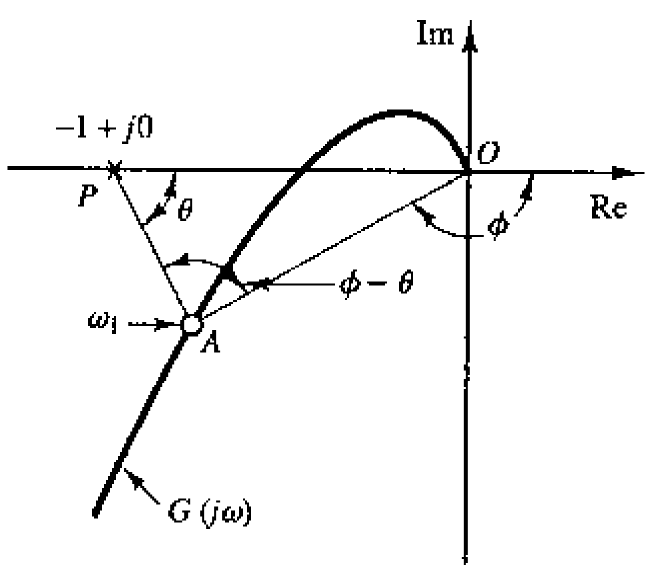
Closed-Loop Frequency Response

Open-loop system: G(s)

Stable closed-loop system: $$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$



$$\frac{|G(s)|}{|1+G(s)|} = \frac{\left|\overrightarrow{OA}\right|}{\left|\overrightarrow{PA}\right|}$$



Closed-Loop Frequency Response:

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1+G(j\omega)} = M \cdot e^{j\alpha}$$

Constant Magnitude Loci:

$$G(j\omega) = X + jY$$

$$M = \frac{|X + jY|}{|1 + X + jY|} = const$$

$$\left(X + \frac{M^2}{M^2 - 1}\right)^2 + Y^2 = \frac{M^2}{\left(M^2 - 1\right)^2}$$

Constant Phase-Angle Loci

$$G(j\omega) = X + jY \;\rightarrow\; \angle e^{j\alpha} = \angle \frac{X + jY}{1 + X + jY} = const$$

$$\left(X + \tfrac{1}{2}\right)^2 + \left(Y + \tfrac{1}{2N}\right)^2 = \frac{1}{4} + \left(\frac{1}{2N}\right)^2, \quad N = \tan\alpha$$

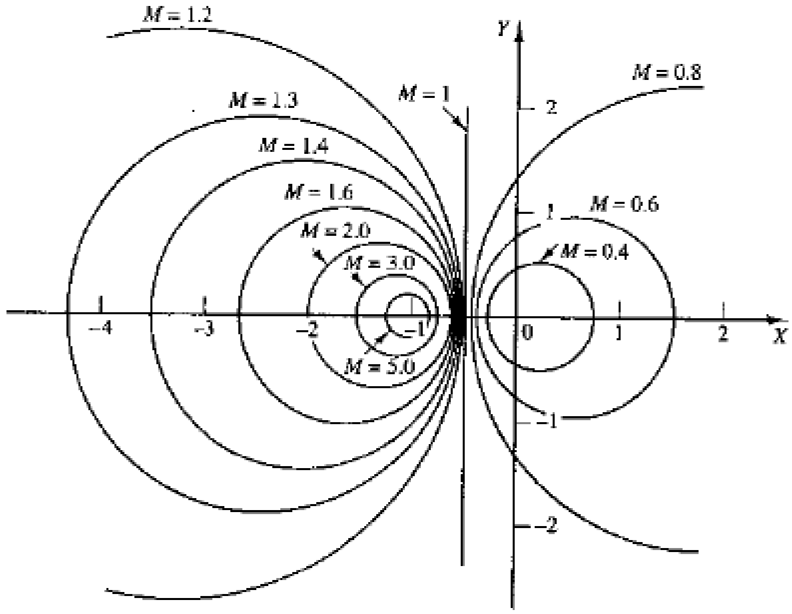


Figure: A family of constant M circles.

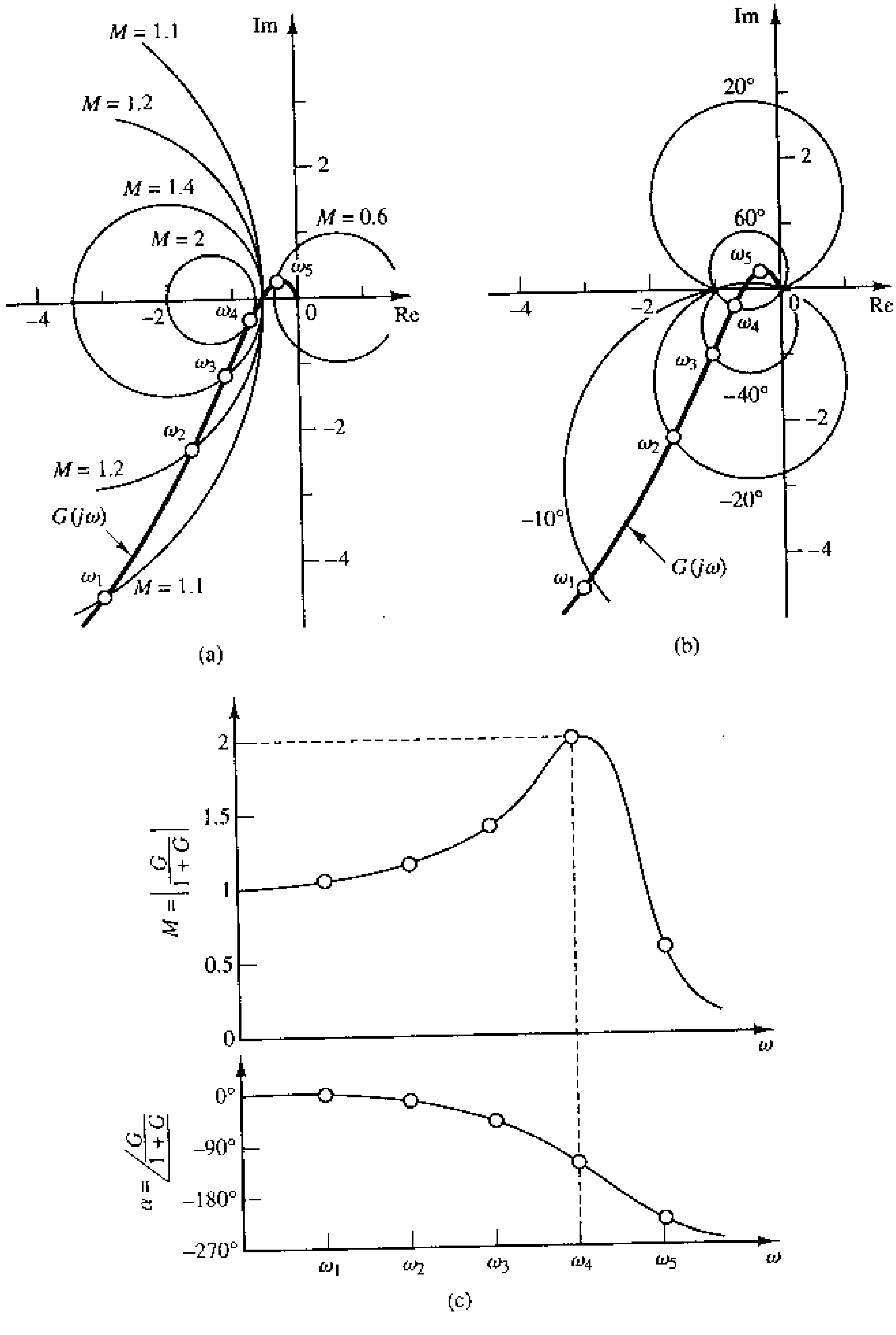


Figure:

(a) G(jω) locus superimposed on a family of M circles;

(b) G(jω) locus superimposed om a family of N circles;

(c) Closed-loop frequency-response curves

Experimental Determination of Transfer Function

- Derivation of mathematical model is often difficult and may involve errors.

- Frequency response can be obtained using sinusoidal signal generators.

  Measure the output and obtain:
  - Magnitudes (quite accurate)
  - Phase (not as accurate)

Use the Magnitude data and asymptotes to find:

- Type and error coefficients
- Corner frequencies
- Orders of numerator and denominator
- If second order terms are involved, ζ is obtained from the resonant peak.

Use phase to determine if system is minimum phase or not:

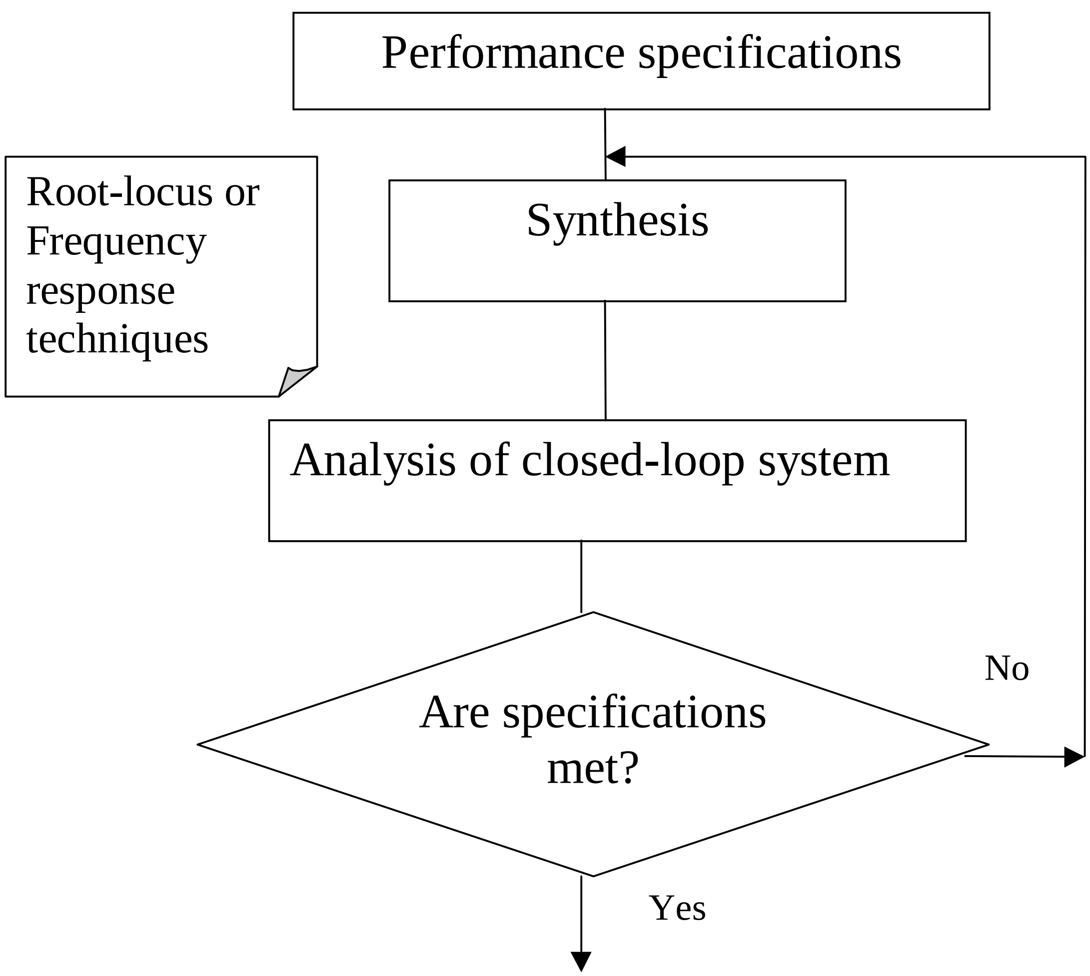
- Minimum phase: $\omega \rightarrow \infty$ phase = -90 (n - m)
  (n-m) difference in the order of denominator and numerator.

Compensation Techniques

- Performance specifications for the closed-loop system

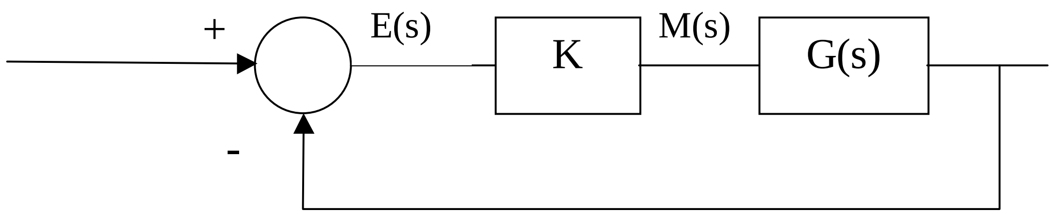
  - Stability
  - Transient response → T_s, M_s (settling time, overshoot) or phase and gain margins
  - Steady-state response → e_{ss} (steady state error)

- Trial and error approach to design



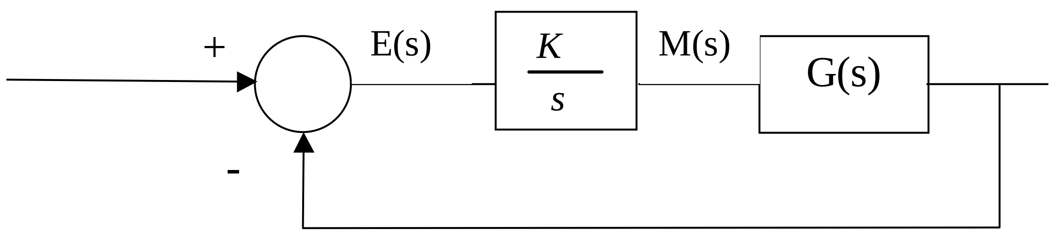
Basic Controls

1. Proportional Control



$$\frac{M(s)}{E(s)} = K \qquad\qquad m(t) = K \cdot e(t)$$

2. Integral Control

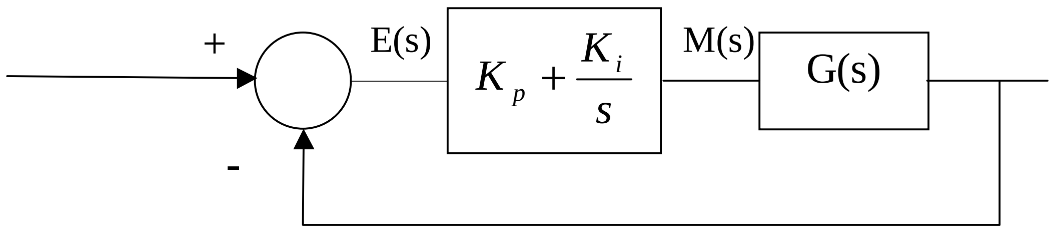


$$\frac{M(s)}{E(s)} = \frac{K}{s} \qquad\qquad m(t) = K \int e(t)dt$$

Integral control adds a pole at the origin for the open-loop:

- Type of system increased, better steady-state performance.
- Root-locus is "pulled" to the left tending to lower the system's relative stability.

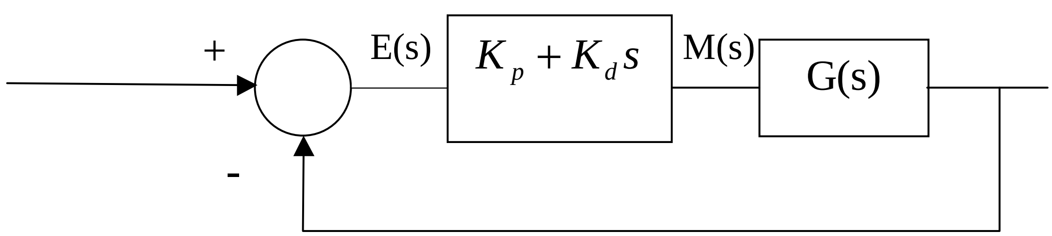
3. Proportional + Integral Control



$$\frac{M(s)}{E(s)} = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \qquad m(t) = K_p e(t) + \int K_i e(t) dt$$

A pole at the origin and a zero at $-\frac{K_i}{K_p}$ are added.

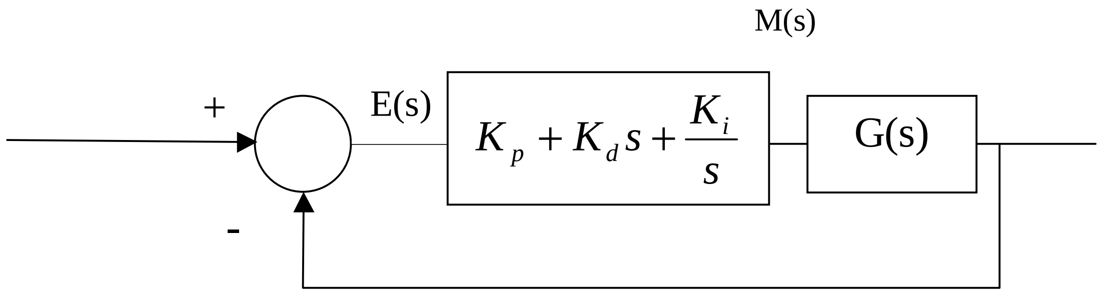
4. Proportional + Derivative Control



$$\frac{M(s)}{E(s)} = K_p + K_d s \qquad m(t) = K_p e(t) + K_d \frac{de(t)}{dt}$$

- Root-locus is “pulled” to the left, system becomes more stable and response is sped up.
- Differentiation makes the system sensitive to noise.

5. Proportional + Derivative + Integral (PID) Control

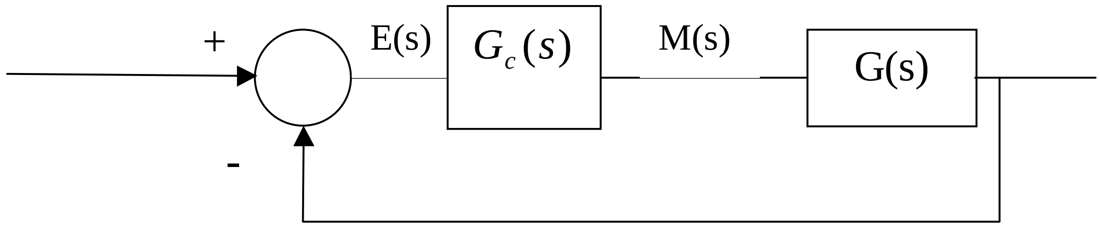


$$\frac{M(s)}{E(s)} = K_p + K_d d + \frac{K_i}{s}$$

$$m(t) = K_p e(t) + K_d \frac{de(t)}{dt} + K_i \int e(t) dt$$

- More than 50% of industrial controls are PID.
- More than 80% in process control industry.
- When G(s) of the system is not known, then initial values for K_p, K_d, K_i can be obtained experimentally and than fine-tuned to give the desired response (Ziegler-Nichols).

6. Feed-forward compensator



Design $G_c(s)$ using Root-Locus or Frequency Response techniques.

Frequency response approach to compensator design

Information about the *performance of the closed-loop system*, obtained from the *open-loop frequency response*:

- *Low frequency* region indicates the steady-state behavior.
- *Medium frequency* (around -1 in polar plot, around gain and phase crossover frequencies in Bode plots) indicates relative stability.
- *High frequency* region indicates complexity.

Requirements on open-loop frequency response

- The gain at low frequency should be large enough to give a high value for error constants.
- At medium frequencies the phase and gain margins should be large enough.
- At high frequencies, the gain should be attenuated as rapidly as possible to minimize noise effects.

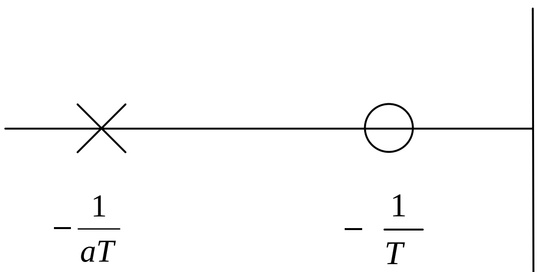
Compensators

- *lead*:improves the transient response.
- *lag*: improves the steady-state performance at the expense of slower settling time.
- *lead-lag*: combines both

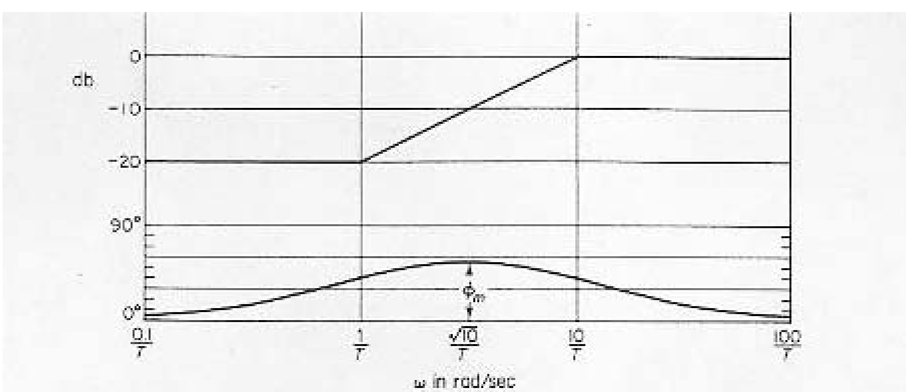
Lead compensators

$$G_c(s) = K_c a \frac{Ts+1}{aTs+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{aT}} \qquad T > 0 \quad \text{and} \quad 0 < \alpha < 1$$

- Poles and zeros of the lead compensator:



- Frequency response of $G_c(j\omega)$:



The maximum phase-lead angle ϕ_m occurs at ω_m , where:

$$\sin\phi_m = \frac{1-\alpha}{1+\alpha} \qquad \text{and}$$

$$\log\omega_m = \frac{1}{2}\left[\log T + \log\frac{1}{aT}\right] \quad \rightarrow \quad \omega_m = \frac{1}{\sqrt{a}\,T}$$

Since

$$\left|\frac{1+j\omega T}{1+j\omega aT}\right|_{\omega=\omega_m} = \frac{1}{\sqrt{a}}$$

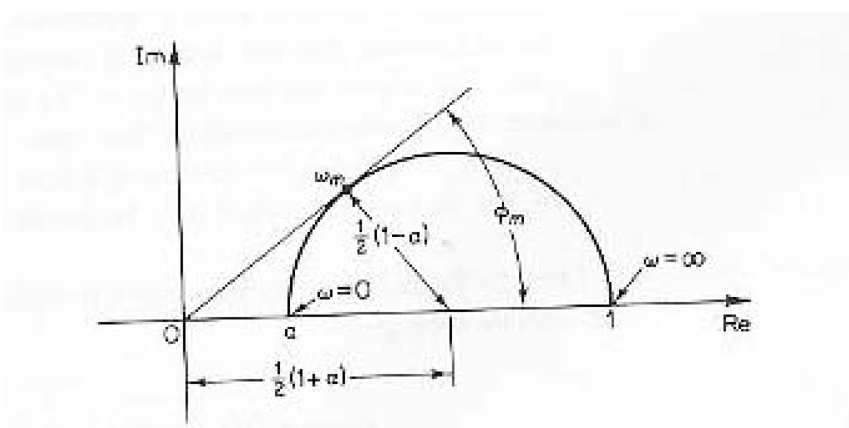
the magnitude of $G_c(j\omega)$ at ω_m is given by:

$$|G_c(j\omega_m)| = K_c\sqrt{a}$$

Polar plot of a lead network

$$\frac{a(j\omega T+1)}{(j\omega aT+1)}$$ where $0 < a < 1$

is given by



Lead compensation based on the frequency response

Procedure:

1. Determine the compensator gain $K_c\alpha$ satisfying the given error constant.
2. Determined the additional phase lead ϕ_m required (+ 10%~15%) for the gain adjusted ($K_c\alpha G(s)$) open-loop system.
3. Obtain α from $\sin\phi_m = \dfrac{1-a}{1+a}$
4. Find the new gain cross over frequency ω_c from

$$K_c a\left|G(j\omega_c)\right| = 10\log a$$

5. Find T from ω_c and transfer function of $G_c(s)$

$$T = \frac{1}{\sqrt{a}\,\omega_c}$$ and

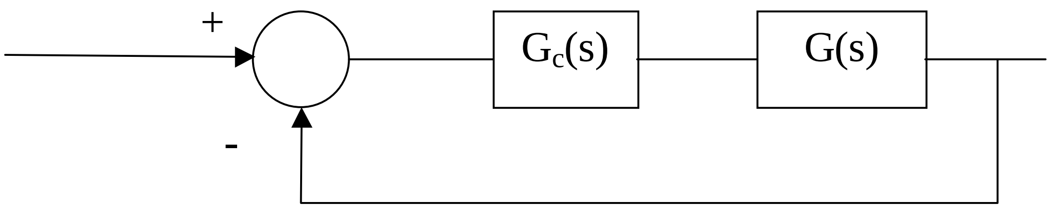
$$G_c(s) = K_c a \frac{Ts+1}{aTs+1}$$

General effect of lead compensator:

- Addition of phase lead near gain crossover frequency.
- Increase of gain at higher frequencies.
- Increase of system bandwidth.

Example:

Consider



where $$G(s) = \frac{4}{s(s+2)}$$

Performance requirements for the system:

Steady-state: $K_v = 20$
Transient response: phase margin >50°
gain margin >10 dB

Analysis of the system with $G_c(s) = K$

For $K_v = 20$ → $K = 10$

This leads to: phase margin ≈ 17°
gain margin ≈ +∞ dB

Design of a lead compensator:

$$G_c(s) = K_c a \frac{Ts + 1}{aTs + 1}$$

1. $K_v = \lim_{s\to 0} sG_c(s)G(s) = \frac{4K_c a}{2} = 2K_c a = 20$ → $K_c\alpha = 10$

2. From the Bode plot of $K_c\alpha G(j\omega)$, we obtain that the additional phase-lead required is: 50° - 17° = 33°.

   We choose 38° (~33° + 15%)

3. $\sin\phi_m = \sin 38^\circ = \dfrac{1-a}{1+a}$ → $\alpha = 0.24$

4. Since for ω_m, the frequency with the maximum phase-lead angle, we have:

$$\left|\frac{1+j\omega_m T}{1+j\omega_m aT}\right| = \frac{1}{\sqrt{a}}$$

We choose ω_c, the new gain crossover frequency so that

$$\omega_m = \omega_c \quad \text{and} \quad \left|G_c(s)G(s)\right|_{s=j\omega_c} = 1$$

This gives that:

$$\left|K_c aG(j\omega_c)\right| = \left|\frac{40}{j\omega_c(j\omega_c+2)}\right|$$

has to be equal:

$$\left(\frac{1}{\sqrt{a}}\right)^{-1} = -6.2dB$$

From the Bode plot of $K_c\alpha G(j\omega)$ we obtain that

$$\left|\frac{40}{j\omega_c(j\omega_c+2)}\right| = -6.2dB \quad \text{at } \omega_c = 9 \text{ rad/sec}$$

5. This implies for T

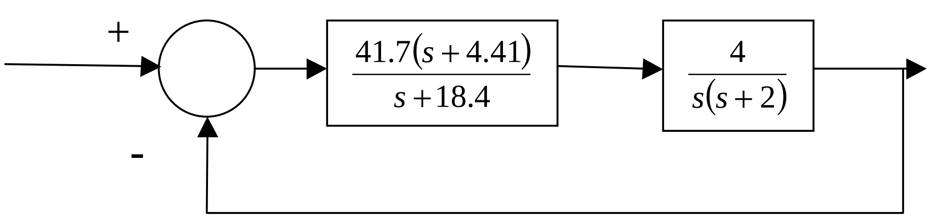
$$\omega_c = \frac{1}{\sqrt{aT}} = \frac{1}{\sqrt{0.24T}} = 9 rad/\sec \rightarrow \frac{1}{T} = 4.41$$

and

$$K_c = \frac{20}{2a} = 41.7$$

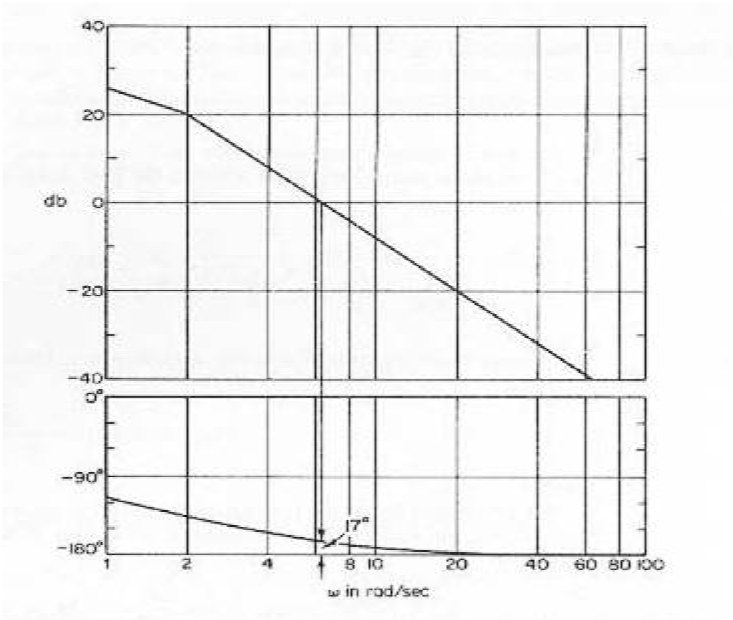
$$G_c(s) = 41.7 \frac{s + 4.41}{s + 18.4}$$

The compensated system is given by:

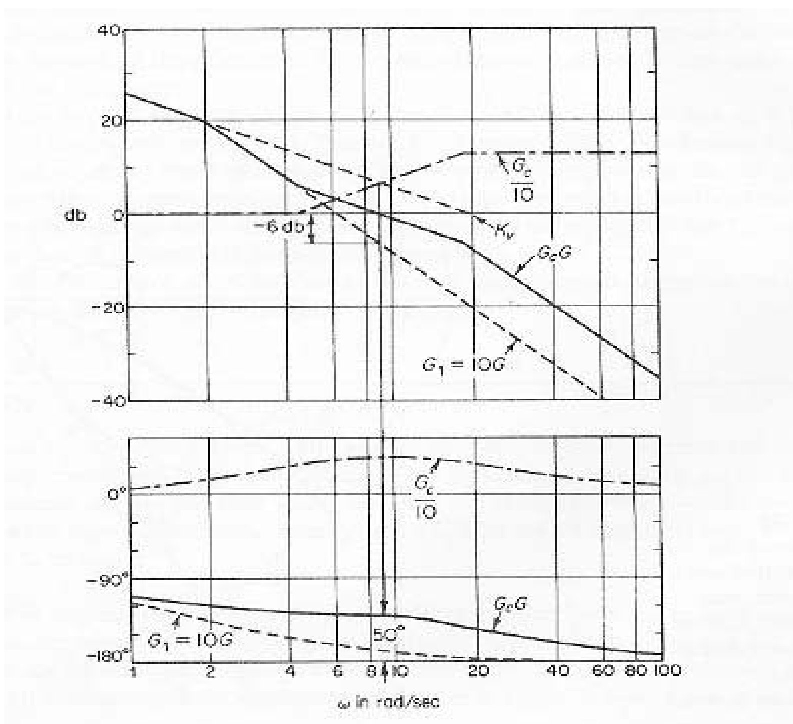


The effect of the lead compensator is:

- Phase margin: from 17° to 50° → better transient response with less overshoot.
- ω_c : from 6.3rad/sec to 9 rad/sec → the system response is faster.
- Gain margin remains ∞ .
- K_v is 20, as required → acceptable steady-state response.



Bode diagram for $K_c \cdot a \cdot G(j\omega) = \dfrac{40}{\mathrm{j}\omega(\mathrm{j}\omega + 2)}$



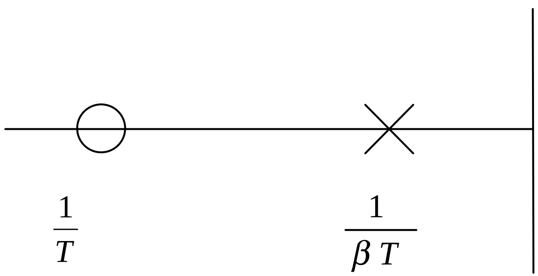
Bode diagram for the compensated system

$$G_c(j\omega)G(j\omega) = 41.7\,\frac{j\omega + 4.41}{j\omega + 18.4} \cdot \frac{4}{\mathrm{j}\omega(\mathrm{j}\omega + 2)}$$

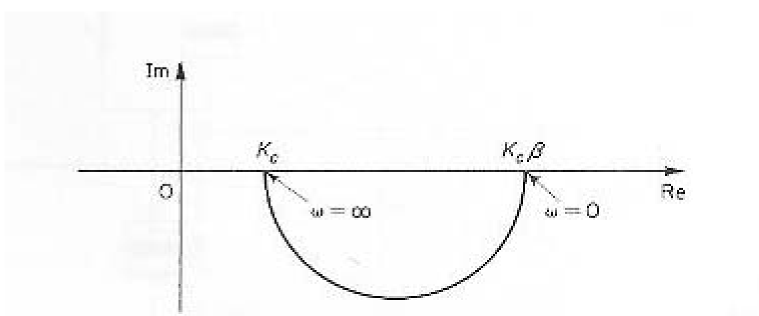
Lag compensators

$$G_c(s) = K_c\beta\,\frac{Ts+1}{\beta Ts+1} = K_c\,\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad T > 0\ ,\ \beta > 1$$

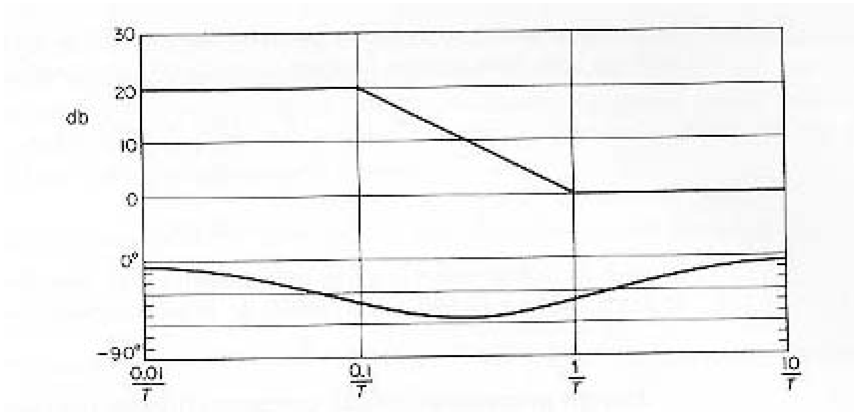
Poles and zeros:



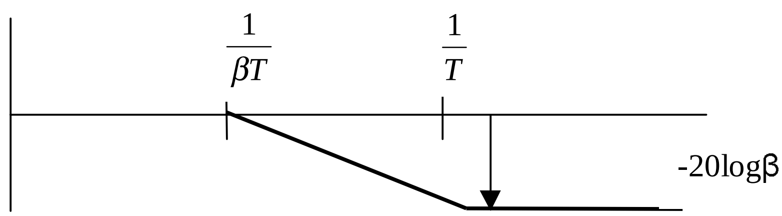
Frequency response:



Polar plot of a lag compensator $K_c\beta(j\omega T+1)/(j\omega\beta T+1)$



Bode diagram of a lag compensator with K_c=1, β = 10



Magnitude of *(jωT+1)/(jωβT+1)*

Lag compensation based on the frequency response

Procedure:

1. Determine the compensator gain $K_c\beta$ to satisfy the requirement for the given error constant.
2. Find the frequency point where the phase of the gain adjusted open-loop system ($K_c\beta G(s)$) is equal to -180° + the required phase margin + 5°~ 12°.

   This will be the new gain crossover frequency ω_c.
3. Choose the zero of the compensator $\omega = 1/T$ at about 1 octave to 1 decade below ω_c .
4. Determine the attenuation necessary to bring the magnitude curve down to 0dB at the new gain crossover frequency

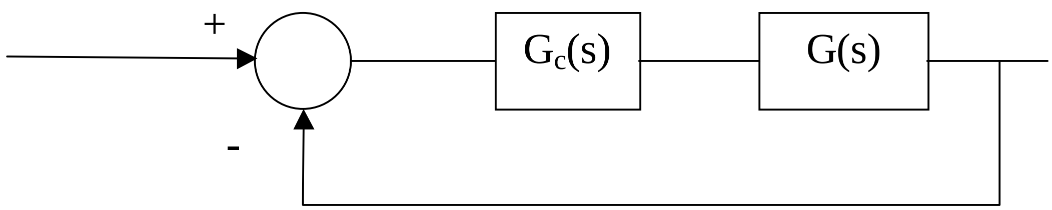
   $$-K_c\beta\left|G(j\omega_c)\right| = -20\ \log\ \beta \quad \rightarrow \beta$$
5. Find the transfer function $G_c(s)$.

General effect of lag compensation:

- Decrease gain at high frequencies.
- Move the gain crossover frequency lower to obtain the desired phase margin.

Example:

Consider



where

$$G(s) = \frac{1}{s(s+1)(0.5s+1)}$$

Performance requirements for the system:

Steady state:	$K_v = 5$
Transient response:	Phase margin > 40°
	Gain margin > 10 dB

Analysis of the system with $G_c(s) = K$

$$K_v = \lim_{s\to 0} KG(s) = K = 5$$

for K = 5, the closed-loop system is unstable

Design of a lag compensator:

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = K_c \beta \frac{Ts+1}{\beta Ts+1}$$

1. $K_v = \lim_{s \to 0} G_c(s)G(s) = K_c\beta = 5$
2. Phase margin of the system 5G(s) is -13°

   → the closed-loop system is unstable.

   From the Bode diagram of $5G(j\omega)$ we obtain that the additional required phase margin of 40° + 12° = 52° is obtained at $\omega = 0.5$ rad/sec.

   The new gain crossover frequency will be:

   $\omega_c = 0.5$ rad/sec

3. Place the zero of the lag compensator at $\omega = 1/T = 0.1$ rad/sec(at about 1/5 of ω_c).

4. The magnitude of $5G(j\omega)$ at the new gain crossover frequency ω_c =0.5 rad/sec is 20 dB. In order to have ω_c as the new gain crossover frequency, the lag compensator must give an attenuation of -20db at ω_c.

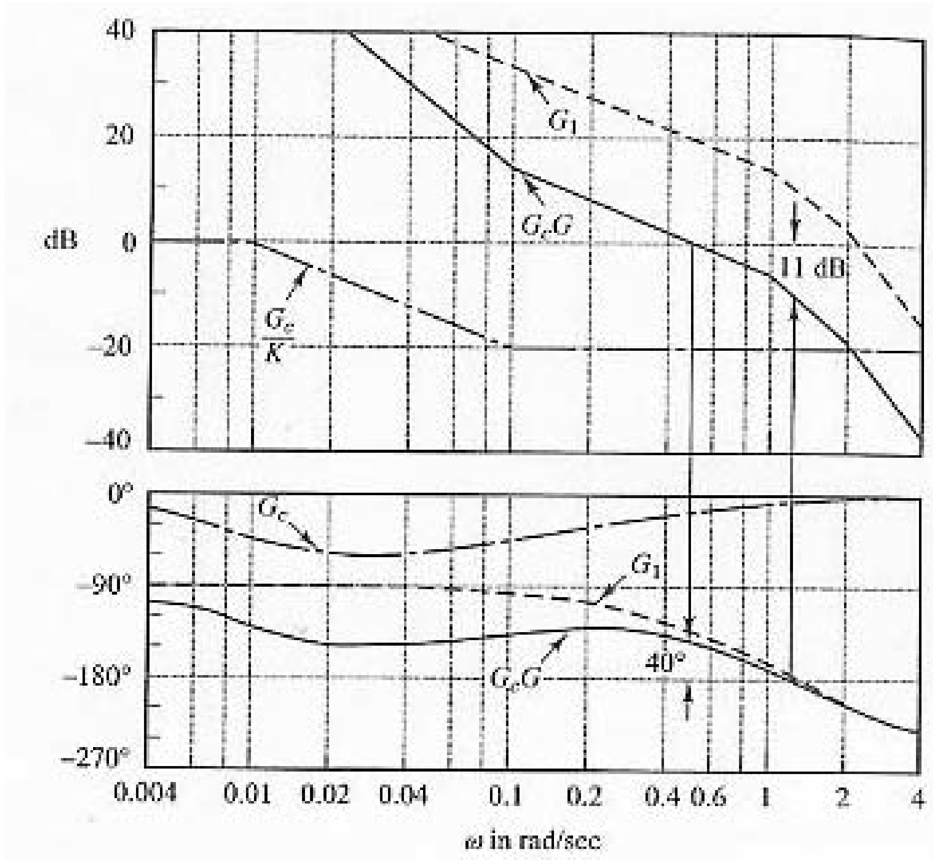
   Therefore

   $$-20\log\beta = -20 \text{ dB} \quad \rightarrow \quad \beta = 10$$

5. $K_c = \dfrac{5}{\beta} = 0.5$, $\quad$ *pole* : $\dfrac{1}{\beta T} = 0.01$

   and

   $$G_c(s) = 0.5\frac{s+0.1}{s+0.01}$$



Bode diagrams for:

- $G_1(j\omega) = 5G(j\omega)$ (gain-adjusted $K_c\beta G(j\omega)$ open-loop transfer function),
- $G_c(j\omega)/K = G_c(j\omega)/5$ (compensator divided by gain $K_c\beta$ = 5),
- $G_c(j\omega)G(j\omega)$ (compensated open-loop transfer function)

The effect of the lag compensator is:

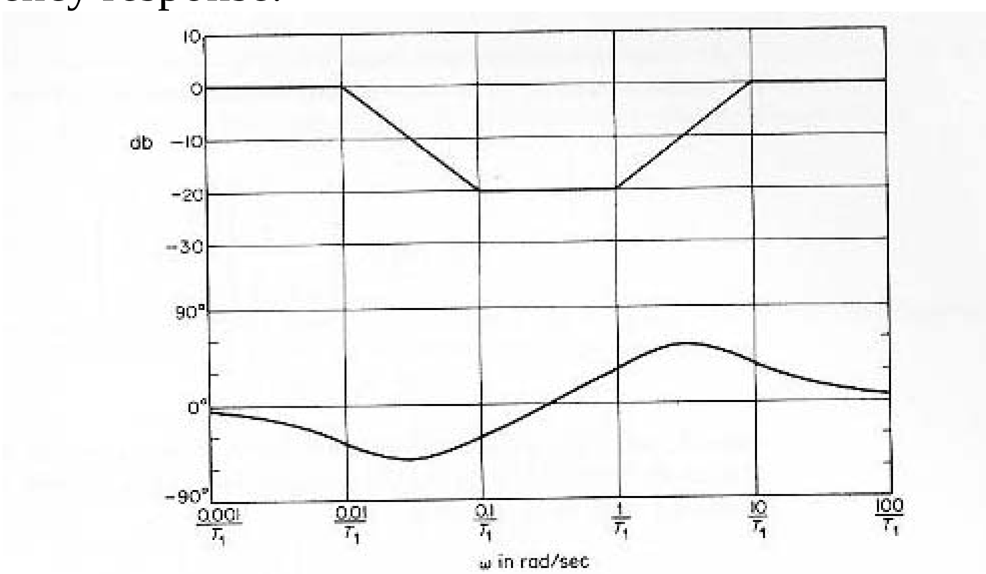
- The original unstable closed-loop system is now stable.
- The phase margin ≈ 40° → acceptable transient response.
- The gain margin ≈ 11dB → acceptable transient response.
- K_v is 5 as required → acceptable steady-state response.
- The gain at high frequencies has been decreased.

Lead-lag compensators

$$G_c(s) = K_c \frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{aT_1}} \cdot \frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} = K_c \frac{\beta}{\gamma} \frac{sT_1+1}{s\frac{T_1}{\gamma}+1} \cdot \frac{sT_2+1}{s\beta T_2+1}$$

$$T_1, T_2 > 0 , \quad \beta > 1 \quad \text{and} \quad \gamma > 1$$

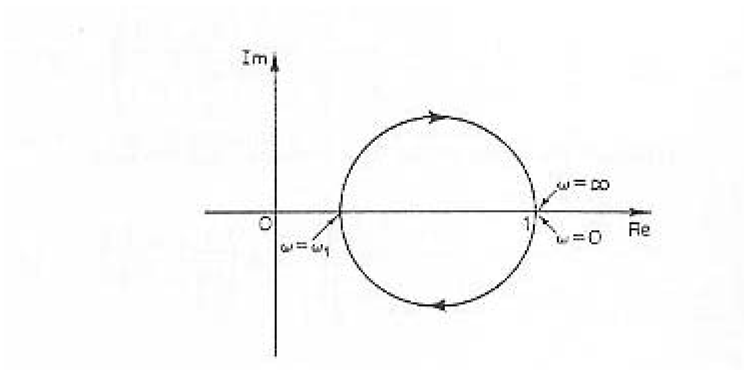
Frequency response:



Bode diagram of a lag-lead compensator given by

$$G_c(s) = K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}} \right) \left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}} \right)$$

with $K_c = 1$, $\gamma = \beta = 10$ and $T_2 = 10\, T_1$



Polar plot of a lag-lead compensator given by

$$G_c(s) = K_c \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}} \right) \left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}} \right)$$

with $K_c = 1$ and $\gamma = \beta$

Comparison between lead and lag compensators

Lead compensator	**Lag** compensator
o High pass	o Low pass
o Approximates derivative plus proportional control	o Approximates integral plus proportional control
o Contributes phase lead	o Attenuation at high frequencies
o Increases the gain crossover frequency	o Moves the gain-crossover frequency lower
o Increases bandwidth	o Reduces bandwidth