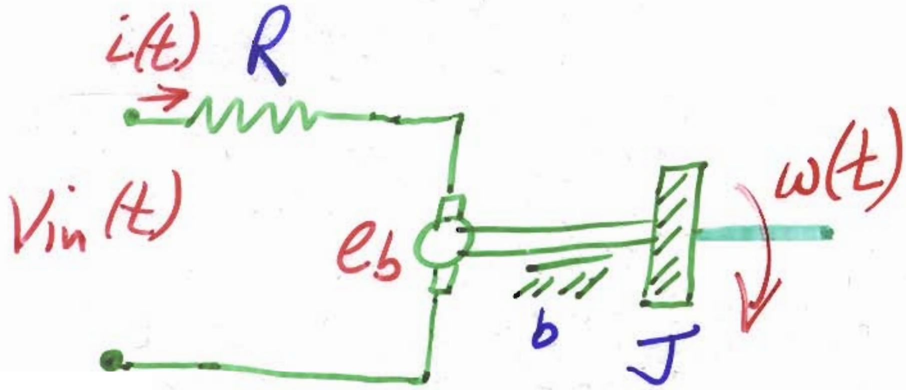


J - moment of inertia

$b\omega$ - friction (bearing)

k_t - torque constant

$$\tau = k_t i$$

$$\dot{\omega}(t) + a\,\omega(t) = b_0 V_{in}(t)$$

$$a = \frac{b + k_t^2/R}{J}$$

$$b_0 = \frac{k_t}{JR}$$

Example

$$\dot{\omega}(t) + a\,\omega(t) = b_0\, V_{in}(t)$$

take L.T.

$$s\,W(s) - W(0) + a\,W(s) = b_0\, V_{in}(s)$$

$$W(s) = \frac{W(0)}{s+a} + \frac{b_0}{s+a} V_{in}(s)$$

Case 1: $\omega(0) = \omega_0 \quad V_{in}(t) = 0$

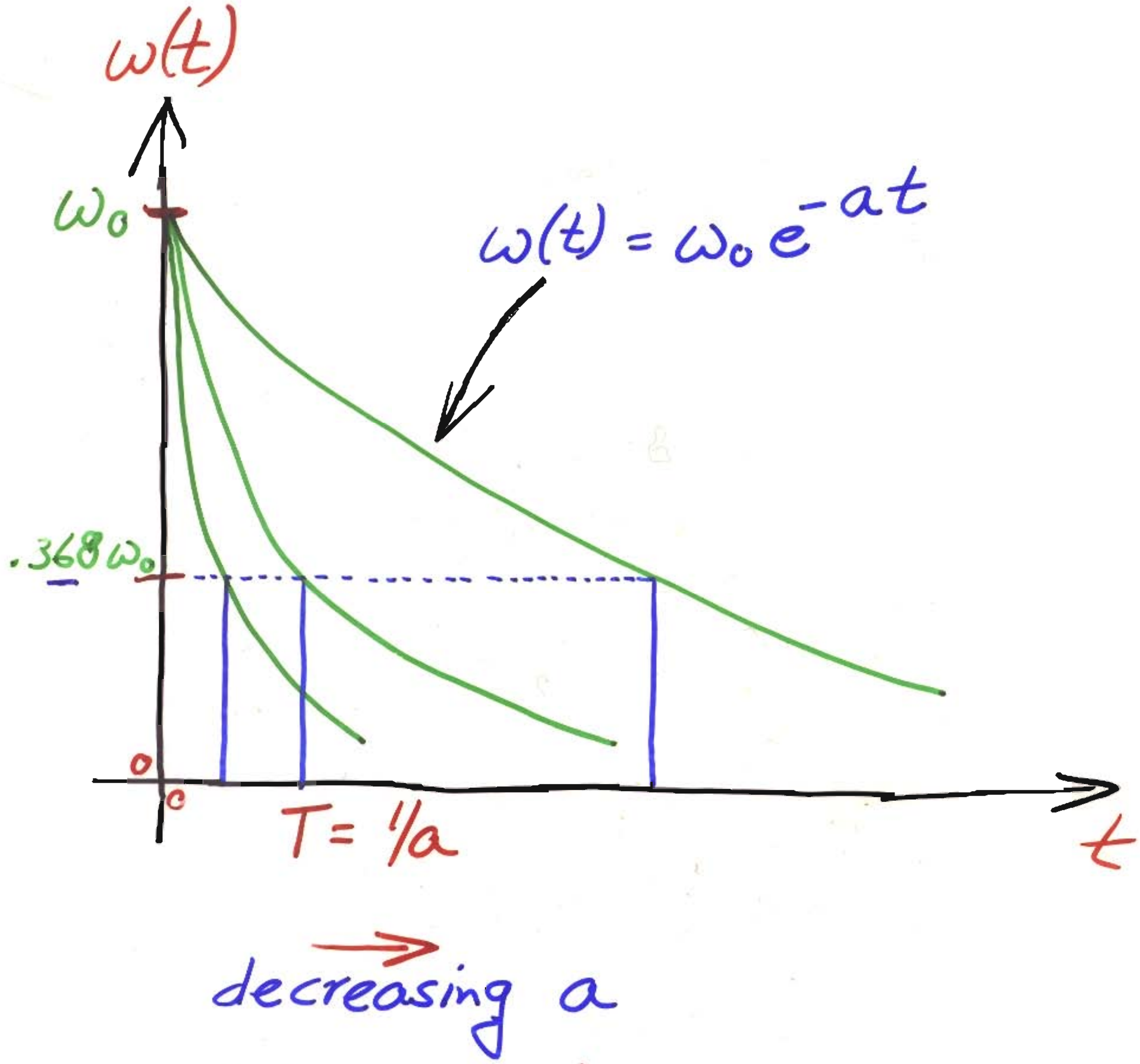
$$\omega(s) = \frac{\omega_0}{s+a} \qquad \mathcal{L}^{-1}\{\omega(s)\} = e^{-at}\omega_0$$

Case 2: $\omega(0) = 0 \quad V_{in}(t) = \delta(t)$

$$\omega(s) = \frac{b_0}{s+a} V_{in}(s) = \frac{b_0}{s+a} = H(s)$$

$$\omega(t) = \mathcal{L}^{-1}\{\omega(s)\} = b_0\, e^{-at}$$

Natural Response



$$\omega(T) = \omega_0 e^{-1} = .368\,\omega_0$$

Case 3: $\omega(0)=0 \quad V_{in}(t) = V_0\, u(t)$

$$\omega(s) = \frac{b_0}{s+a} V_{in}(s) = \frac{b_0 V_0}{s(s+a)}$$

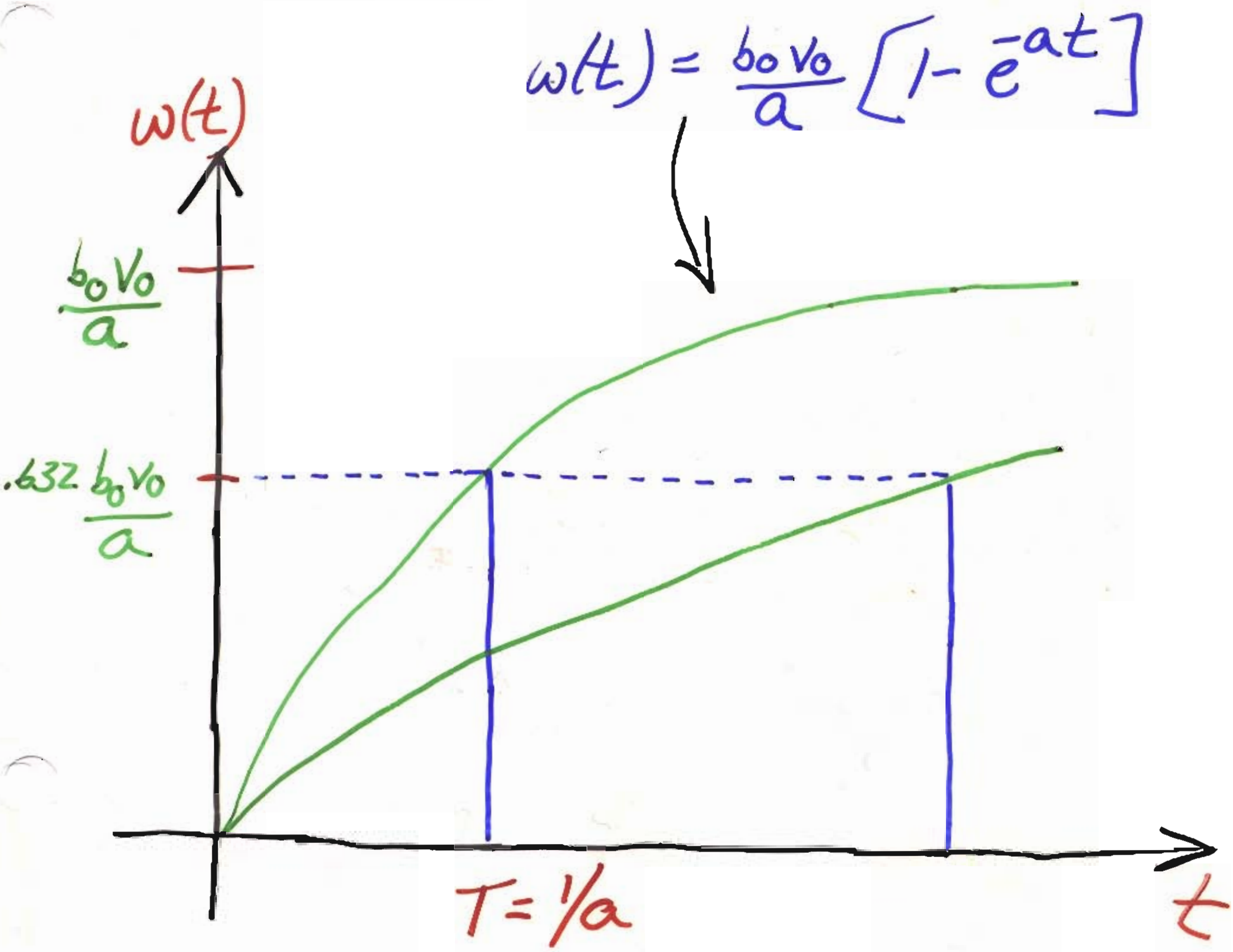
$$\omega(t) = \mathcal{L}^{-1}\left\{ \frac{b_0 V_0}{s(s+a)} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{c_1}{s} + \frac{c_2}{s+a} \right\}$$

$$\omega(t) = \frac{b_0 V_0}{a}\left[1 - e^{-at}\right] \qquad t \geq 0$$

Case 4: $\omega(0) = \omega_0 \quad V_{in}(t) = V_0\, u(t)$

$$\omega(s) = \frac{\omega_0}{s+a} + \frac{b_0 V_0}{s(s+a)}$$



Step Response

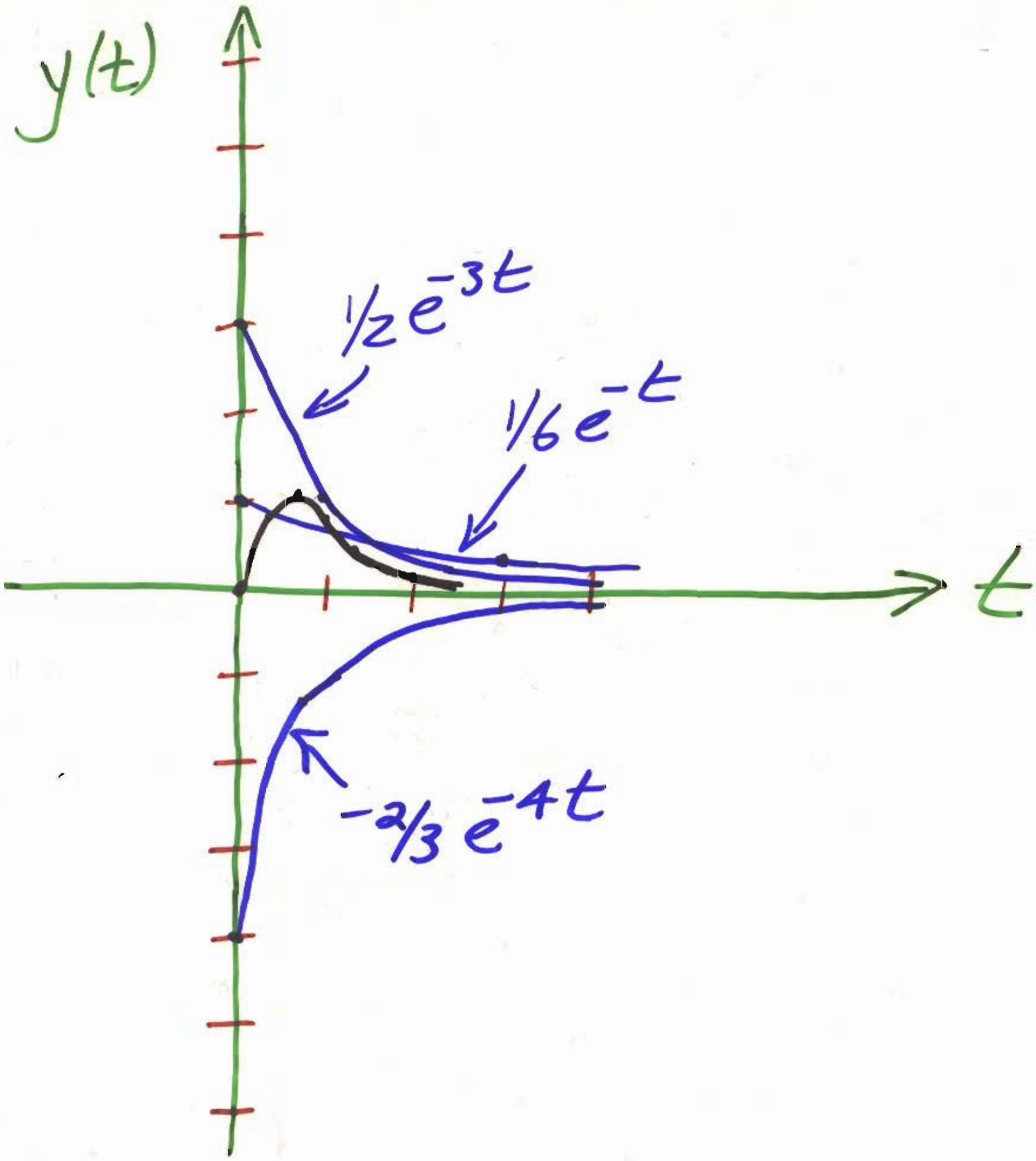
$\omega(0) = 0 \qquad V_{in}(t) = V_0 \sin zt$

$$V_{in}(s) = \frac{V_0 z}{s^2 + z^2}$$

$$\omega(s) = H(s)\, V_{in}(s) = \frac{b_0 V_0 z}{(s+a)(s^2+z^2)}$$

$$= \frac{c_1}{s+a} + \frac{c_2 s + c_3}{s^2 + z^2}$$

- find the c_i
- find $\mathcal{L}^{-1}\{\omega(s)\}$ to get $\omega(t)$



y(t)
t
1/2 e^{-3t}
1/6 e^{-t}
-2/3 e^{-4t}

Solving an n-th order LTI Differential Equation

- zero IC's

$$y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_0 y(t) =$$
$$b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t)$$

$y(t)$ input
$x(t)$ output
$n \geq m$

Take LT: $y^{(n)}(t) \Leftrightarrow s^n Y(s)$

$$\left(s^n + a_{n-1} s^{n-1} + \cdots + a_0\right) Y(s) =$$
$$\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right) X(s)$$

Transfer Function

$$\frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots a_0} = H(s)$$

$$Y(s) = H(s)\,X(s) = \frac{N(s)}{D(s)}\,X(s)$$

let $x(t) = \delta(t) \Rightarrow X(s) = 1$

step 1: factor $N(s)$ and $D(s)$

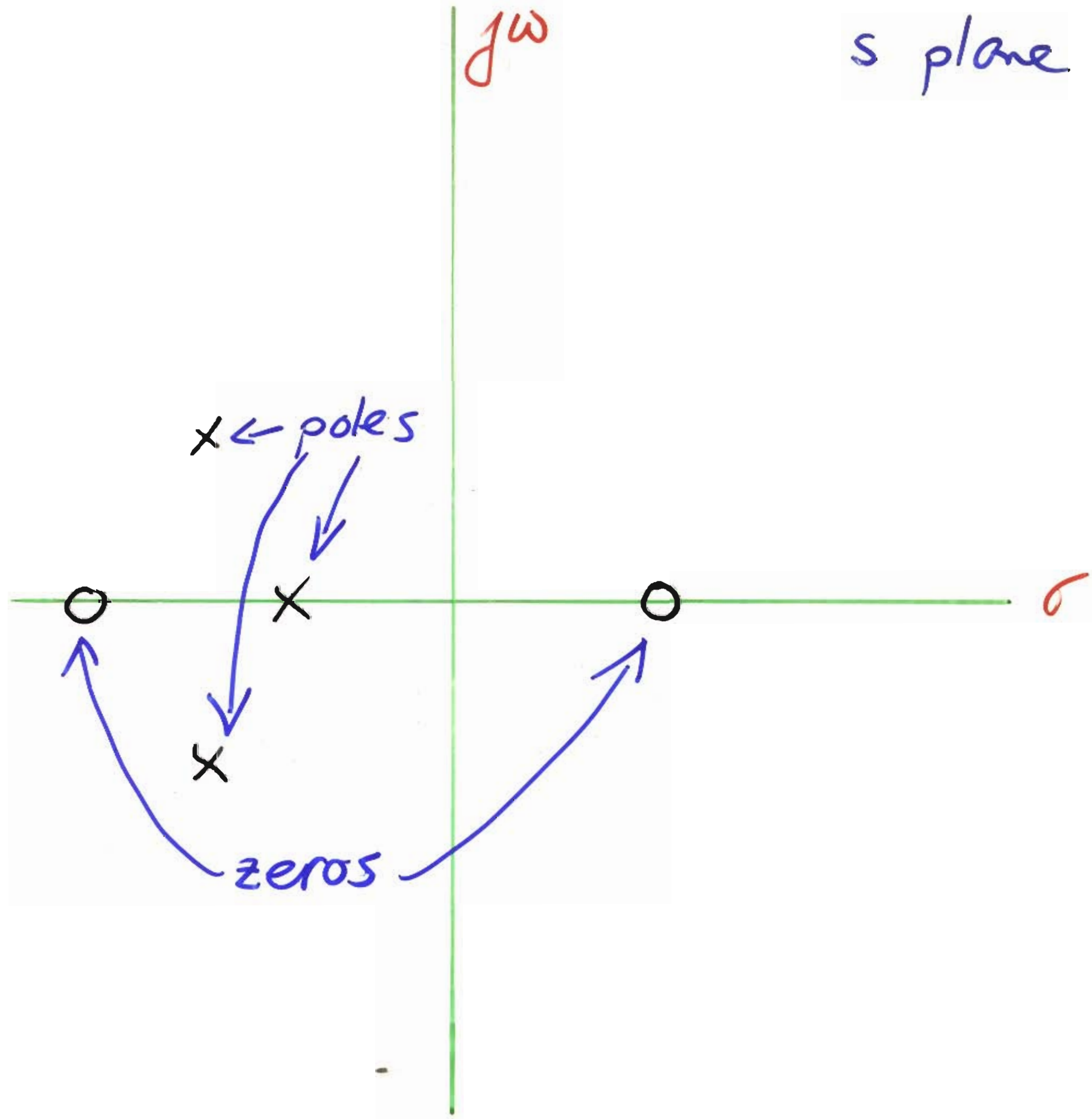
$$N(s) = b_m (s - z_1)(s - z_2) \cdots (s - z_m)$$

the z_i are called ZEROS

$$D(s) = (s - p_1)(s - p_2) \cdots (s - p_n)$$

the p_i are called POLES

for an nth order DE there are n POLES


jω
s plane
poles
σ
zeros

step 2: expand into partial fractions

$$Y(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \cdots \frac{c_n}{s-p_n}$$

use the residue method to find the c_i

step 3: find the inverse transform

$$y(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \cdots + c_n e^{p_n t}$$

if IC's are present treat as an additional term in $Y(s)$

$$Y(s) = H(s) X_i(s) + \frac{C(s)}{D(s)}$$

forced response

natural response

Example

$$\dddot{y}(t) + 8\ddot{y}(t) + 19\dot{y}(t) + 12y(t) = \dot{x}(t) + 2x(t)$$

all IC's zero $\quad x(t) = \delta(t)$

step 1: take Laplace transform

$$(s^3 + 8s^2 + 19s + 12)\,Y(s) = (s+2)\,X(s)$$

$$Y(s) = \frac{s+2}{s^3 + 8s^2 + 19s + 12} = H(s)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

step 2: factor $D(s)$

$$D(s)\Big|_{s=-1} = -1 + 8 - 19 + 12 = 0$$

$$D(s) = (s+1)(s^2 + 7s + 12)$$

$$= (s+1)(s+3)(s+4)$$

$$\therefore \quad Y(s) = \frac{c_1}{s+1} + \frac{c_2}{s+3} + \frac{c_3}{s+4}$$

STEP 3: find the c_i

$$c_1 = (s+1)\, Y(s)\Big|_{s=-1} = \frac{1}{6}$$

$$c_2 = (s+3)\, Y(s)\Big|_{s=-3} = \frac{1}{2}$$

$$c_3 = (s+4)\, Y(s)\Big|_{s=-4} = -2/3$$

$$Y(s) = \frac{1/6}{s+1} + \frac{1/2}{s+3} - \frac{2/3}{s+4}$$

STEP 4: do inverse L.T.

$$y(t) = 1/6\, e^{-t} + 1/2\, e^{-3t} - 2/3\, e^{-4t}$$

Example

$$\dddot{y}(t) + 8\ddot{y}(t) + 19\dot{y}(t) + 12y(t) = 0$$

$$y(0) = 0 \quad \dot{y}(0) = 1 \quad \ddot{y}(0) = -6$$

STEP 1: take Laplace Transform

$$s^3 Y(s) - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0)$$
$$+8\left(s^2 Y(s) - sy(0) - \dot{y}(0)\right)$$
$$+19\left(sY(s) - y(0)\right)$$
$$+12\, Y(s) = 0$$

$$\left(s^3 + 8s^2 + 19s + 12\right) Y(s) = s + 2$$

$$Y(s) = \frac{s+2}{s^3 + 8s^2 + 19s + 12} = \frac{C(s)}{D(s)}$$

$$y(t) = \mathcal{L}^{-1}\left\{ Y(s) \right\}$$

STEP 2: factor $D(s)$

$$D(s) = (s+1)(s+3)(s+4)$$

Repeated Roots

$$Y(s) = \frac{s+2}{(s+1)^2(s+3)} = \frac{C1}{s+1} + \frac{C2}{(s+1)^2} + \frac{C3}{s+3}$$

$$c_2 = (s+1)^2\, Y(s)\Big|_{s=-1} = \frac{(-1+2)}{(-1+3)} = 1/2$$

$$c_3 = (s+3)\, Y(s)\Big/_{s=-3} = \frac{(-3+2)}{(-3+1)^2} = -1/4$$

$$C_1 = \frac{d}{ds}\left[(s+1)^2\, Y(s)\right]\Big|_{s=-1}$$

RHS $$\frac{d}{ds}\left[(s+1)C_1 + C_2 + \frac{(s+1)^2}{s+3}C_3\right]\Big|_{s=-1} = C_1$$

LHS $$C_1 = \frac{d}{ds}\left[\frac{s+2}{s+3}\right]\Big|_{s=-1} = \frac{1}{s+3} - \frac{s+2}{(s+3)^2}\Big|_{s=-1}$$

$$= 1/4$$

$$Y(s) = \frac{1/4}{s+1} + \frac{1/2}{(s+1)^2} - \frac{1/4}{s+3}$$

$$y(t) = 1/4\, e^{-t} + 1/2\, t\, e^{-t} - 1/4\, e^{-3t}$$

In General

$$\frac{c_1}{s+p_1} + \frac{c_2}{(s+p_1)^2} + \cdots + \frac{c_r}{(s+p_1)^r}$$

the c_i are

$$c_{r-i} = \frac{1}{i!}\frac{d^i}{ds^i}\left[(s+p_1)^r Y(s)\right]_{s=-p_1}$$

for the example $r=2$

$$\frac{c_1}{s+1} + \frac{c_2}{(s+1)^2}$$

$i=1$: $c_1 = \frac{d}{ds}\left[(s+1)^2 Y(s)\right]_{s=-1}$

$= \frac{d}{ds}\left[\frac{s+2}{s+3}\right]_{s=-1} = 1/4$

$i=0$: $c_2 = \left[(s+1)^2 Y(s)\right]_{s=-1} = \left.\frac{s+2}{s+3}\right|_{s=-1} = 1/2$

Find the input-output transfer function from:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

take Laplace transform:

$$sX(s) - X(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

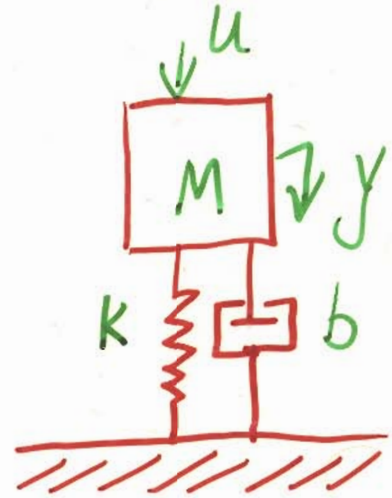
$$(sI - A)X(s) = BU(s) + \cancelto{0}{X(0)}$$

$$X(s) = (sI-A)^{-1} B\,U(s)$$

$$Y(s) = C(sI-A)^{-1} B\,U(s)$$

$$H(s) = \frac{Y(s)}{U(s)} = C(sI-A)^{-1}B$$

Example



$$M\ddot{y} + b\dot{y} + ky = u$$

choose state variables

$x_1 = y \quad x_2 = \dot{y}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$H(s) = \frac{Y(s)}{U(s)} = C\,(sI - A)^{-1} B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}{s^2 + \frac{b}{m} s + \frac{k}{m}}$$

$$H(s) = \frac{\frac{1}{m}}{s^2 + \frac{b}{m} s + \frac{k}{m}}$$

$$= \frac{1}{ms^2 + bs + k}$$

Direct Construction of the Transfer Function

Resistor	$v(t) = R\,i(t)$	$V(s) = R\,I(s)$
Capacitor	$\dot{v}(t) = \frac{1}{C}\,i(t)$	$s\,V(s) = \frac{1}{C}\,I(s)$
Inductor	$v(t) = L\,\dot{i}(t)$	$V(s) = sL\,I(s)$

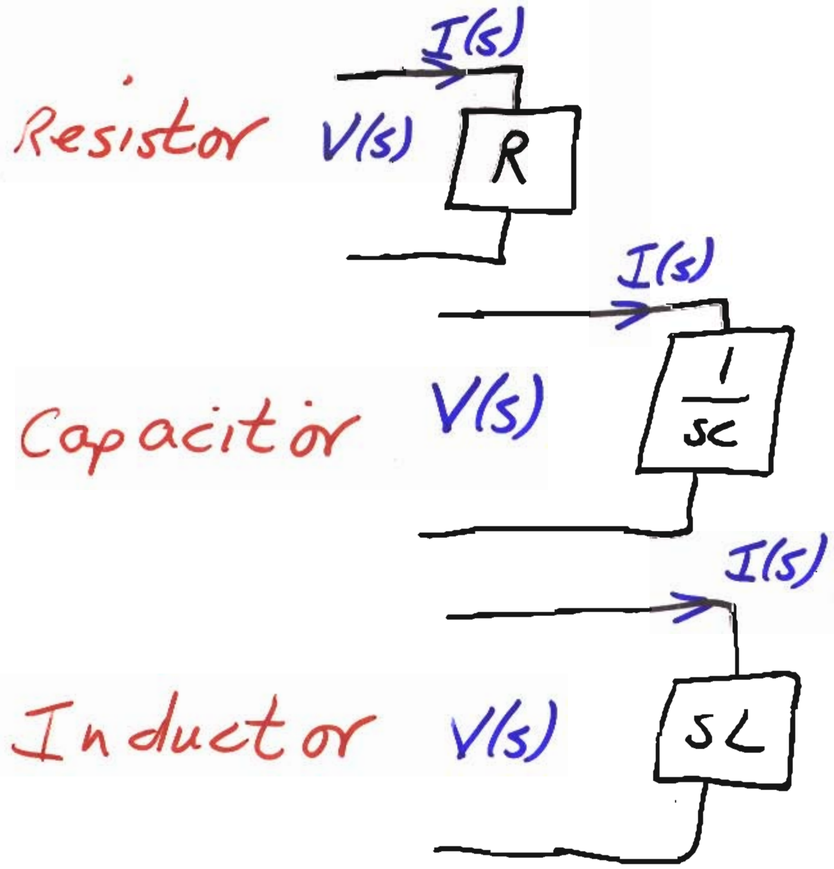
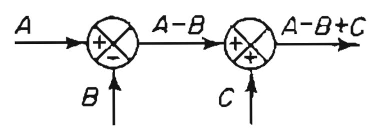
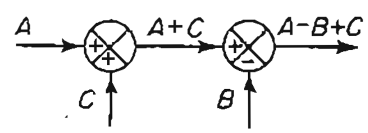
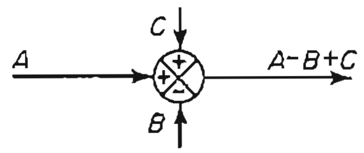
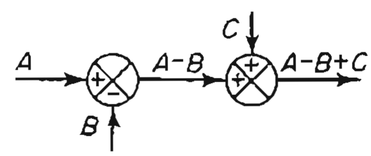
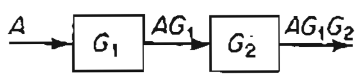
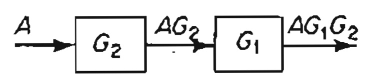
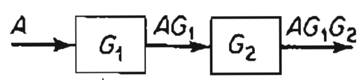
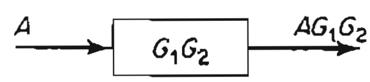
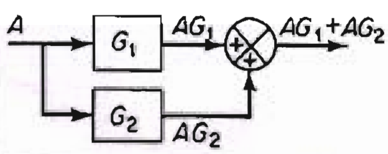
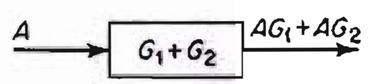
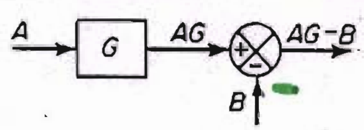
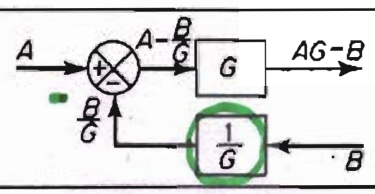
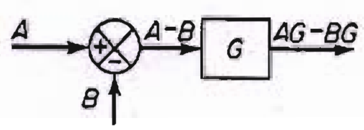
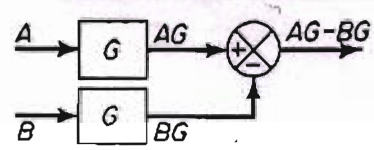
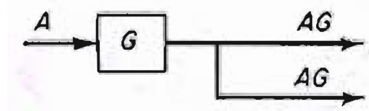
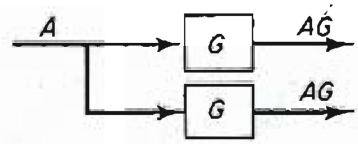
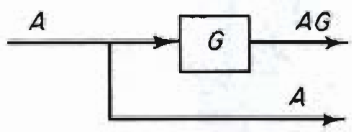
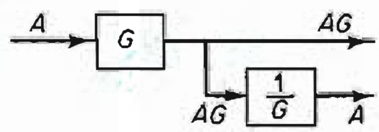
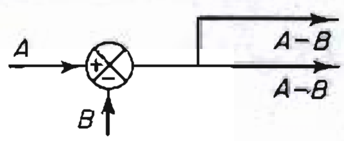
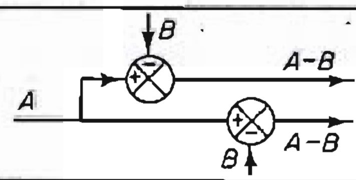
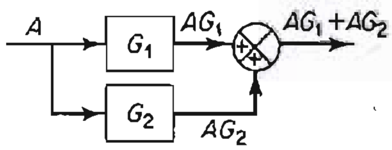
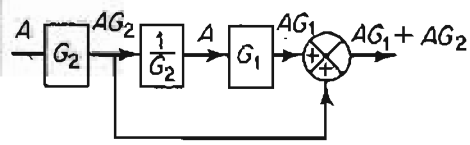
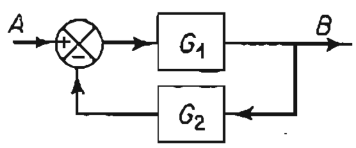
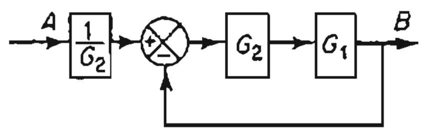


TABLE 9–1 RULES OF BLOCK DIAGRAM ALGEBRA

	Original block diagrams	Equivalent block diagrams
1		
2		
3		
4		
5		
6		
7		

	Original block diagrams	Equivalent block diagrams
8		
9		
10		
11		
12		
13