ELEC 631 Digital Filters: II

Review of Prerequisite Material Part II

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- This presentation will deal with Part II of the review.
  - The Transfer Function
  - Time-Domain Analysis

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  - The Transfer Function
  - Time-Domain Analysis
  - Frequency-Domain Analysis
  - Amplitude and Delay Distortion

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$$y(nT) = \sum_{k=-\infty}^{\infty} x(kT)h(nT - kT)$$

- If we apply the convolution theorem of the z transform, we get

$$Y(z) = \mathcal{Z}y(nT) = \mathcal{Z}h(nT)\mathcal{Z}x(nT) = H(z)X(z)$$

or

$$\frac{Y(z)}{X(z)} = H(z)$$

...

$$\frac{Y(z)}{X(z)} = H(z)$$

- In effect, *the transfer function is numerically equal to the z transform of the impulse response* $h(nT)$.

- An Nth-order causal, linear, time-invariant, recursive digital filter can be represented by the difference equation

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- The z transform gives

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- Hence

$$Y(z) = \sum_{i=0}^{N} a_i z^{-i} X(z) - \sum_{i=1}^{N} b_i z^{-i} Y(z)$$

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- Solving for $Y(z)/X(z)$ and then multiplying the numerator and denominator polynomials by z^N, we get

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{1 + \sum_{i=1}^{N} b_i z^{-i}} = \frac{\sum_{i=0}^{N} a_i z^{N-i}}{z^N + \sum_{i=1}^{N} b_i z^{N-i}} \\ &= \frac{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}{z^N + b_1 z^{N-1} + \cdots + b_N} \end{aligned}$$

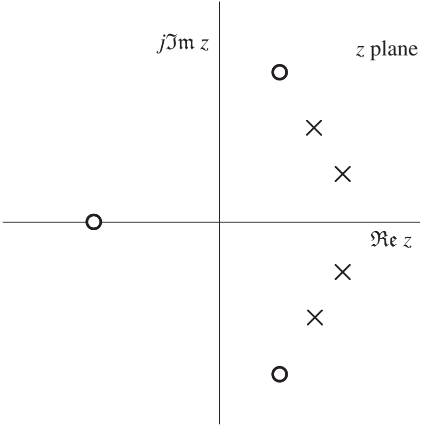
- By factorizing the numerator and denominator polynomials, the transfer function of a causal digital filter can be expressed as

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{Z}(z - z_i)^{m_i}}{\prod_{i=1}^{P}(z - p_i)^{n_i}}$$

  where

  - $z_1, z_2, \ldots, z_Z$ are the zeros of $H(z)$
  - $p_1, p_2, \ldots, p_P$ are the poles of $H(z)$
  - m_i is the order of zero z_i
  - n_i is the order of pole p_i
  - $N = \sum_{i=1}^{Z} m_i$ is the order of $N(z)$
  - $N = \sum_{i=1}^{P} n_i$ is the order of $D(z)$
  - H_0 is a multiplier constant

- Digital filters can be represented by zero-pole plots like the one shown.



- A nonrecursive digital filter can be represented by the difference equation

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Transfer Function *Cont'd*

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$$= \frac{\sum_{i=0}^{N} a_i z^{N-i}}{z^N}$$

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- Evidently, *the poles of nonrecursive digital filters are all located at the origin* of the z plane.

Transfer Function *Cont'd*

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  - then show that the impulse response is absolutely summable.

Transfer Function *Cont'd*

- It turns out that the stability of a digital filter depends entirely on the locations of the poles.

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- If the poles of the digital filter are given by

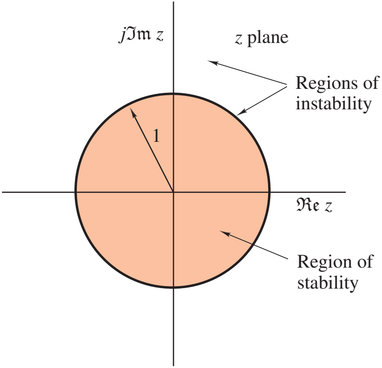
$$p_i = r_i e^{j\psi_i} \quad \text{for} \quad i = 1, 2, \ldots, N$$

then the condition that all the poles are inside the unit circle $|z| = 1$, i.e.,

$$r_i \leq r_{\max} < 1 \quad \text{for} \quad i = 1, 2, \ldots, N$$

is a necessary and sufficient condition that the impulse response be *absolutely summable*.

- Therefore, we conclude that *a digital filter is stable if and only if all its poles are inside the unit circle of the z plane.*



Note: Nonrecursive digital filters are always stable since their poles are always located at the *origin* of the z plane.

Time-Domain Response

- A digital filter with excitation $x(nT)$, response $y(nT)$, and impulse response $h(nT)$ can be represented by the equation

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- The inverse z transform can be obtained by using any one of the standard inversion techniques described in Chap. 3.

- The impulse response of a digital filter is obtained by finding the response produced by an input $x(nT) = \delta(nT)$.

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- Since

$$X(z) = \mathcal{Z}x(nT) = \mathcal{Z}\delta(nT) = 1$$

  we get

$$h(nT) = \mathcal{Z}^{-1}[H(z)]$$

  as is to be expected.

- Similarly, the unit-step response is obtained by finding the response produced by an input $x(nT) = u(nT)$.

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- Since the z transform of a unit step is given by

$$X(z) = \mathcal{Z}u(nT) = \frac{z}{z-1}$$

  we get

$$y_u(nT) = \mathcal{Z}^{-1}\frac{H(z)z}{z-1}$$

- The z transform of a sinusoidal signal that starts at time zero is given by

$$
\begin{aligned}
X(z) = \mathcal{Z}[u(nT)\sin\omega nT] &= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \\
&= \frac{z\sin\omega T}{(z - e^{j\omega T})(z - e^{-j\omega T})}
\end{aligned}
$$

Time-Domain Response *Cont'd*

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- Hence the response of a digital filter to a sinusoidal excitation can be obtained as

$$y(nT) = \mathcal{Z}^{-1}[H(z)X(z)] = \mathcal{Z}^{-1}\frac{H(z)(z\sin\omega T)}{(z - e^{j\omega T})(z - e^{-j\omega T})}$$

- For a transfer function with simple poles

$$H(z) = \frac{N(z)}{D(z)} = \frac{H_0 \prod_{i=1}^{N}(z - z_i)}{\prod_{i=1}^{N}(z - p_i)}$$

the sinusoidal response is given by

$$y(nT) = u(nT) \sum_{i=1}^{N+2} \text{res}_{\ z=p_i} [H(z)X(z)z^{n-1}]$$

- After some fairly extensive analysis, the sinusoidal response of a digital filter can be expressed as a sum of two components as

$$y(nT) = y_{\mathrm{TR}}(nT) + \tilde{y}(nT)$$

where

$$\begin{aligned} y_{\mathrm{TR}}(nT) &= \sum_{i=1}^{N} X(p_i) p_i^{n-1} \mathrm{res}_{\,z=p_i} H(z) \\ \tilde{y}(nT) &= \frac{1}{2j}[H(e^{j\omega T})e^{j\omega nT} - H(e^{-j\omega T})e^{-j\omega nT}] \end{aligned}$$

- For a stable filter, we have

$$\lim_{n\to\infty} y_{\mathrm{TR}}(nT) = \lim_{n\to\infty} \sum_{i=1}^{N} X(p_i)[r_i^{n-1} e^{j(n-1)\psi_i}]\mathrm{res}_{\,z=p_i} H(z) \to 0$$

i.e., *$y_{TR}(nT)$ is a transient component* which tends to zero as $n \to \infty$.

- For a stable filter, the second component, $\tilde{y}(nT)$, can be expressed as

$$\begin{aligned}\tilde{y}(nT) &= \lim_{n\to\infty} y(nT) = \frac{1}{2j}[H(e^{j\omega T})e^{j\omega nT} - H(e^{-j\omega T})e^{-j\omega nT}] \\ &= M(\omega)\sin[\omega nT + \theta(\omega)]\end{aligned}$$

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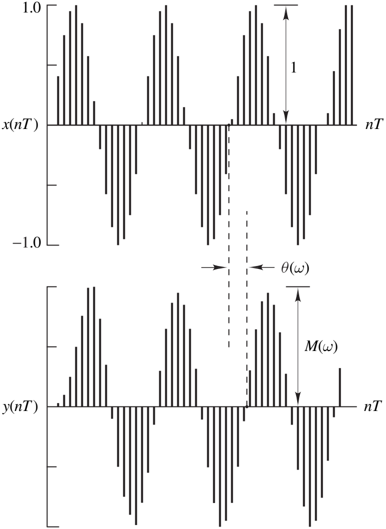
- In effect, $\tilde{y}(nT)$ is the *steady-state sinusoidal response* of the filter.
- Since the sinusoidal input was assumed to have unit amplitude and zero phase angle, we conclude that *a stable digital filter will multiply the amplitude of a sinusoidal input by $M(\omega)$ and increase its phase angle by $\theta(\omega)$*.


1.0
−1.0
$x(nT)$
nT
1
$\theta(\omega)$
$M(\omega)$
$y(nT)$
nT

- The quantity

$$M(\omega) = |H(e^{j\omega T})|$$

is said to be the *gain* of the digital filter at frequency ω.

The gain can vary over many orders of magnitude and is often expressed in *decibels (dB)* as $20 \log M(\omega)$.

Time-Domain Response *Cont'd*

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- The quantity

$$\theta(\omega) = \arg H(e^{j\omega T})$$

is said to be the *phase shift* at frequency ω.

It is measured in degrees or radians.

Frequency-Domain Response

- As a function of frequency, ω, the quantity

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- The complex quantity $H(e^{j\omega T})$, which includes the gain and phase angle of the filter, is said to be the *frequency response*.

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- Frequency-domain analysis is the *process of obtaining the frequency, amplitude, or phase response* of a digital filter.

Frequency-Domain Response *Cont'd*

- The frequency response of a digital filter can be obtained by letting $z = e^{j\omega T}$ in the discrete-time transfer function $H(z)$.

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- This amounts to evaluating the transfer function on the unit circle $|z| = 1$ of the z plane.

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- This amounts to evaluating the transfer function on the unit circle $|z| = 1$ of the z plane.
- If we let $z = e^{j\omega T}$ in the transfer function

$$H(z) = \frac{H_0 \prod_{i=1}^{N}(z - z_i)^{m_i}}{\prod_{i=1}^{N}(z - p_i)^{n_i}}$$

we obtain

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} = \frac{H_0 \prod_{i=1}^{N}(e^{j\omega T} - z_i)^{m_i}}{\prod_{i=1}^{N}(e^{j\omega T} - p_i)^{n_i}}$$

...

$$H(e^{j\omega T}) = M(\omega)e^{j\theta(\omega)} = \frac{H_0 \prod_{i=1}^{N}(e^{j\omega T} - z_i)^{m_i}}{\prod_{i=1}^{N}(e^{j\omega T} - p_i)^{n_i}}$$

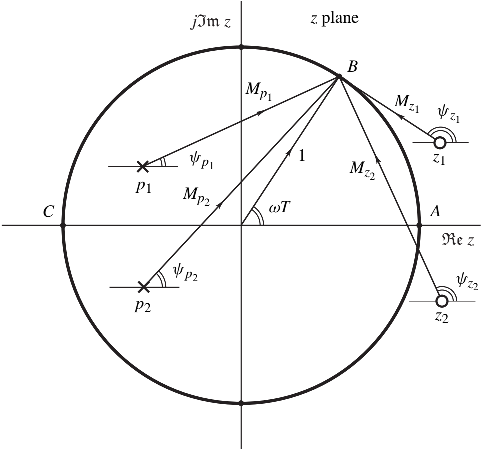
- By letting $e^{j\omega T} - z_i = M_{z_i} e^{j\psi_{z_i}}$ and $e^{j\omega T} - p_i = M_{p_i} e^{j\psi_{p_i}}$

  we get $$M(\omega) = \frac{|H_0| \prod_{i=1}^{N} M_{z_i}^{m_i}}{\prod_{i=1}^{N} M_{p_i}^{n_i}}$$

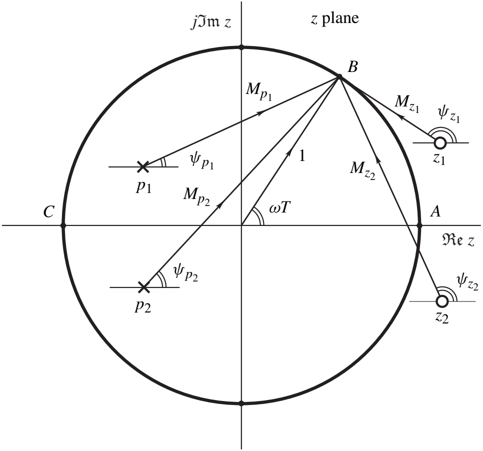
  $$\theta(\omega) = \arg H_0 + \sum_{i=1}^{N} m_i \psi_{z_i} - \sum_{i=1}^{N} n_i \psi_{p_i}$$

  where $\arg H_0 = \pi$ if H_0 is negative and is zero otherwise.

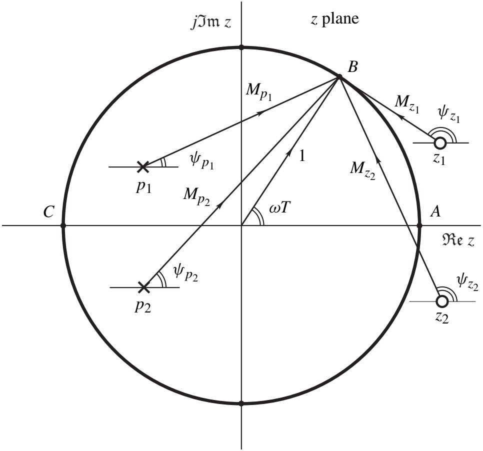
- Frequency response of second-order digital filter:



- Point A corresponds to zero frequency.



- One complete revolution from point A in the counterclockwise sense back to point A corresponds to $\Delta\omega = \omega_s = 2\pi/T$.

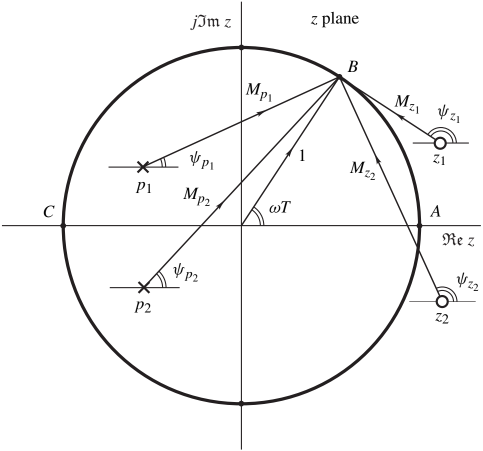


...

- One complete revolution from point A in the counterclockwise sense back to point A corresponds to $\Delta\omega = \omega_s = 2\pi/T$.

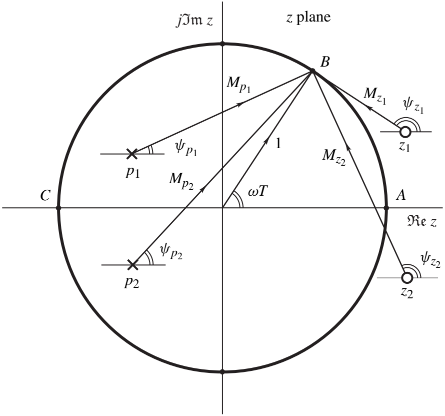
  Since T is the period between samples, ω_s is called the *sampling frequency* in rad/s.

- Point C corresponds to frequency $\pi/T = \frac{1}{2}\omega_s$ which is commonly referred to as the *Nyquist frequency*.



- If $e^{j\omega T}$ is rotated k complete revolutions, the values of $M(\omega)$ and $\theta(\omega)$ will obviously remain unchanged and so

$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$



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$$H(e^{j(\omega+k\omega_s)T}) = H(e^{j\omega T})$$

- We conclude that *the frequency response of a digital filter is periodic with period* ω_s.

- Since the frequency response of a digital filter is periodic with period ω_s, it follows that it is completely specified if it is known over the frequency range $-\omega_s/2 < \omega \leq \omega_s/2$.

  This frequency range is known as the *baseband*.

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- It can be easily shown that the amplitude response is an *even* function and the phase response is an *odd* function of ω, i.e.,

$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

Frequency-Domain Response *Cont'd*

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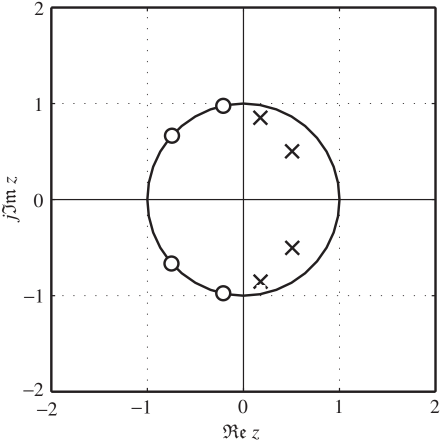
$$M(-\omega) = M(\omega) \quad \text{and} \quad \theta(-\omega) = -\theta(\omega)$$

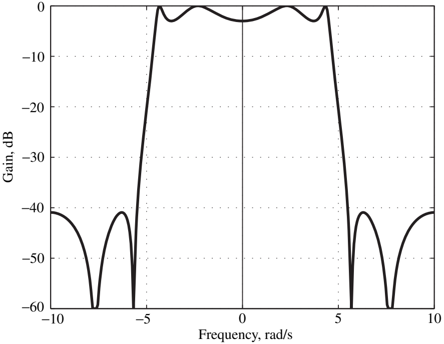
- Therefore, the frequency response is completely specified if it is known over the positive half of the baseband, i.e., $0 \leq \omega \leq \omega_s/2$.

- Elliptic Lowpass Filter:



(a)

- Amplitude response:



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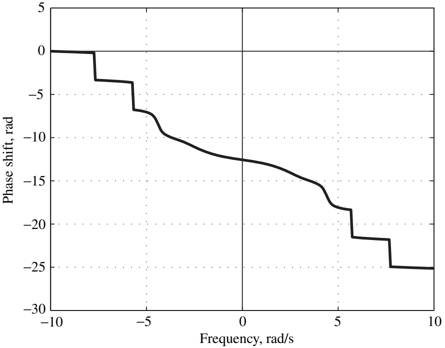
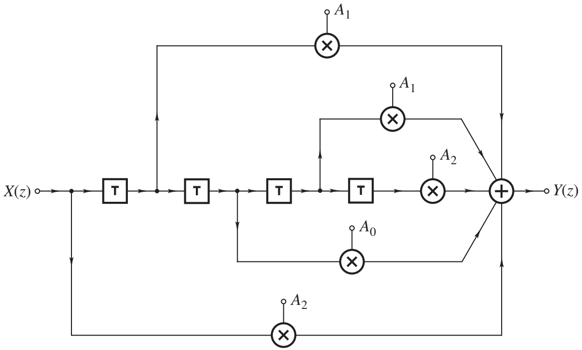
Frequency-Domain Response *Cont'd*

- Phase response:



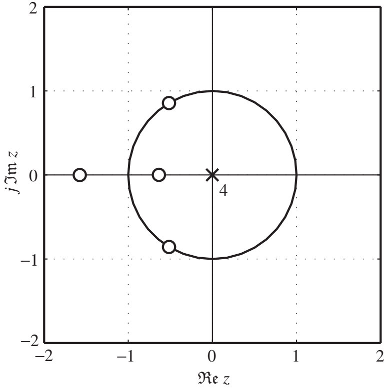
(*i*)

- Nonrecursive lowpass filter:



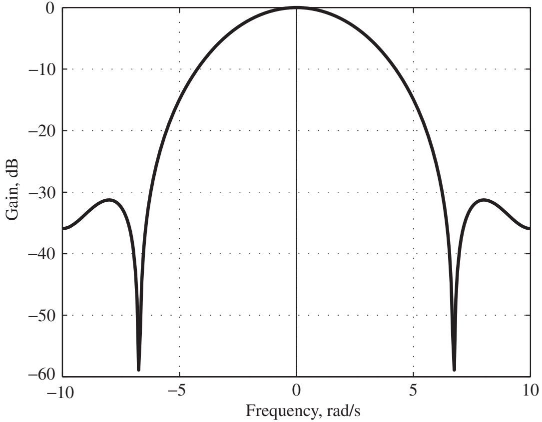
$$A_0 = 0.3352, \quad A_1 = 0.2540, \quad A_2 = 0.0784$$

- Zero-pole plot of nonrecursive lowpass filter:



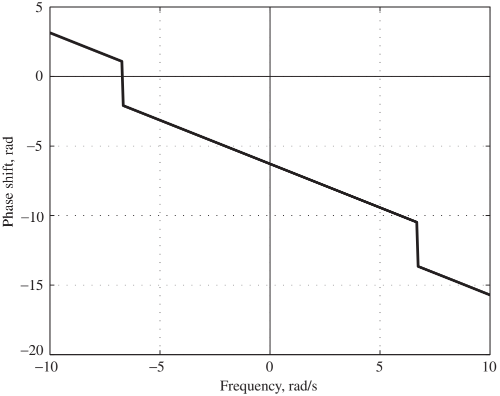
(*a*)

- Amplitude response on nonrecursive lowpass filter:



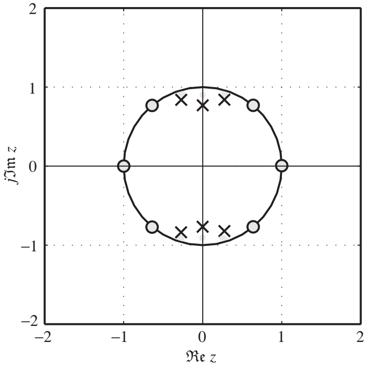
(*e*)

- Phase response on nonrecursive lowpass filter:



(f)

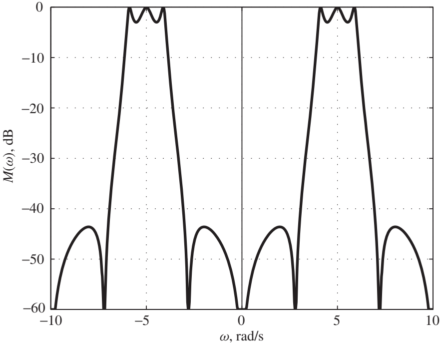
- Zero-pole plot of elliptic bandpass filter:



(*a*)

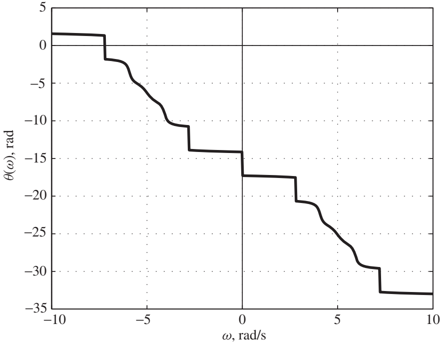
Frequency-Domain Response *Cont'd*

- Amplitude response of elliptic bandpass filter:



(*e*)

- Phase response of elliptic bandpass filter:



(f)

- Another characteristic of a digital filter is its group-delay characteristic which is a plot of the *group delay* of the filter versus frequency.

  The group-delay is defined as

$$\tau(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

- In theory, the amplitude response of a filter is required to be constant with respect to passbands and zero in stopbands.

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Frequency-Domain Response *Cont'd*

- In theory, the amplitude response of a filter is required to be constant with respect to passbands and zero in stopbands.
- Ideally, the group delay of the filter should be constant with respect to passbands but it could assume any value in stopbands.
- What amounts to the same thing, the phase response of the filter should be linear function of frequency with respect to passbands.

- If the amplitude response of a digital filter is not constant in one or more passbands, then so-called *amplitude distortion* will be introduced since different frequency components of the signal will be amplified by different amounts.

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- If the group delay is not constant or, equivalently, if the phase response is not linear, with respect to one or more passbands, different frequency components will be delayed by different amounts, and *delay (or phase) distortion* will be introduced.

This slide concludes Part II of the review.
Thank you for your attention.