Robust Signal Recovery Approach for Compressive Sensing Using Unconstrained Optimization

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Outline

1. Compressive Sensing
   - Wavelet basis and sparse representations
   - Sensing problem and incoherent sampling
   - Robust signal recovery

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   - The ℓ_1 norm and beyond
   - The smoothly clipped absolute deviation

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5. Conclusions

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  - Gradient projection for sparse reconstruction (GSPR).

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- We propose a robust signal recovery approach for compressive sensing using unconstrained optimization.
- We employ a convex and differentiable quadratic approximation of the smoothly clipped absolute deviation (SCAD) as the sparsity promoting function.

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- The coefficient vector **a** is assumed to be sparse in the sense that it has only $\mathcal{S}$ *nonzero values* and $\mathcal{S} < \mathcal{N}$.

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- The incoherence $\mu\left(\boldsymbol{\Theta}, \boldsymbol{\Psi}\right)$ is *low*.

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- We would then be inclined to replace the ℓ_1 by the ℓ_0 norm in robust signal recovery process.
- The solution of the new optimization problem could recover sparse solutions with *much fewer* measurements $\mathcal{Q}$.
- This is of little practical use as the optimization problem becomes nonconvex requiring an intractable combinatorial search.

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  - The SCAD performs as well as the *oracle* estimator, i.e., as if the coefficients which are zero were known.
  - Fan and Li proposed a local quadratic approximation (LQA) of the SCAD which renders the problem *computationally tractable*.

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  - We ran 500 recovery trials with each approach for several sparsity values $\mathcal{S}$.

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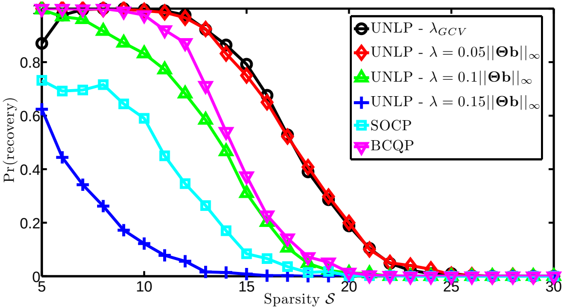
Proposed Robust Recovery ($\sigma = 10^{-4}$)



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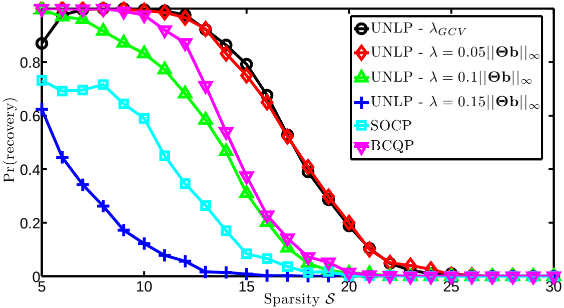

- Solid lines correspond to the probability of "perfect" signal recovery, such that $||\mathbf{a} - \mathbf{a}^*||_{\ell_\infty} \leq 10^{-3}$.

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Numerical Simulations (cont.)

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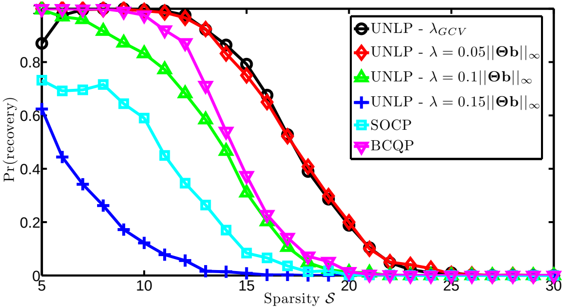

- *Marked improvement* in signal recovery with proposed UNLP over the SOCP and BCQP formulations for a *good choice* of λ.

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Numerical Simulations (cont.)

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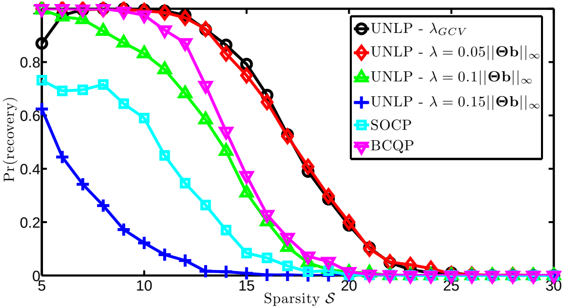

- These *improvements* come with an *added computational cost* of roughly 2 to 3 times that required for the SOCP and BCQP.

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Conclusions

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Thank you for your attention.