On the Roots of Digital Signal Processing 300 BC to 1770 AD

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Introduction

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- By the 1500s, collections of numbers in the form of numerical tables began to be published, which were used to facilitate the calculations required in business and commerce, in the emerging new sciences, and in navigation.
- Simultaneously, mathematical techniques began to evolve that could be used to generated numerical tables or to enhance their usefulness.

Introduction *Cont'd*

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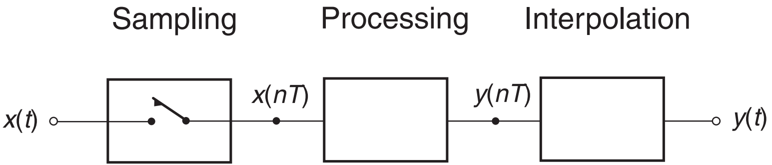
Notes:

1. This presentation is based on an article published in the IEEE Circuits and Systems Magazine [Antoniou, 2007].
2. References appear at the end of the slide presentation.

What is DSP?

In order to process a continuous-time signal, we need to perform three operations:

- Sample and digitize the signal.



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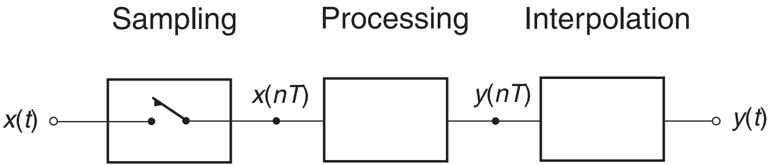

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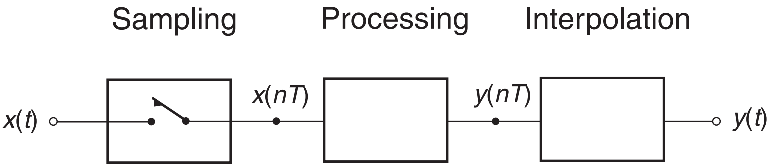
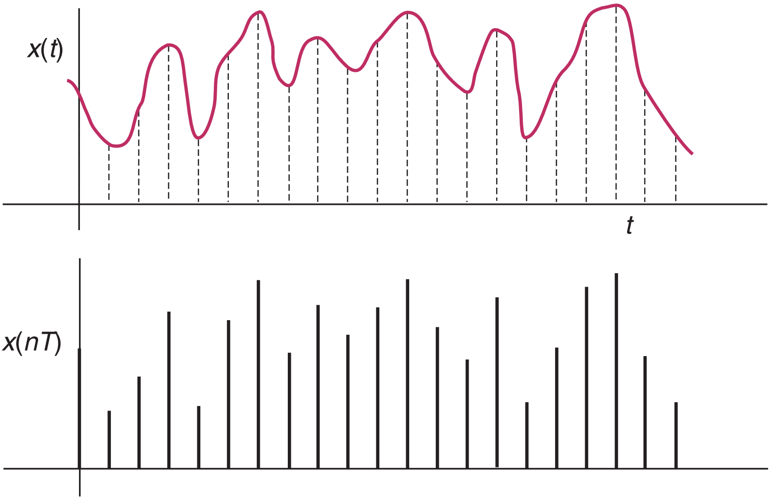
In order to process a continuous-time signal, we need to perform three operations:

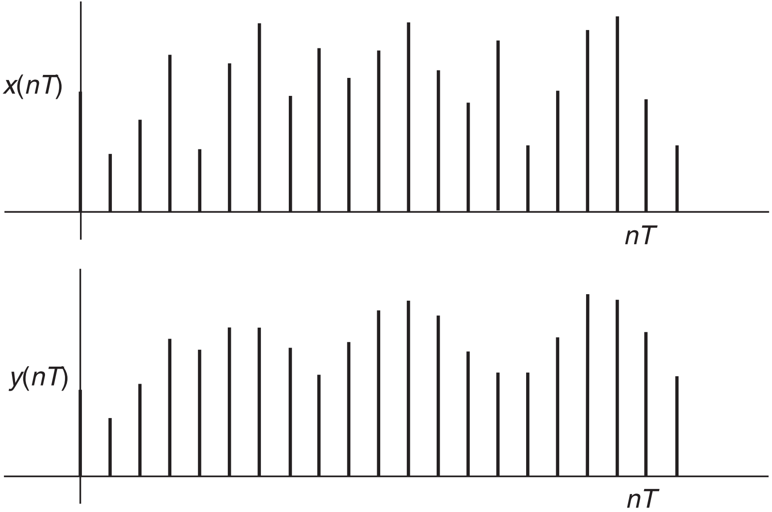
- Sample and digitize the signal.
- Process the digitized signal.
- Apply interpolation to the processed digitized signal to generate a processed version of the continuous-time signal.



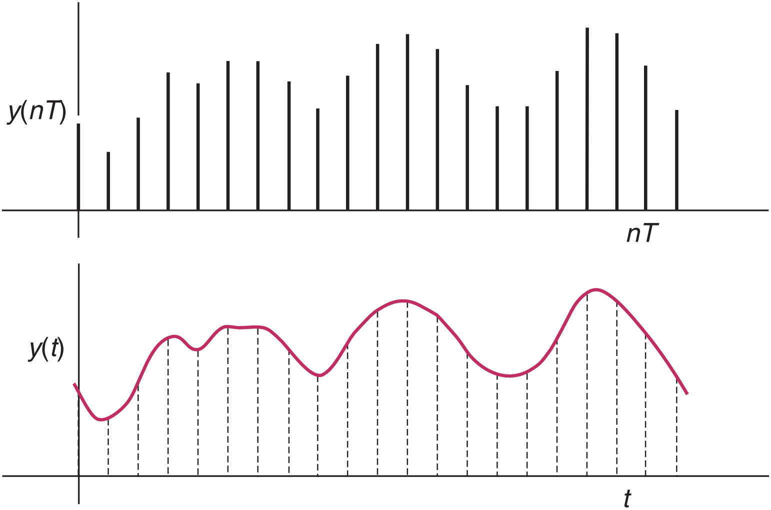
Sampling and Digitization



Processing



Interpolation



What is DSP? *Cont'd*

To trace the origins of DSP, we must, therefore, trace the origins of the fundamental processes that make up DSP, namely,

- Sampling
- Processing
- Interpolation

Archimedes of Syracuse

- Archimedes was born in Syracuse, Sicily, and lived during the period 287-212 BC.

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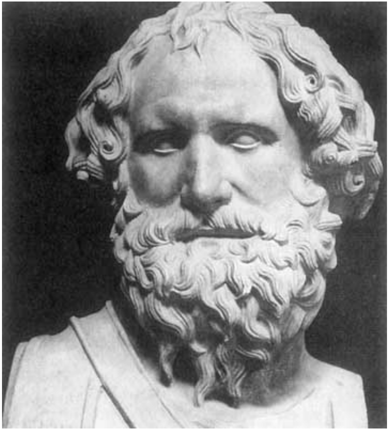
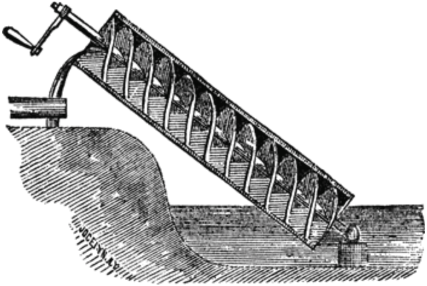
- He is most famous for the *the Archimedes principle* which gives the weight of a body immersed in a liquid.



Note: This image and some others to follow originate from The MacTutor History of Mathematics Archive [Indexes of Biographies].

Archimedes of Syracuse

- He was a great mathematician, developed fundamental theories for mechanics, and is credited for many inventions, like the Archimedes screw, and other things.



Note: This image originates from Wikipedia [Archimedes Screw].

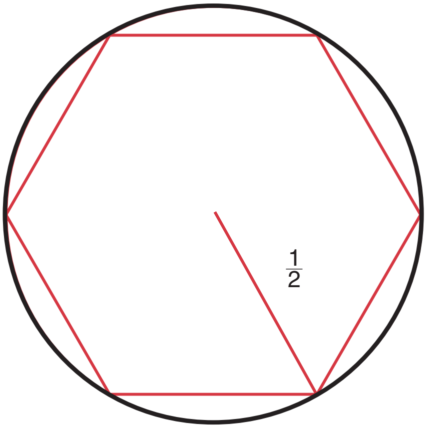
Archimedes of Syracuse

- Archimedes was the first person to propose a formal method for the calculation of π.

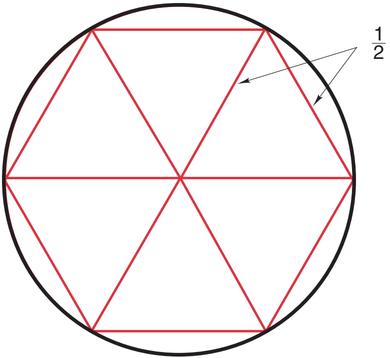
  As will be demonstrated in the slides that follow, *Archimedes' method entails both sampling as well as interpolation*.

Archimedes' Evaluation of π

- A lower bound for π can be readily obtained by inscribing a hexagon inside a circle of radius $\frac{1}{2}$.



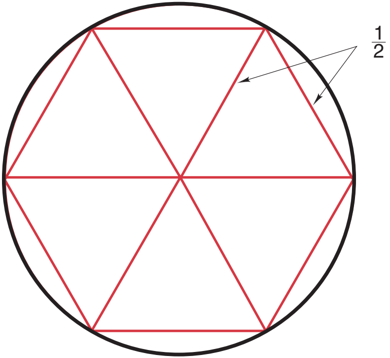
- The regular hexagon can be broken down into 6 equilateral triangles; hence the perimeter of the hexagon, denoted as p_6, is $6 \times \frac{1}{2} = 3$, i.e, $p_6 = 3$.



p_6=3

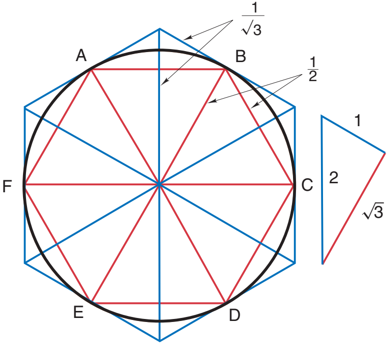
- The perimeter of the inscribed hexagon is obviously smaller than the circumference of the circle, which is $2\pi \times$ radius $= \pi$, i.e.,

$$3 < \pi$$



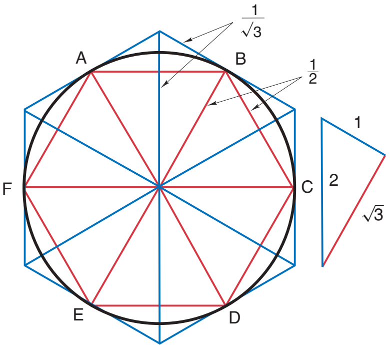
p_6=3

- An upper bound for π can be readily obtained by circumscribing a circle of radius $\frac{1}{2}$ by a hexagon.
- Draw tangents at points A, B, C, D, E, and F as shown.



$P_6 = 2\sqrt{3}$

- The perimeter of the larger hexagon is given by $P_6 = 6 \times 1/\sqrt{3} = 2\sqrt{3} = 3.4641$.



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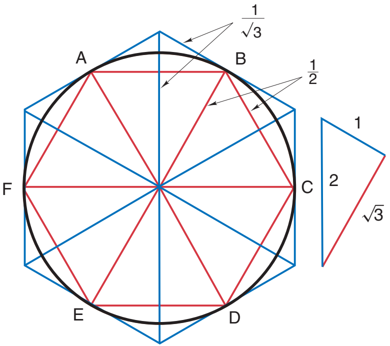
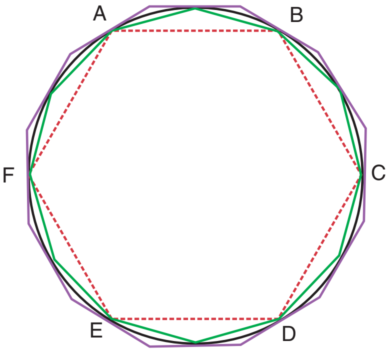
- The circumference of the circle, π, is smaller that the perimeter of the larger hexagon; hence we have

$$p_6 = 3 < \pi < 2\sqrt{3} = P_6$$

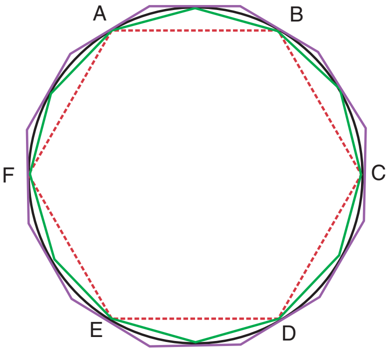


$P_6 = 2\sqrt{3}$

- Tighter lower and upper bounds on π can be readily obtained by using 12-sided regular polygons (dodecagons) instead of 6-sided ones, as shown below.



- The inside dodecagon is obtained by drawing straight lines that divide the arcs AB, BC, etc.
- The outside dodecagon is obtained by drawing tangents at the 12 vertices of the inside dodecagon.



- Geometry will show that the perimeters of the larger and smaller dodecagons are given by

$$P_{12} = \frac{2p_6P_6}{p_6 + P_6} \quad \text{and} \quad p_{12} = \sqrt{p_6P_{12}}$$

respectively, or

$$P_{2\times6} = \frac{2p_6P_6}{p_6 + P_6} = \frac{2 \times 3 \times 3.4641}{3 + 3.4641} = 3.2154$$

and

$$p_{2\times6} = \sqrt{p_6P_{2\times6}} = \sqrt{3 \times 3.2154} = 3.1058$$

Evaluation of π *Cont'd*

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- Therefore, we have

$$3 < 3.1058 < \pi < 3.2154 < 3.4641$$

or

$$p_6 < p_{2\times 6} < \pi < P_{2\times 6} < P_6$$

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- Archimedes found out that the same procedure can be repeated with 24-sided, 48-sided, and 96-sided regular polygons.

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- He also found out that the perimeters of successive outside and inside polygons can be evaluated (in today's mathematical notation) as

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respectively (see [Burton, 2003] for details).

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Note: Archimedes' method was formulated in terms of geometry. Algebra did not emerge as a subject of study until the 800s AD when a man by the name of al-Khwarizmi wrote two books on arithmetic and algebra.

Table 1 Bounds for π

No. of sides	Lower bound	Upper bound
6	3.0000	3.4641
12	3.1058	3.2154
24	3.1326	3.1597
48	3.1394	3.1461
96	3.1410	3.1427

- Archimedes repeated his procedure 5 times but stopped with 96-sided polygons.

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- He concluded that the perimeter of the outside polygon is larger than that of the circle whereas the perimeter of the inner polygon is smaller than that of the circle in each case by *a grain of sand (ε in today's terminology)*.

Evaluation of π *Cont'd*

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  which, fittingly enough, is known as the *Archimedean π*.
- This entails an error of 0.04%.
- The average of the Archimedean upper and lower bounds entails an error of 0.01%!

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- Interest in π remained very strong through the ages. See [History of Pi] for more information.

- In his effort to calculate π, Archimedes was, in effect, the first to apply sampling — the different polygons are discrete approximations of the perimeter of the circle.

Archimedes' Contribution to the roots of DSP

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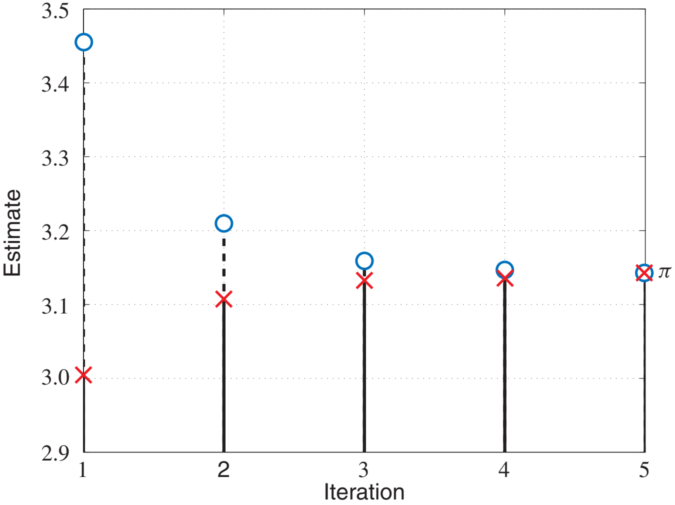
- In obtaining the Archimedean π, i.e., 22/7, he applied interpolation for the first time.
- The procedure he used to obtain progressively tighter lower and upper bounds is in reality a recursive algorithm, most probably the first recursive algorithm described in Western literature.


3.5
3.4
3.3
3.2
3.1
3.0
2.9
Estimate
1
2
3
4
5
Iteration
π

Death of Archimedes

Interpolation

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- Induction and interpolation techniques as we know them today began to emerge during the 1600s and important contributions were made by
  - John Wallis (1616–1673)
  - James Gregory (1638–1675)
  - Isaac Newton (1643–1727)

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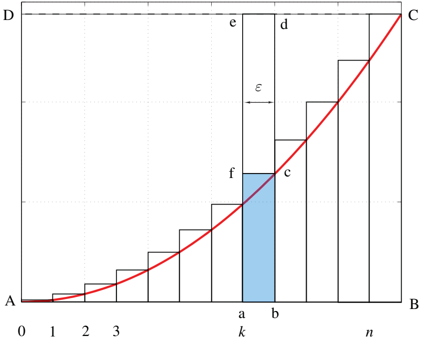
- Wallis was a clergyman but spent most of his life as an accomplished mathematician.
- They say that he was the leading English mathematician before Newton.

- Wallis considered the area under the parabola

$$y = x^2$$

to be made up of a series of elemental rectangles:

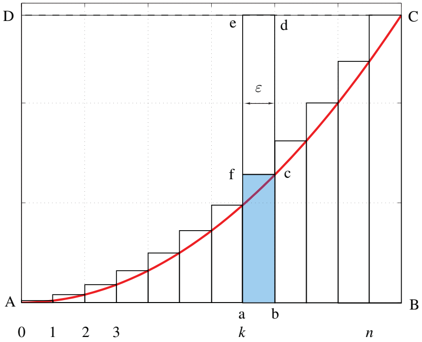


(*a*)

He noted that

$$\text{Area abcfa} \approx (k\varepsilon)^2 \cdot \varepsilon = k^2\varepsilon^3$$

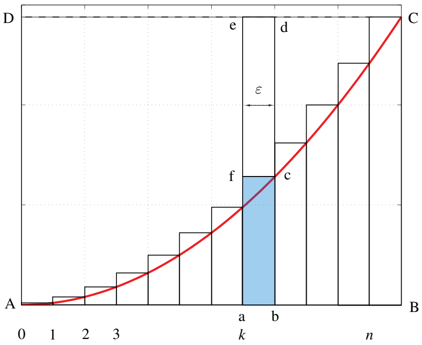
$$\text{Area abdea} = (n\varepsilon)^2 \cdot \varepsilon = n^2\varepsilon^3$$



(a)

- Therefore, the area under parabola AC, designated as A_P, can be expressed in terms of the area of rectangle ABCDA, A_R, as

$$A_P \approx \frac{(0^2 + 1^2 + 2^2 + \cdots + n^2)\varepsilon^3}{(n^2 + n^2 + n^2 + \cdots + n^2)\varepsilon^3} \cdot A_R$$



(*a*)

By applying induction, he deduced the following result:

$$\frac{0^2+1^2}{1^2+1^2} = \frac{1}{2} = \frac{1}{3}+\frac{1}{6}$$
$$\frac{0^2+1^2+2^2}{2^2+2^2+2^2} = \frac{5}{12} = \frac{1}{3}+\frac{1}{12}$$
$$\frac{0^2+1^2+2^2+3^2}{3^2+3^2+3^2+3^2} = \frac{7}{18} = \frac{1}{3}+\frac{1}{18}$$
$$\vdots$$
$$\frac{0^2+1^2+2^2+\cdots+n^2}{n^2+n^2+n^2+\cdots+n^2} = \frac{1}{3}+\frac{1}{6n}$$

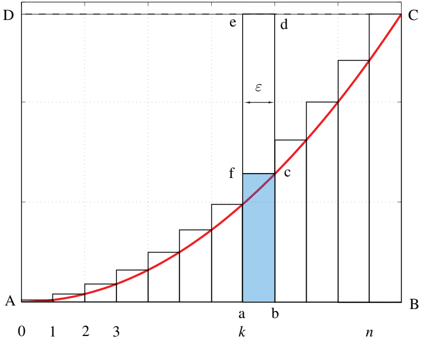
- Then he did something that was never done before:

  He made the base of each of the elemental rectangles infinitesimally small and to compensate for that he made the number of rectangles infinitely large, in today's language, and concluded that

$$\begin{aligned} A_P &= \lim_{n\to\infty} \frac{(0^2+1^2+2^2+\cdots+n^2)\varepsilon^3}{(n^2+n^2+n^2+\cdots+n^2)\varepsilon^3} \cdot A_R \\ &= \lim_{n\to\infty} \left(\frac{1}{3}+\frac{1}{6n}\right) A_R \\ &= \frac{1}{3} A_R \end{aligned}$$

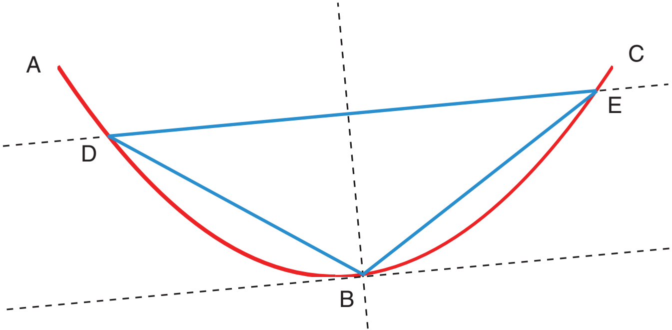
  See [Burton, 2003] for details.

- In effect, *the area below the parabola is one-third the area of the rectangle that contains the parabola.*



(a)

- Actually, the result turned out to be a trivial special case of a result due to the great Archimedes himself, which states that *the area enclosed by parabola ABC and line DE shown below is equal to four-thirds the area of triangle DEB.*



See [Boyer, 1991] for details.

John Wallis *Cont'd*

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John Wallis *Cont'd*

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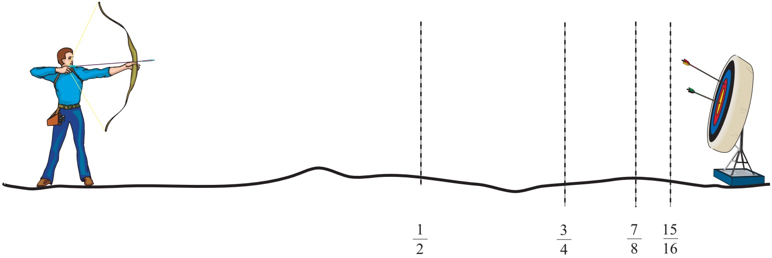
- He introduced the concept of the limit thereby resolving *Zeno's Paradox* once and for all.
- He also coined the term *interpolation* and proposed the symbol for infinity we use today (∞) according to historians.

Zeno's Paradoxes

- Zeno of Elea conceived many paradoxes and a typical example is as follows.
- The arrow below must traverse half the distance to the target before reaching the target and after that it must traverse half of the remaining distance, and so on?



Zeno's Paradox *Cont'd*

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Zeno's Paradox *Cont'd*

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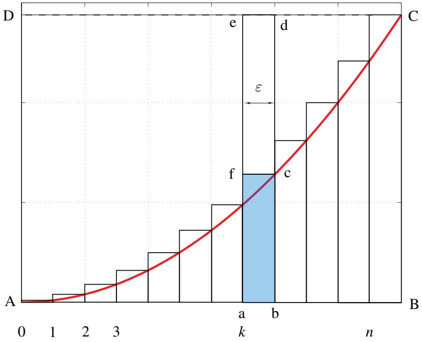
- For the same reason, Achilles, who was the fastest runner in Greece, would never be able to catch up with a tortoise that has been given a head start!

- Therefore, the arrow will never hit the target because a small distance to the target will always remain!
- For the same reason, Achilles, who was the fastest runner in Greece, would never be able to catch up with a tortoise that has been given a head start!
- The riddle is immediately solved by noting that an infinite sum of numbers can have a finite value, for example

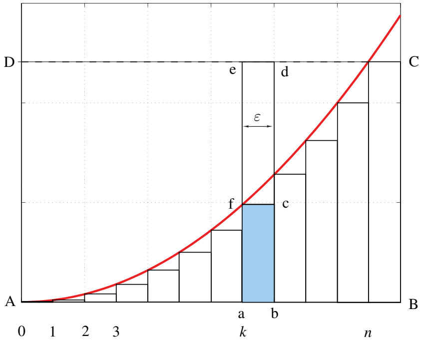
$$\lim_{n\to\infty} \sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n = 1.0$$

- What Wallis did, in effect, was to discretize the parabola by circumscribing it in terms of a piecewise-constant function in the same way as Archimedes had discretized the circle by circumscribing it by an *n*-sided polygon.



(a)

- He could have achieved the same result by inscribing a piecewise-constant function in the parabola as shown below, which is quite analogous to a continuous-time signal that has been subjected to the *sample-and-hold operation*.

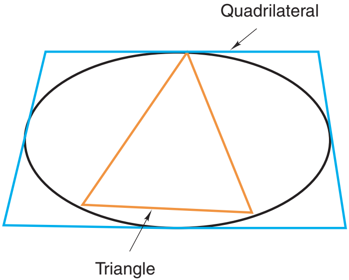


(b)

James Gregory

- James Gregory (1638–1675), a Scot mathematician, extended the results of Archimedes on the area of the circle to the area of the ellipse [Boyer, 1991].

- He inscribed a triangle of area a_0 in the ellipse and circumscribed the ellipse by a quadrilateral of area A_0, as shown.



- By successively doubling the number of sides of the triangles and quadrilaterals, he generated the sequence

$$a_0,\ A_0,\ a_1,\ A_1\ \ldots a_n,\ A_n,\ \ldots$$

using the recursive relations

$$a_n = \sqrt{a_{n-1}A_{n-1}} \quad \text{and} \quad A_n = \frac{2A_{n-1}a_n}{A_{n-1} + a_n}$$

James Gregory *Cont'd*

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- Then he arranged the elements of the sequence obtained into two sequences, as follows:

$$a_0,\ a_1,\ \ldots a_n,\ \ldots \quad \text{and} \quad A_0,\ A_1,\ \ldots A_n,\ \ldots$$

- He concluded that each of the two sequences would, in his words, *converge* to the area of the ellipse if n were made infinitely large.

- He concluded that each of the two sequences would, in his words, *converge* to the area of the ellipse if n were made infinitely large.
- Although he died is his thirties, he is known for several other achievements:
  - He is known for his work on series.

    In fact,

    $$\int_0^x \frac{1}{1+x^2}\,dx = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

    is known as *the Gregory series* [Boyer, 1991].
  - He is known along with Newton for the *Gregory-Newton interpolation formula*.
  - Interestingly, *he discovered the Taylor series*, some 44 years before it was published by Brook Taylor (1685–1731).

Newton

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- His most important contribution to the roots of DSP other than calculus is the binomial theorem.
- He started with the numerical values of the area

$$\int_0^1 (1 - t^2)^n \, dt$$

(in today's notation) for certain integer values of n, which were estimated by Wallis a few years earlier using an induction method (recall that there was no calculus at that time).

• • •

$$\int_0^1 (1 - t^2)^n \, dt$$

- By replacing the upper limit in the integration shown by x, he was able to obtain the following results:

$$\int_0^x (1 - t^2) \, dt = x - \tfrac{1}{3}x^3$$
$$\int_0^x (1 - t^2)^2 \, dt = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5$$
$$\int_0^x (1 - t^2)^3 \, dt = x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7$$

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Then through some laborious interpolation he found out that

$$\int_0^x (1-t^2)^{\frac{1}{2}}\,dt = x - \frac{\frac{1}{2}}{3}x^3 - \frac{\frac{1}{8}}{5}x^5 - \cdots$$

- The amazing regularity of his solutions led him to conclude that

$$\int_0^x (1-t^2)^k\,dt = x - \tfrac{1}{3}\binom{k}{1}x^3 + \tfrac{1}{5}\binom{k}{2}x^5 - \cdots + \frac{1}{2n+1}\binom{k}{n}x^{2n+1} - \cdots$$

where

$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n}$$

• • •

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Good as he was with the method of tangents (differentiation), he differentiated both sides to obtain

$$(1-x^2)^k = 1 - \binom{k}{1}x^2 + \binom{k}{2}x^4 - \cdots + \binom{k}{n}x^{2n} - \cdots$$

• • •

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Finally, if we replace $-x^2$ by x, the binomial series in its standard form is revealed:

$$(1+x)^k = 1 + \binom{k}{1}x + \binom{k}{2}x^2 + \cdots + \binom{k}{n}x^n + \cdots$$

See [Burton, 2003]

Newton *Cont'd*

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Newton *Cont'd*

- The binomial series for integer values of n was known long before Newton in terms of the Pascal triangle but it was not discovered by Pascal.
- It first appeared in a treatise written by a Chinese mathematician by the name of Chu Shih-chieh (circa 1260-1320).

$$\begin{array}{ccccccccccc} \multicolumn{11}{c}{\text{Pascal Triangle}} \\ &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 1 && 2 && 1 &&& \\ && 1 && 3 && 3 && 1 && \\ & 1 && 4 && 6 && 4 && 1 & \\ 1 && 5 && 10 && 10 && 5 && 1 \\ \vdots &&&&& \vdots &&&&& \vdots \end{array}$$

Newton *Cont'd*

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  Note: Some say that Gauss 'discovered' the binomial theorem at the age of 15 without knowledge of Newton's work.

The Binomial Theorem in DSP

- If we replace variable x in the binomial series by z^{-1} and allow z to be a complex variable, then we get

$$(1+z^{-1})^k = 1 + \binom{k}{1} z^{-1} + \binom{k}{2} z^{-2} + \cdots + \binom{k}{n} z^{-n} + \cdots$$

which is referred to in the DSP literature as the z transform of right-sided signal

$$x(nT) = u(nT)\binom{k}{n}$$

where $u(nT)$ is the discrete-time unit-step function.

The Binomial Theorem in DSP

- Now if we expand the function

$$X(z) = \frac{Kz^m}{(z-w)^k}$$

into a binomial series, where m and k are integers, and K and w are real or complex constants, a whole table of z transform pairs can be deduced [Antoniou, 2005].

Table of Standard z Transforms

$x(nT)$	$X(z)$
$u(nT)$	$\frac{z}{z-1}$
$u(nT-kT)K$	$\frac{Kz^{-(k-1)}}{z-1}$
$u(nT)Kw^n$	$\frac{Kz}{z-w}$
$u(nT-kT)Kw^{n-1}$	$\frac{K(z/w)^{-(k-1)}}{z-w}$
$u(nT)e^{-\alpha nT}$	$\frac{z}{z-e^{-\alpha T}}$
$u(nT)nT$	$\frac{Tz}{(z-1)^2}$
$u(nT)nTe^{-\alpha nT}$	$\frac{Te^{-\alpha T}z}{(z-e^{-\alpha T})^2}$

Interpolation

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  The so-called *barycentric form* of the Lagrange interpolation formula is used to facilitate the application of the Remez algorithm for the design of nonrecursive (FIR) filters.
- Interpolation formulas were also proposed by Carl Fredrich Gauss (1777–1855) and Wilhelm Bessel (1784–1846).

Stirling Interpolation Formula

If the values of a discrete-time signal $x(nT)$ are known at 0, T, $2T$, ..., then the value of $x(nT + pT)$ for some value of p in the range $0 < p < 1$ can be determined as

$$\begin{aligned} x(nT + pT) = & \left[1 + \frac{p^2}{2}\delta^2 + \frac{p^2(p^2-1)}{4}\delta^4 + \cdots\right] x(nT) \\ & + \frac{p}{2}\delta x(nT - \tfrac{1}{2}T) + \delta x(nT + \tfrac{1}{2}T) \\ & + \frac{p(p^2-1)}{2(3)}\delta^3 x(nT - \tfrac{1}{2}T) + \delta^3 x(nT + \tfrac{1}{2}T) \\ & + \frac{p(p^2-1)(p^2-2^2)}{2(5)}\delta^5 x(nT - \tfrac{1}{2}T) \\ & + \delta^5 x(nT + \tfrac{1}{2}T) + \cdots \end{aligned}$$

where $$\delta x(nT + \tfrac{1}{2}T) = x(nT + T) - x(nT)$$

is known as the *central difference*.

Neglecting differences of order 6 or higher, letting $p = 1/2$ in the interpolation formula, and then eliminating the central differences, we get (see [Antoniou, 2005] for details)

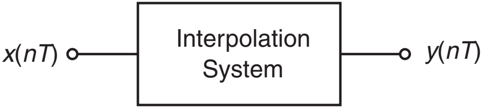
$$y(nT) = x(nT + \tfrac{1}{2}T) = \sum_{i=-3}^{3} h(iT)x(nT - iT)$$

i	$h(iT)$
−3	−5.859375E−3
−2	4.687500E−2
−1	−1.855469E−1
0	7.031250E−1
1	4.980469E−1
2	−6.250000E−2
3	5.859375E−3

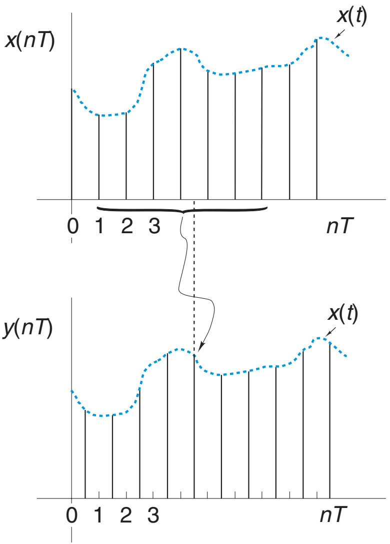
The formula

$$y(nT) = x(nT + \tfrac{1}{2}T) = \sum_{i=-3}^{3} h(iT)x(nT - iT)$$

is a *difference equation* that represents a nonrecursive discrete-time system which can perform interpolation:



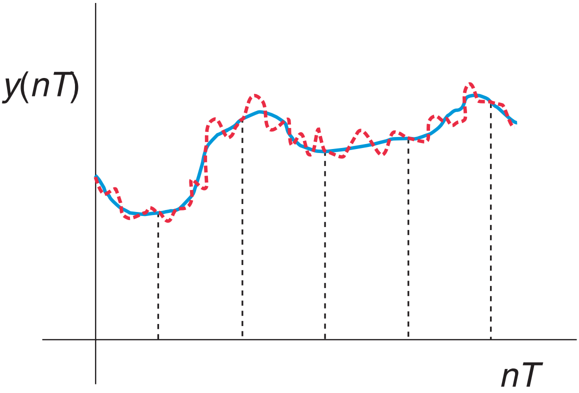
Stirling Interpolation Formula *Cont'd*



Stirling Interpolation Formula *Cont'd*

- Interpolation is a process that will fit a smooth curve through a number of sample points.

  In effect, interpolation is essentially *lowpass filtering*.



Stirling Interpolator

- To check this out, we obtain the transfer function of the interpolator as

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=-3}^{3} h(iT)z^{-k}$$

by applying the z transform to the difference equation.

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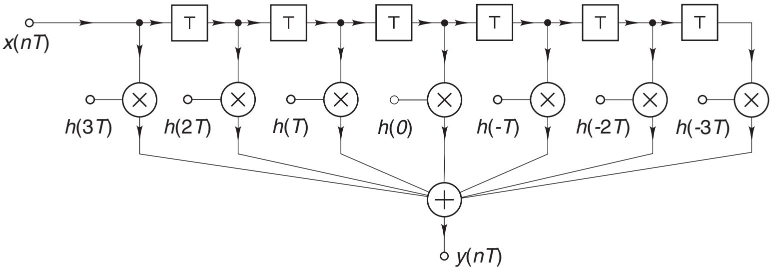
by applying the z transform to the difference equation.

- Like other methods for the design of nonrecursive systems, Stirling's interpolation formula gives a noncausal system.

However, a *causal interpolator* can be obtained by multiplying the transfer function by z^{-3} which amounts to introducing a delay of 3 sampling periods.

Hence, we get

$$H(z) = \frac{Y(z)}{X(z)} = z^{-3} \sum_{k=-3}^{3} h(iT)z^{-k}$$


x(nT)
T
T
T
T
T
T
h(3T)
h(2T)
h(T)
h(0)
h(-T)
h(-2T)
h(-3T)
y(nT)

- Evaluating the transfer function on the unit-circle of the z plane, we get the amplitude response, phase response, and group delay as

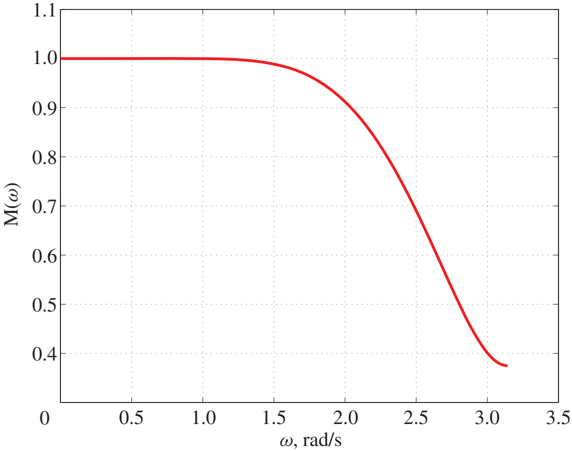
$$M_c(\omega) = \left| \sum_{i=0}^{6} h(iT) e^{-jk\omega T} \right|$$

$$\theta_c(\omega) = -3\omega + \arg \sum_{i=-3}^{3} h(iT) e^{-jk\omega T}$$

$$\tau_c(\omega) = -\frac{d\theta_c(\omega)}{d\omega}$$

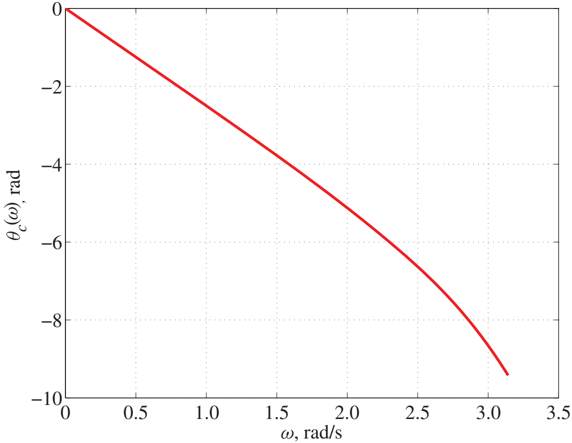
respectively.

Amplitude Response of Stirling Interpolator



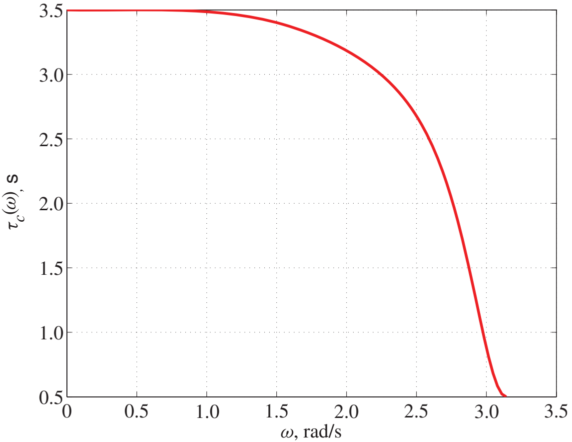
(*a*)

Phase Response of Stirling Interpolator



(b)

Group-Delay Characteristic of Stirling Interpolator



(c)

Group-Delay Characteristic of Stirling Interpolator *Cont'd*

- As anticipated, the Stirling interpolator is a nonrecursive lowpass digital filter.
- In fact it has nearly linear phase or constant group delay with respect a fairly well-defined passband.

Conclusions

- It has been demonstrated that the basic processes of DSP, namely, discretization (or sampling) and interpolation have been part of mathematics in one form or another since classical times.

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Conclusions

- It has been demonstrated that the basic processes of DSP, namely, discretization (or sampling) and interpolation have been part of mathematics in one form or another since classical times.
- By introducing the concepts of infinity, the limit, and convergence and then extending the classical methods of Archimedes, Wallis and Gregory rendered the emergence of calculus almost inevitable.
- In addition to consolidating the methods of tangents and quadrature under the unified theory of calculus, Newton discovered the binomial theorem which can be deemed to be the z transform of a certain class of signals.

Conclusions *Cont'd*

- Stirling discovered in the 1700s an interpolation method that can be used to design nonrecursive filters which were not invented until the 1960s.

  The same method can be used to design differentiators and integrators.

Conclusions *Cont'd*

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  The same method can be used to design differentiators and integrators.

- In short, mathematical discoveries made since the early 1600s are very much a part of the toolbox of a modern DSP practitioner.

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This slide concludes the presentation.
Thank you for your attention.