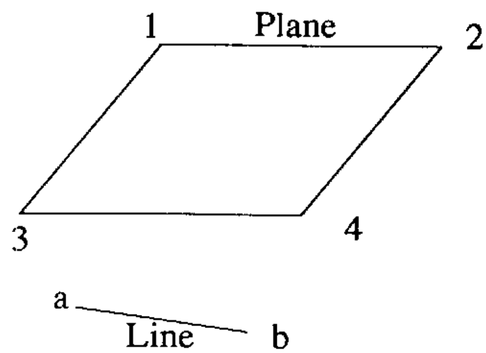
Line and Plane Interactions

This lecture:

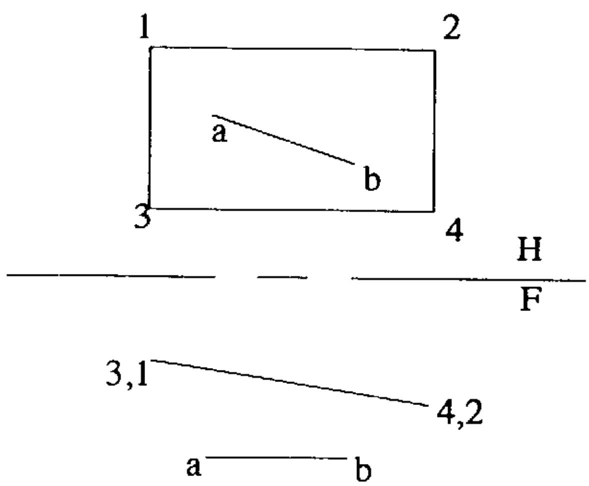
1. Visibility (continued from last class)
2. The true angle between planes
3. Intersection of lines and planes by the cutting plane method

Visibility

Consider the following 3-D pictorial of a line and a plane:



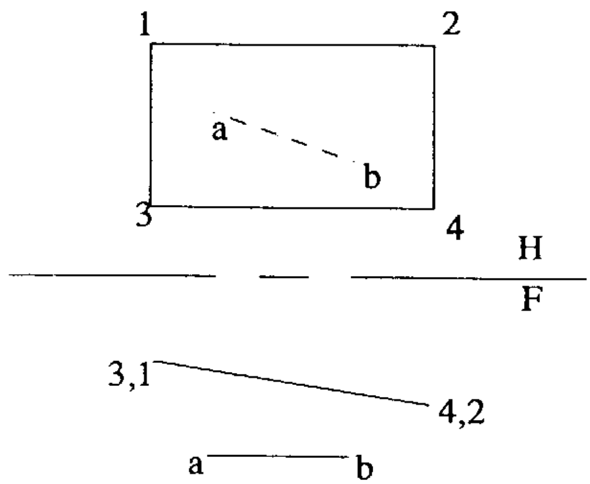
If we look at the top view we can not tell which is higher - need to look at a front view.



Cannot tell which is higher from this view.

From this view we can see that the plane is above the line ∴ the line should show hidden in the H view

The orthographic projection of the line and the plane should therefore be:



Without correct visibility, the H view is ambiguous.

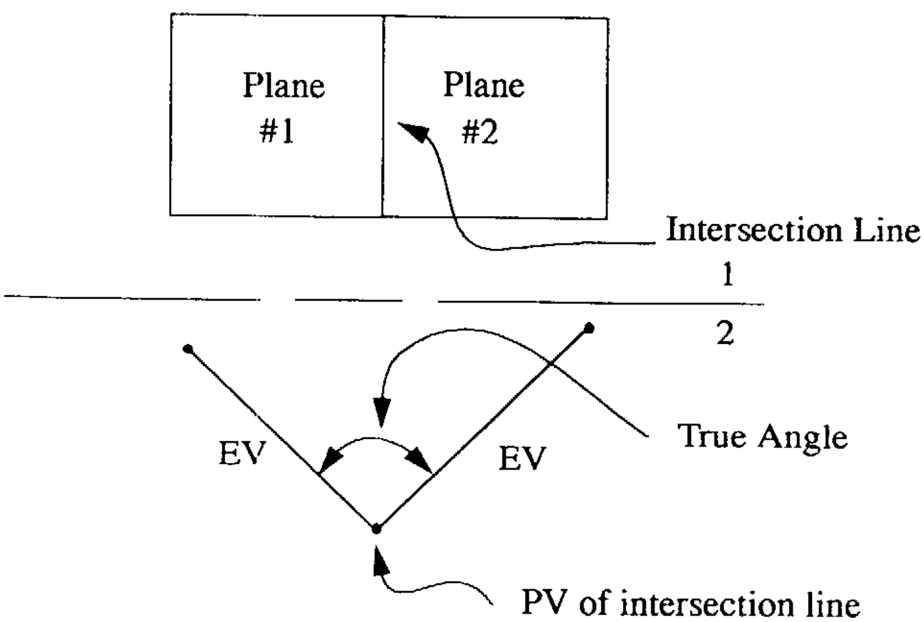
Angles between two Planes

To find the true angle between two planes (such as a roof and an air conditioning duct) we need:

- a view showing both planes in EV

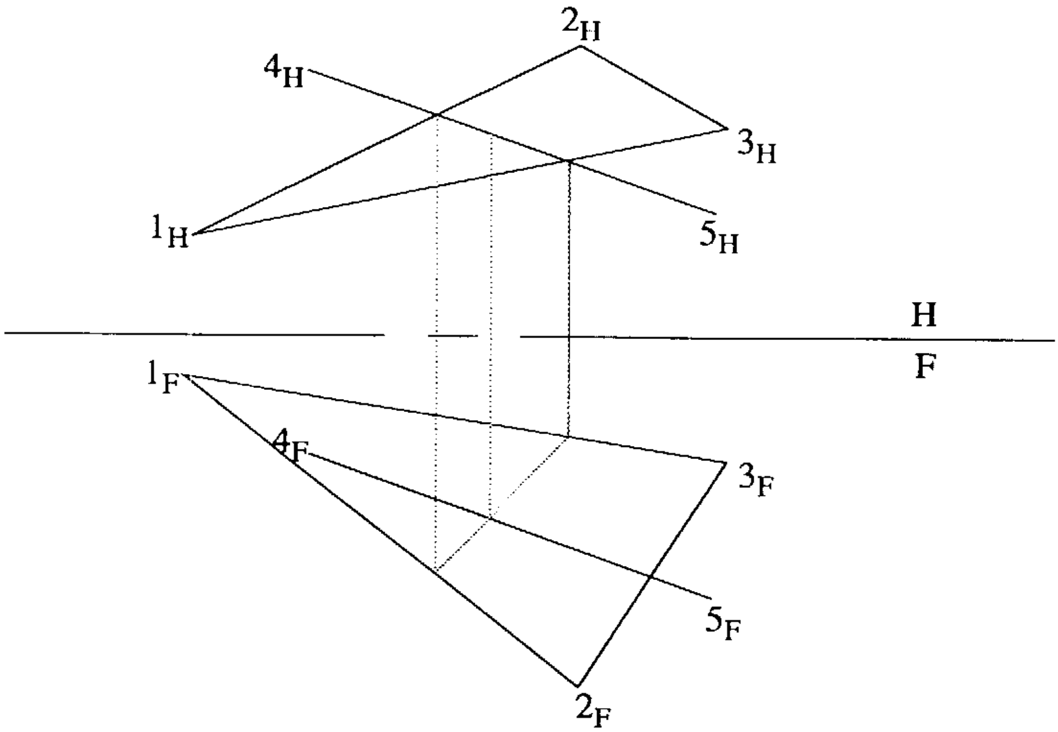
Or, equivalently,

- The line of intersection between the planes must be seen in PV



Intersection of Panes and Lines by "Cutting Plane" Method

- Before we determined the intersection point by getting the plane EV and the line TL. (auxiliary views)
- Here, we only use the existing views



1. For example, start in H view - Line 4-5 intersects plane 123 along line XY. Project XY into the F view.
2. XY is now our "cut" across the plane. The intersection, Z, is then found.
3. Project Z back into the H view
4. Final step - Determine **Visibility** (as before)

Exercises: #77, #78

Plane-Plane Intersections and Solid Line Intersections

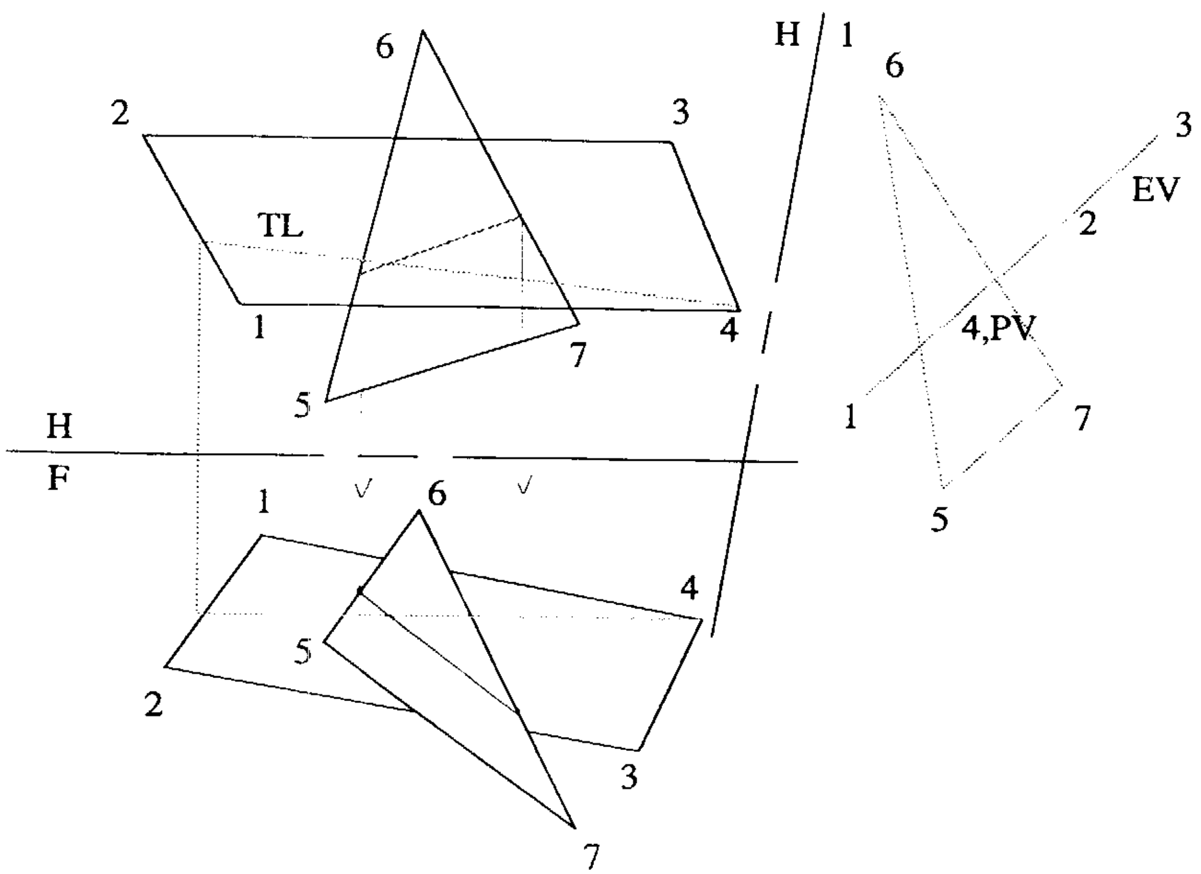
Plane-Plane Intersections

Two methods can be used:

1. Edge View method
2. Cutting Plane Method

Edge View Method

1. Create an edge view of one of the planes by using an auxiliary view.
2. Project the other plane into the auxiliary view.
3. Locate the intersection points in the auxiliary view and project them back into the other views.
4. Find the **visibility**.

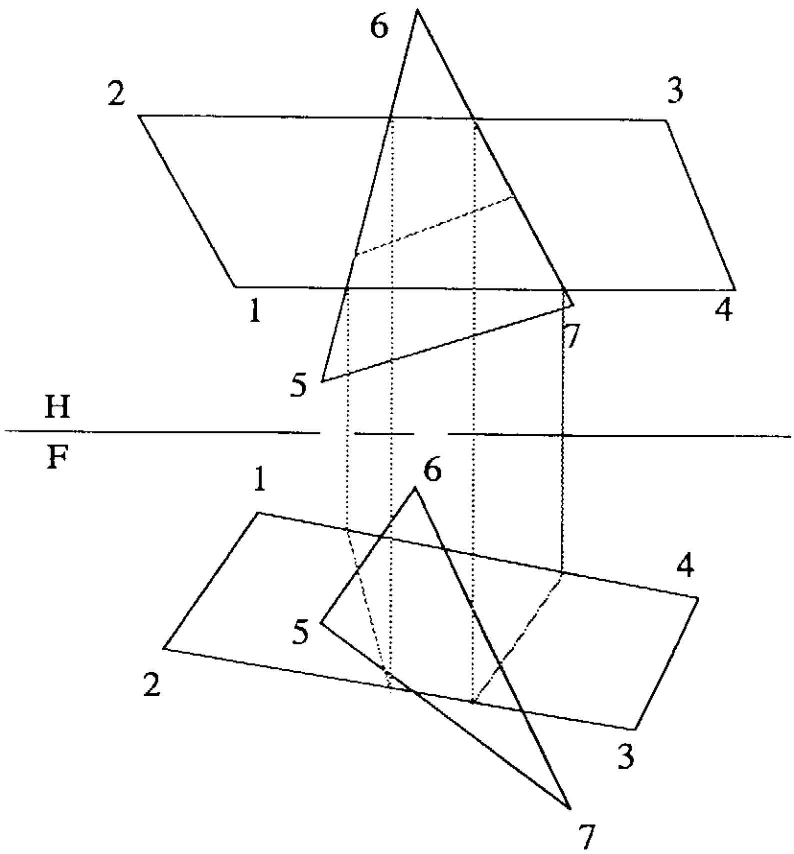


Cutting Plane Method

- Use the same cutting plane method we have been using to find the intersection of lines and plane

but,

- This time perform the method twice (once for each intersection point)

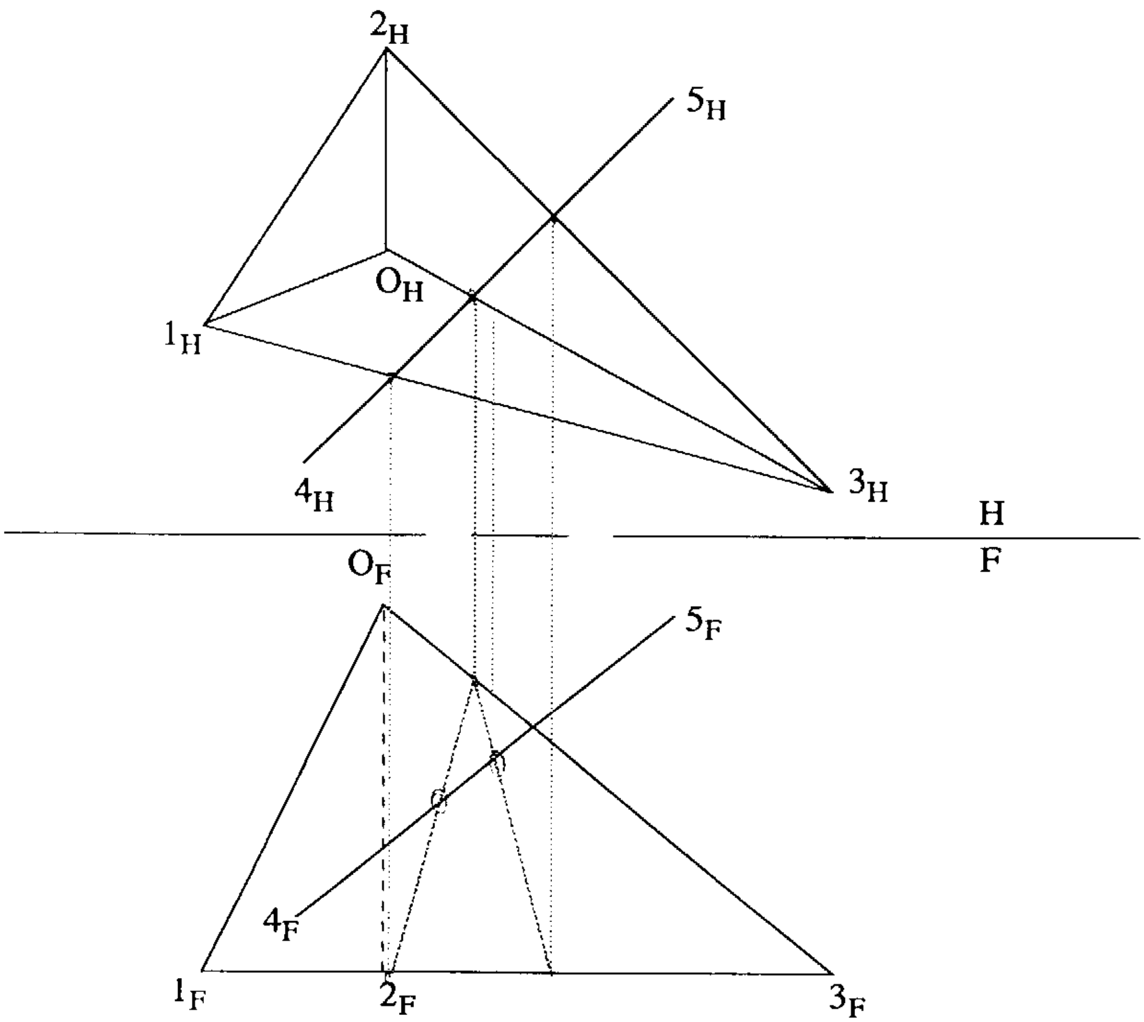


- Advantages: No auxiliary view is needed.

Line-Solid Intersections

- As before, use the cutting plane method to determine the points where the line pierces the solid.
- Remember to check **visibility**.

Example: Line piercing a Pyramid



Exercises: #79, #80

2-D Transformations of Points and Lines

- Examine how CAD operations such as rotation, scaling, reflection and translation are performed on points and line entities.
- Mathematical underpinnings of CAD software.

Background

1. A 2 (row) x 2 (col.) matrix: $\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$ or $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

   1 x 2 matrix: $\begin{bmatrix} x & y \end{bmatrix}$

2. Transpose of a 1x2 matrix: $\begin{bmatrix} x & y \end{bmatrix}^T = \begin{bmatrix} x \\ y \end{bmatrix}$

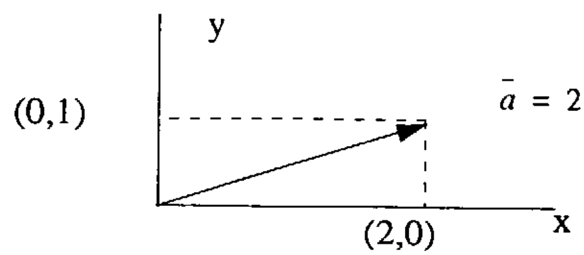
3. Inverse of a 2x2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   A^{-1} A I

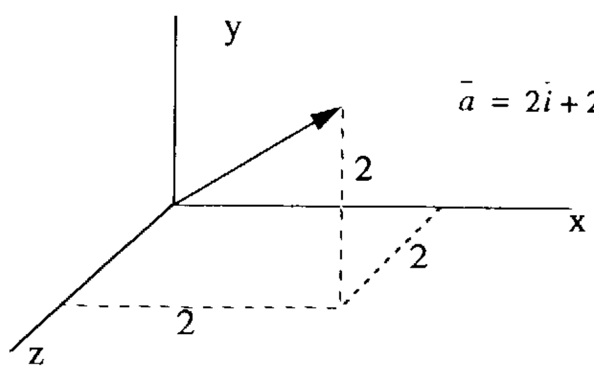
   inverse matrix identity

   if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \frac{1}{ad - bc}$

4. Vector - represent direction and magnitude



$\bar{a} = 2\bar{i} + \bar{j} = \begin{bmatrix} 1 & 2 \end{bmatrix}$



$\bar{a} = 2\bar{i} + 2\bar{j} + 2\bar{k} = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}$

2D Transformations of Points

- Take a point (x, y) in cartesian coordinates and represent it by a 1x2 matrix $\begin{bmatrix} x & y \end{bmatrix}$
- Now multiply $\begin{bmatrix} x & y \end{bmatrix}$ by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} (ax + cy) & (bx + dy) \end{bmatrix}$$

$\therefore$ x coordinate is transformed to $ax + by$

y coordinate is transformed to $bx + dy$

- The point is "moved" or transformed in space.

Now, let's put some special values into our 2x2 transformation matrix:

1. $a = d = 1, b = c = 0$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$\therefore$ Identity matrix has no effect on $\begin{bmatrix} x & y \end{bmatrix}$.

2. $d = 1, b = c = 0$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & y \end{bmatrix}$$

Original coordinates stretched (or shrunk) in x-direction (**scaling**)

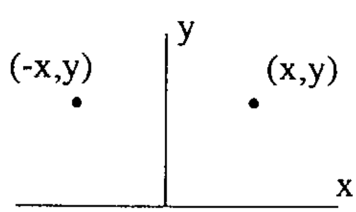
3. $b = c = 0$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ax & dy \end{bmatrix}$$

Scaled in x & y. If $a = d$ then scaling is equal in x & y directions.

4. If a and/or d are negative, we get **reflections**

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -x & y \end{bmatrix}$$



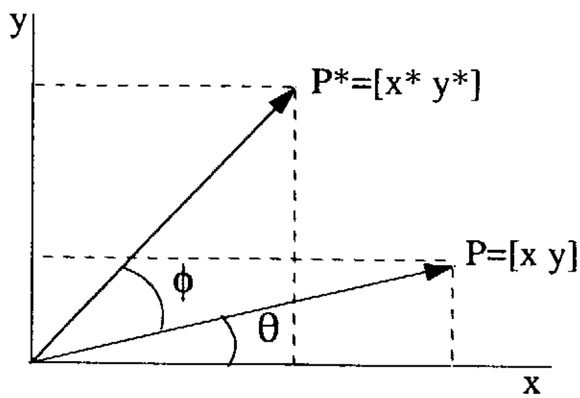
Reflection about the y-axis.

5. $a = d = 1, c = 0$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & (bx + y) \end{bmatrix}$$

The y-coordinate is **sheared** from its original coordinate.

6. **Rotation**



- Consider a vector P that is given by

$$P = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} r\cos\theta & r\sin\theta \end{bmatrix}$$

- Rotate the vector counter-clockwise by ϕ
- The new point P^* is given by

$$P^* = \begin{bmatrix} x^* & y^* \end{bmatrix} = \begin{bmatrix} r\cos(\theta + \phi) & r\sin(\theta + \phi) \end{bmatrix}$$

- Using the sum of angles formula this rotated vector can be written in terms of the old vector P,

$$P^* = \begin{bmatrix} (r\cos\theta\cos\phi - r\sin\theta\sin\phi) & (r\cos\theta\sin\phi + r\sin\theta\cos\phi) \end{bmatrix}$$

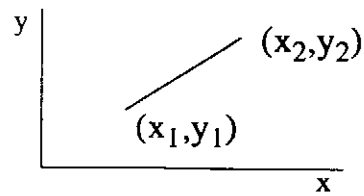
$$= \begin{bmatrix} (x\cos\phi - y\sin\phi) & (x\cos\phi + y\sin\phi) \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} = PR$$

where $R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is the 2D rotation transform matrix

2D Transformations of Lines

- Place the end coordinates of the line in a matrix

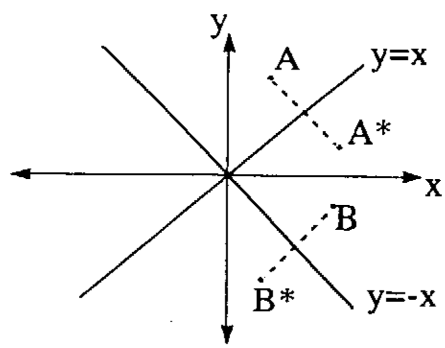


$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

- Multiply this matrix by any of the above transforms, and the line is moved to a new position.
- See handout for the set of 2x2 transformation matrices.

Example 1:

- Reflections about the $y = x$ or $y = -x$



- The point $A = (1, 3)$ is transformed to $A^* = (3, 1)$

$$\therefore\ T_{y=x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

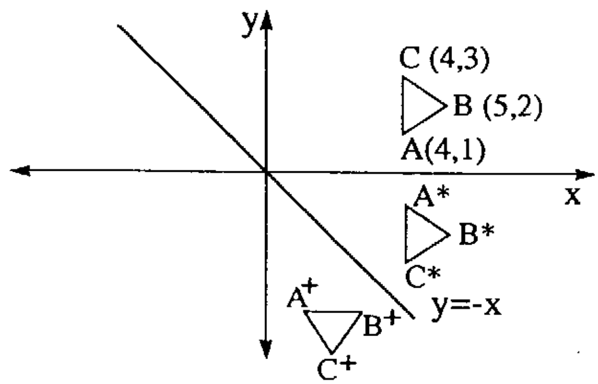
- Similarly, reflection about $y = -x$ yields

$$T_{y=-x} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \end{bmatrix}$$

Example 2:

- Create the composite transform for consecutive reflections of ABC about $y = 0$ and then $y = -x$



1. Reflection about $y = 0$

$$[T_{y=0}] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$[x] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\therefore\ [x^*] = [x]\,[T_{y=0}] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix}$$

2. Now, reflect $[x^*]$ about $y = -x$

$$[x^+] = [x^*]\,[T_{y=-x}] = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix}$$

- A **composite** transformation matrix is easily found

$$T_{comp} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- This transformation matrix would map $[x]$ directly to $[x^+]$

Important Notes: (regarding Composite Transforms)

1. The matrices must be multiplied in the correct order, matrix multiplication is **noncumulative.** ($AB \neq BA$)
2. The area of any transformed figure is related to the area of the initial figure by:

$$\text{Area Transfromed} = (\text{Area Initail})\, det\,[T]$$

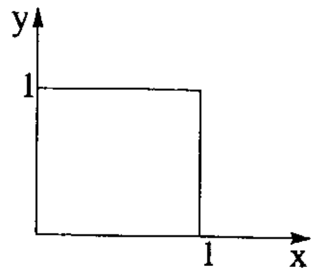
where $det\,[T] \equiv$ determinate of matrix $[T]$

if $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $det\,[T] = ad - bc$

In-Class Problems:

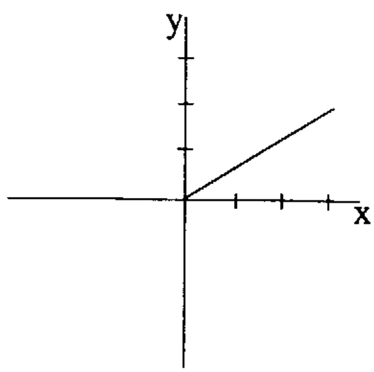
Problem 1:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$



Build a 4x2 location matrix for the unit square. Shear it to a new location using A.

Problem 2:



- Rotate the line 90° CCW about 0, then
- reflect point about x-axis, then
- reflect point about y-axis