2D Transformations Cont.

- Last class we looked at transforms for scaling, skewing, reflection, and rotation
- We did not look at translation matrices

Why?

- To implement translation of a figure, matrix addition or subtraction must be performed
- All the other operations require matrix multiplication

Therefore, to make computer algorithms easier for CAD lets introduce a coordinate system that allows matrix multiplication for all operations.

Homogeneous Coordinate System:

- a cartesian pt. (x,y) becomes (x,y,1) in homogeneous coordinates.
- Unfortunately, a third components in the point position vector means alteration of all our transformation matrices from 2x2 to 3x3.
- The general 3x3 matrix for transforming 2-D points using h.c. notation is:

$$\begin{bmatrix} a & b & p \\ c & d & q \\ m & n & s \end{bmatrix}$$

  - the upper left section (a,b,c,d) performs the rotations, scaling, reflections, and shearing
  - the lower left section (m,n) performs translations
  - generally $p = q = 0$, $s = 1$

Example 1:

Translate the point (4,3) by 6 x-units and 3 y-units.

$$\begin{bmatrix} 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 1 \end{bmatrix}$$

Example 2:

Find the new coordinates of point $P = (\sqrt{2}, 0)$ if it is rotated 90° CCW about the origin and then translated by $(-0.5, 0.5)$

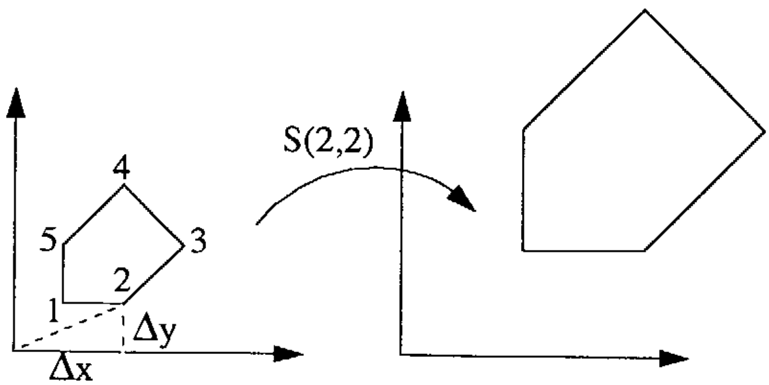
$$P' = \begin{bmatrix} \sqrt{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 1.914 & 1 \end{bmatrix}$$

Composite Transformations about Arbitrary Points/Axes

1. Scaling a Plane Object

- If we simply scale the polygon below by 2 then the polygon's position will also change



- This apparent translation occurs because we are scaling the position vectors (i.e a point at (1,1) is scale to the point (2,2))

Question: How do we scale the polygon without having it translated?

- Well, if one of the points on the polygon is at the origin then that point will not be translated.

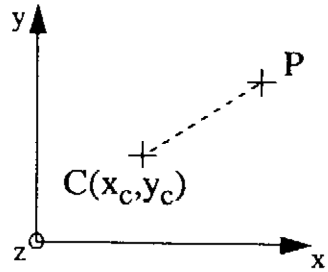
∴ The steps required to scale the polygon without translation are:

1. Translate a point on the polygon to the origin (i.e vertex 2)
2. Scale the polygon by 2x in the x and y directions
3. Translate the vertex 2 back to its original position.

- The composite transform is: $P' = T(-\Delta x, -\Delta y)\, S(2, 2)\, T(+\Delta x, +\Delta y)$

2. Rotation about an Arbitrary Axis

- Rotate a point about an axis parallel to the z-axis & passing through the point $C(x_c,y_c)$



- We already know how to do rotations about the origin, therefore,

1. Move point C to (0,0) $T(-x_c, -y_c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_c & -y_c & 1 \end{bmatrix}$

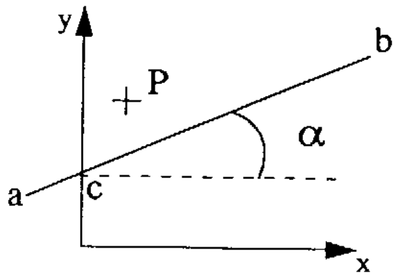
2. Rotate by angle a about the z-axis $R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Move C back to the original position $T(+x_c, +y_c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_c & y_c & 1 \end{bmatrix}$

$\therefore$ the composite transform matrix is $T = T(-x_c, -y_c)R(\alpha)T(x_c, y_c)$

3. Reflections About an Arbitrary Line

- Reflect point P about the line ab which intercepts the y-axis at c

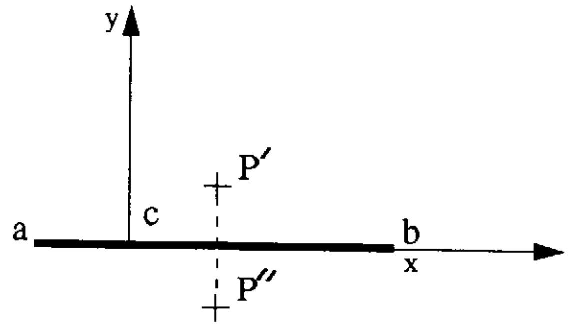


- We know how to do reflections about the x and y axes, therefore transform the line ab onto one of the axes, perform the reflection, and then translate back.

1. Translate ab to pass through the origin: $T(0,-c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}$

2. Rotate ab through $-\alpha$ about origin so it coincides with the x-axis

$$R(-\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



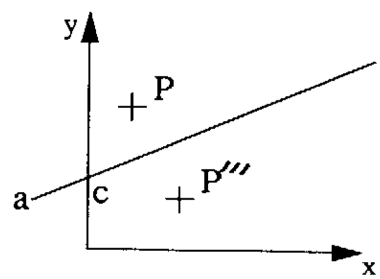
3. Reflect about the x-axis: $M(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. Rotate ab back to its original slope: $R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. Move ab back up by c: $T(0,c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$

∴ the composite transform matrix is $T = T(0,-c)\,R(-\alpha)\,M(x)\,R(\alpha)\,T(0,c)$

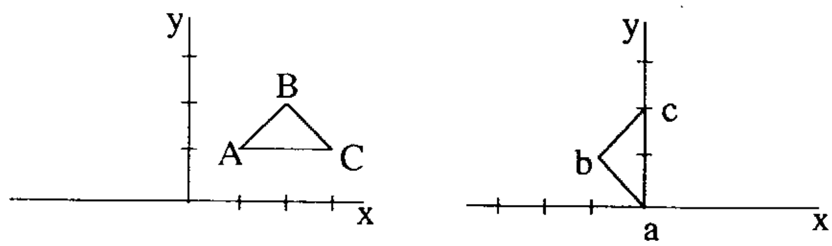
and so, $P''' = PT$



In-Class Problems:

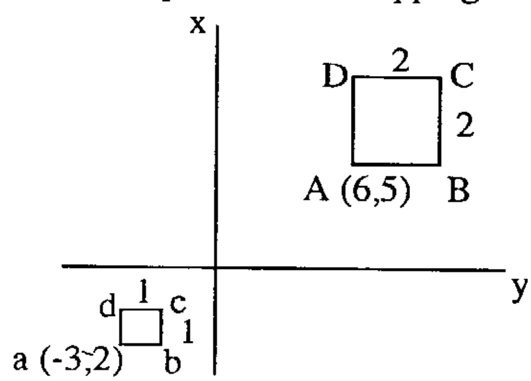
Problem 1:

- Find the composite matrix which maps planar object ABC to abc



Problem 2:

- A box is mapped from position 1 to position 2. Find a composite transformation matrix to perform the mapping. Verify it.



3D Transformations

- Now we increase the complexity by adding a dimension
- However, the transformation matrices are similar

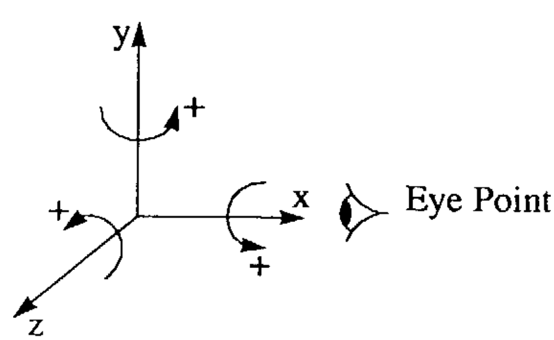
1. Translation

$$\{x', y', z', 1\} = [x, y, z, 1]\, T(\Delta x, \Delta y, \Delta z)$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \Delta x & \Delta y & \Delta z & 1 \end{bmatrix}$$

2. Rotation

- about x-axis $R(x, \alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- about y-axis $R(y, \beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- about z-axis $R(z, \gamma) = \begin{bmatrix} \cos\gamma & \sin\gamma & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- In all the above rotations, **POSITIVE** angles are CCW as viewed from the positive end of the subject axis



3. Scaling

$$S(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- about an arbitrary point use:

$$[X_2] = [X_1]\,[T_1]\,[S]\,[T_2]$$

$[T_1]$ - translates object to the origin

$[T_2]$ - translates the object back to its original position

4. Reflection

- Here we reflect points (objects) about planes (e.g. xy, xz, yz)

- about xy plane $M(x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- about xz plane $M(x, z) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- about yz plane $M(y, z) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note: 3-D transforms are concatenated just as before

5. Rotations about Arbitrary Axes

Case 1: Axis of Rotation is Parallel to a Coordinate Axis

- Simple - translate rotation axis to the coordinate axis it is parallel with, perform the rotation, and translate back

$$[X_2] = [X_1]\,[T_1]\,[R]\,[T_2]$$

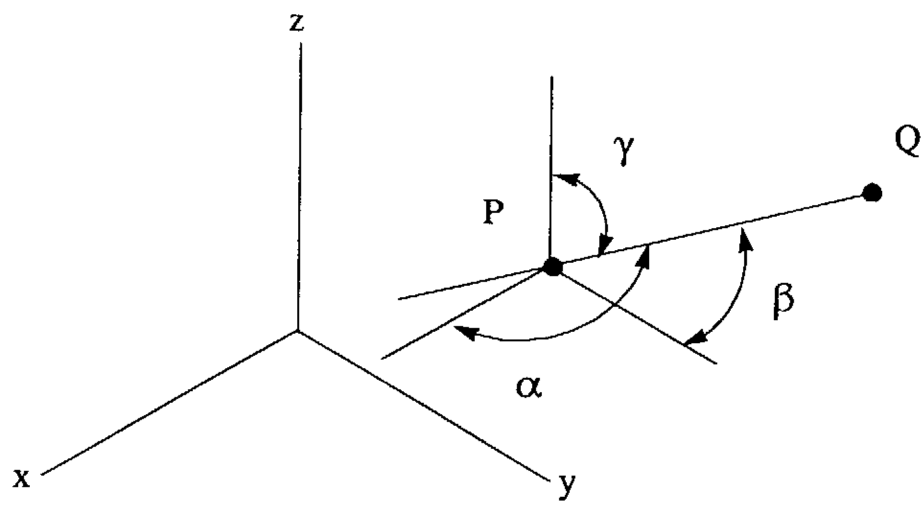
$[T_1]$ - translates body until axis of rotation is coincident with the desired coordinate axis (i.e x, y or z axis)

$[R]$ - rotates the body about the x, y or z axis

$[T_2]$ - translates the object back

Case 2: Axis of Rotation is not Parallel to a Coordinate Axis

- This is a bit more complicated case since we now must rotate the given axis to make it coincident with one of the coordinate axes.



- Identify the axis $P(p_x, p_y, p_z)$ $Q(q_x, q_y, q_z)$ by the **direction cosines**. The cosines of the angles the line PQ makes with the principle axes.

$$\cos\alpha = \frac{q_x - p_x}{L} \qquad \cos\beta = \frac{q_y - p_y}{L} \qquad \cos\gamma = \frac{q_z - p_z}{L}$$

$$L = \sqrt{(q_x - p_x)^2 + (q_y - p_y)^2 + (q_z - p_z)^2}$$

- Assume we wish to rotate point $R(x, y, z)$ around line PQ by θ CCW

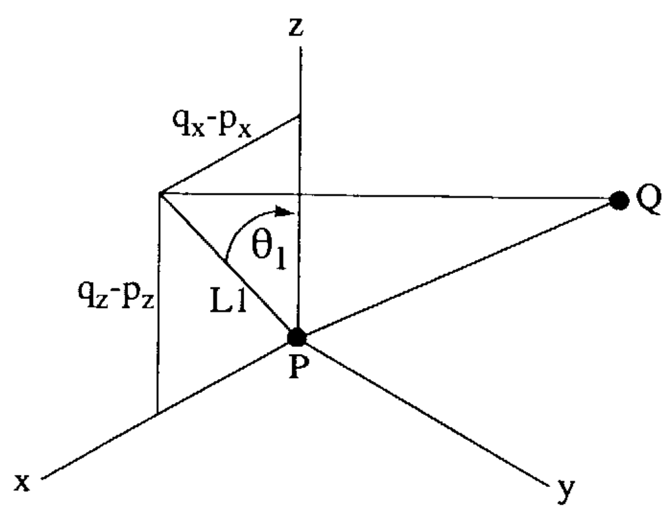
1. Move P, Q and R so that P is at the origin

$$TR1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p_x & -p_y & -p_z & 1 \end{bmatrix}$$

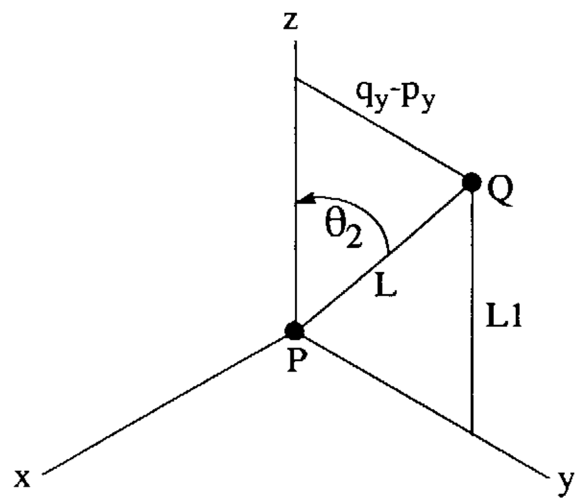
2. Rotate about y-axis in a CW sense through θ_1 so that PQ lies in the yz plane

$$R(y, -\theta_1) = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sin\theta_1 = \frac{q_x - p_x}{L1} \qquad \cos\theta_1 = \frac{q_z - p_z}{L1} \qquad L1 = \sqrt{(q_x - p_x)^2 + (q_z - p_z)^2}$$



3. Now rotate CCW about x axis through θ_2



$$R(x,\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & \sin\theta_2 & 0 \\ 0 & -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sin\theta_2 = \frac{q_y - p_y}{L} \qquad \cos\theta_2 = \frac{L1}{L}$$

- Line PQ is now coincident with the z axis

4. Now perform the CCW rotation by θ about the line PQ

$$R(z,\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The rotation is now affected. However we should now move PQ back to where it was originally. Essentially reverse steps 3, 2, & 1

  Briefly, $R(x, -\theta_2)$

  $R(y, \theta_1)$

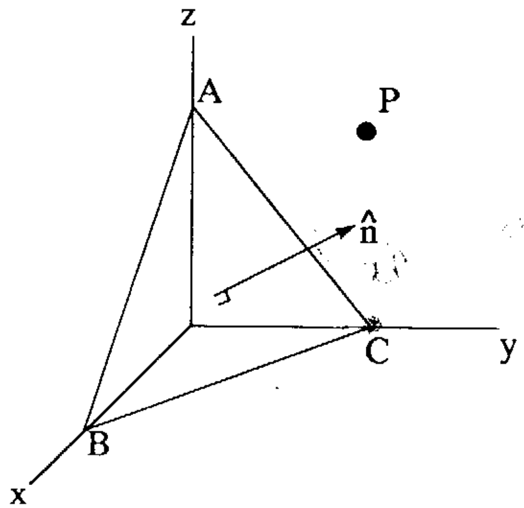
  $TR2(p_x, p_y, p_z)$

- In total,

$$T = [TR1]\,[R(y,-\theta_1)]\,[R(x,\theta_2)]\,[R(z,\theta)]\,[R(x,-\theta_2)]\,[R(y,\theta_1)]\,[TR2]$$

6. Reflection about an Arbitrary Plane

- Assume we want to reflect point P about the plane ABC



- We know how to do reflections about a principle planes (xy, xz, or yz). So we need to make ABC coincident with one of these principle planes.
- One way to do this is to make the **normal** to ABC coincident with one of the principle axes.
- The normal of a plane can be found by taking the **cross product** of two vectors that are on the plane

$$n = \overrightarrow{AB} \times \overrightarrow{BC}$$

Normalizing n, $\hat{n} = \frac{n}{\|n\|} = \frac{n}{\sqrt{n_x^2 + n_y^2 + n_z^2}}$

Note: $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \times \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \end{bmatrix}$

- Once we have found the normal vector, the problem is now one of making an arbitrary axis (given by vertex B and $\hat{n}$) coincident with one of the principle axes. This is what we did in the previous section (Revolution about an arbitrary axis - Case 2).

∴ The steps involved in reflecting P about ABC are:

1. Translate vertex B to the origin - $TR1$
2. Rotate $\hat{n}$ by $-\theta_1$ to place it in the zy plane - $R(y, -\theta_1)$
3. Rotate $\hat{n}$ by θ_2 to make it coincident with the z-axis - $R(x, \theta_2)$
4. Now perform the refection - $M(x, y)$
5. Now reverse steps 3, 2 & 1 to put ABC back to its original position

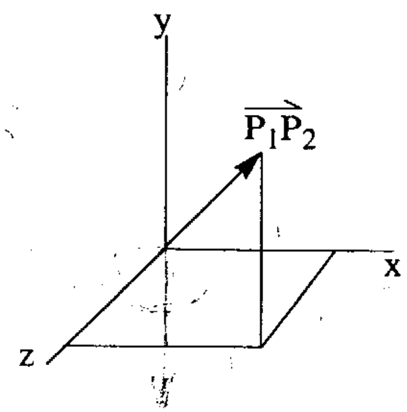
∴ The composite transform is:

$$T = [TR1]\,[R(y, -\theta_1)]\,[R(x, \theta_2)]\,[M(x, y)]\,[R(x, -\theta_2)]\,[R(y, \theta_1)]\,[TR2]$$

In-Class Problems:

Problem 1:

- Given vector $\overrightarrow{P_1P_2}$, $P_1 = (0, 0, 0)$ and $P_2 = (1, 1, 1)$. Find the composite transform that maps $\overrightarrow{P_1P_2}$ onto the negative y-axis. show all intermediate matrices.



Problem 2:

- Reflect the sphere about the given plane