Transformations

Two Dimensional (Planar) Transformations

Given a point, **p**, in 2D space represented by the simple row matrix as:

$$\begin{bmatrix} x & y \end{bmatrix}$$

it is a simple matter to argue that the most general linear transformation of the point can be expressed as the following linear operation:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ax+cy & bx+dy \end{bmatrix}$$

where a,b,c,d are arbitrary constants. This can be further simplified using matrix notation as:

$$\begin{bmatrix} x & y \end{bmatrix}\mathbf{T} = \begin{bmatrix} x^* & y^* \end{bmatrix}$$

or

$$\mathbf{p}\ \mathbf{T} = \mathbf{p}^*$$

This simple form yields a new point, **p***, with components that are linear combinations of the original points. The matrix, **T**, is called a transformation matrix. This kind of transformation can be readily applied to a polyline or to any number of points simply by adding the additional points as successive rows to the point matrices, **p**, and **p***. The role of the arbitrary constants in **T** can be deduced from some simple examples as follows.

Case 1 – a=d=1, b=c=0

In this case, **T**=[**I**] (unit matrix) and it follows that **pT**=**p I** = **p**. Thus there is no change in the point.

Case 2 – a ≠d, b=c=0

This is a simple extension of Case 1 but it yields some interesting results as shown in the following equations:

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ax & dy \end{bmatrix}$$

The result is an anisotropic transformation that stretches the x coordinate by the factor, a, and the y coordinate by the factor, d. If a and d are both positive and greater than unity, the transformed point will move farther away from the origin. If a=d, the result is an isotropic transformation that stretches both x and y coordinates equally. On the other hand, if a<0, the result is to switch the sign of the new x component (similar result for d<0). For the special case where a=-1 and d=1, this amounts to a reflection across the y axis. Similarly, if a=1 and d=-1, the result is a reflection across the x axis. If a=d=-1, then the reflection is across both axes (e.g., across the origin). It should be noted that det(**T**) is negative for the mirroring transformations across the x or y axes. It can be shown that transformations with negative determinants are transformations that cannot be accomplished by simply stretching or shrinking an object; instead, some kind of inside-out or mirroring operation must be involved.

Case 3. – a=d=1, c=0

When either b or c are nonzero, the resulting transformation causes a “shearing” kind of displacement as will be shown in the following simple example.

$$
\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y + bx \end{bmatrix}
$$

In this case the new (transformed) y coordinate is not simply a stretch or contraction of the original y coordinate, but instead it also includes a component that depends on the x coordinate of the point. Figure 1 shows the effect on a series of points lying along a line at y=1. Note that as the x coordinate of the points increases, the transformed x coordinate remains unchanged but the transformed y coordinate in increased by the factor, bx. Note also that if b=0, the transformation reduces to the unit matrix and there is no effect at all. Finally, a similar argument can be made for the case where b=0 and c is nonzero.

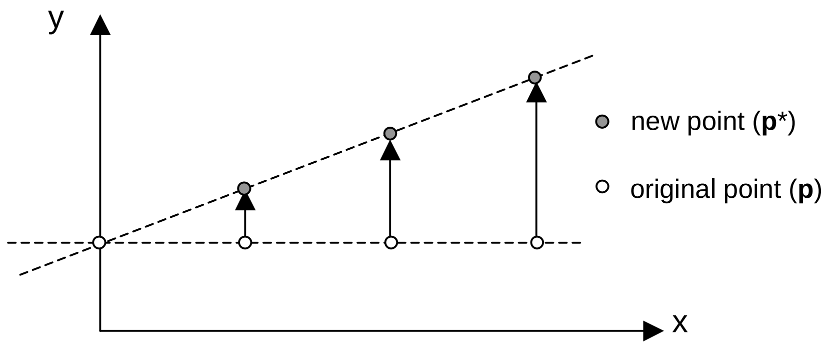

Figure 1. Shear Transformation (a=d=1, c=0)

Rotations

From the above cases, it is clear that arbitrary values of the constants in T will cause a combination of stretching, shrinking, mirroring and shear action. An interesting question is to ask what kind of transformation will not produce any shear action. It turns out that the answer is a rotation about the origin. This can be argued simply by visualizing the transformation of points lying along two perpendicular lines. The new points transformed by such a rotation will still lie along perpendicular lines.

The rotation transformation can be readily developed by considering the rotation of a point about the origin by an amount, θ, as shown in Figure 2.

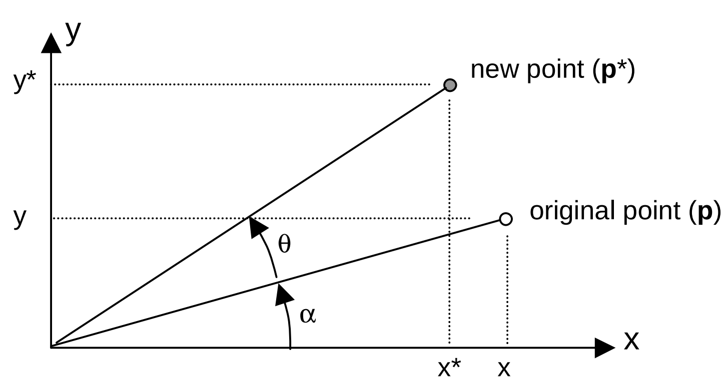

Figure 2. Rotation of Point (p) about Origin

It follows from Figure 2 that if the initial point is:

$$\mathbf{p} = \begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} d\cos\alpha & d\sin\alpha \end{bmatrix}$$

then the transformed point will be:

$$\begin{aligned}\mathbf{p}^* = \begin{bmatrix} x^* & y^* \end{bmatrix} &= \begin{bmatrix} d\cos(\alpha+\theta) & d\sin(\alpha+\theta) \end{bmatrix} \\ &= \begin{bmatrix} d(\cos\alpha\cos\theta - \sin\alpha\sin\theta) & d(\cos\alpha\sin\theta + \sin\alpha\cos\theta) \end{bmatrix} \\ &= \begin{bmatrix} x\cos\theta - y\sin\theta & x\sin\theta + y\cos\theta \end{bmatrix}\end{aligned}$$

This result can be expressed as a matrix multiplication in the same form as used for T above:

$$\begin{bmatrix} x^* & y^* \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \mathbf{T}$$

Thus the resulting rotation transformation is given by:

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

It should be noted that this represents a counterclockwise rotation of the point about the origin. The same transformation could also represent the clockwise rotation of the coordinate axes with respect to the point. The only difference is whether one keeps the axes fixed and rotates the point or fixes the point and rotates the coordinate axes. It is only a matter of perspective! We can conclude that:

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

represents a counterclockwise rotation of the point about the origin or a clockwise rotation of the coordinate axes with respect to the point, while

$$\mathbf{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

represents just the opposite!

Several observations are worth noting:

1. The transformation is a rotation about the invisible z axis and therefore the origin is unaffected by the transformation.
2. The det(T) is +1. This is a characteristic of a rotation matrix.
3. Given a $T(\theta)$, then $T(-\theta)=T^T$ where $()^T$ means matrix transpose.
4. Given a $T(\theta)$, then $T(-\theta)$ simply reverses the rotation, so if we apply $T(\theta)$ followed by $T(-\theta)$, the result should be an unchanged point. That is, $\mathbf{p}^*=\mathbf{p}\,\mathbf{T}(\theta)\,\mathbf{T}(-\theta)=\mathbf{p}$. This means that $\mathbf{T}(\theta)\mathbf{T}(-\theta)=\mathbf{I}$ or that $\mathbf{T}(-\theta)=\mathbf{T}^{-1}(\theta)$. That is, the inverse transformation is simply the transformation with the sign of θ reversed.
5. From #3 and #4, it follows that $\mathbf{T}^{-1}(\theta)=\mathbf{T}^{T}(\theta)$ or the inverse is the same as the transpose. This defines what is called an “orthogonal” matrix.
6. $\mathbf{T}(\theta)$ is also an “orthonormal” matrix because it is orthogonal and det($\mathbf{T}$)=1.

One very practical consequence is that one should never compute inverse rotation matrices numerically but instead should use the transpose operation which is much faster and more efficient in a computer program!

Translation – the Case for Homogeneous Coordinates

The linear transformation, T, by itself cannot produce a simple translation of the point, and this presents a serious problem for geometric modeling applications. Homogeneous coordinates provide a simple and very elegant way round this problem. While there are many ways to introduce homogeneous coordinates, the simplest approach is to just define the process of projecting a point back and forth from 2D space to homogeneous coordinate space. When we add a homogeneous coordinate to a 2D problem, it is just like adding a 3rd Euclidean coordinate (e.g., the z coordinate). Fundamentally, there is no way we can infer the z value from knowledge of the (x,y) coordinates alone. Additional information about the z coordinate (the "depth") must be provided. Similarly, if we project a 3D point into the z=0 plane, we simply discard its z coordinate. In this same spirit, let us extend our 2D space to a 3D homogeneous space with the addition of a homogeneous coordinate as follows:

$$\begin{bmatrix} x & y \end{bmatrix} \Rightarrow \begin{bmatrix} hx & hy & h \end{bmatrix}$$

What we are doing is simply adding a 3rd coordinate, h, in a homogeneous fashion, that is, we are multiplying the x and y coordinates by the same h value. This will create a new point in 3D homogeneous space. The reverse of this process is the "projection" of the point from 3D homogeneous space back into 2D (Euclidean) space and this is obviously done by first homogeneously dividing the coordinates by the homogeneous value, h:

$$\begin{bmatrix} hx & hy & h \end{bmatrix} \Rightarrow \begin{bmatrix} x & y & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} x & y \end{bmatrix}$$

The process of making the homogeneous division is generally described as "normalizing" and it must be done in order to determine the resulting (x,y) coordinates. In other words, points in homogeneous space with any value of h in the above equation all project back to the same (x,y) point, and this is just like in Euclidean space where 3D points with given x and y values but any value of z all project back to the same (x,y). To illustrate:

Euclidean Space:

$$\begin{bmatrix} 2 & 3 & 5 \end{bmatrix} or \begin{bmatrix} 2 & 3 & -7 \end{bmatrix} or \begin{bmatrix} 2 & 3 & 0.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \end{bmatrix}$$

That is, all these points project back to the same point (2,3) in the (x,y) plane.

Homogeneous Space:

$$\begin{bmatrix} 4 & 6 & 2 \end{bmatrix} or \begin{bmatrix} 8 & 12 & 4 \end{bmatrix} or \begin{bmatrix} -2 & -3 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \end{bmatrix}$$

And similarly, all these points project from homogeneous space to the same 2D point (2,3).

Finally, it should be noted that the simplest way to go from 2D to 3D homogeneous coordinates is to just add the homogeneous coordinate as unity, that is: $[x\ y] \rightarrow [x\ y\ 1]$.

This is all well and good from a purely mathematical point of view, but to an engineer, the question is "what can we do with this result?" The result is surprising. Since we are now using 3 coordinates, the dimensions of T must be increased from 2x2 to 3x3 so that we have:

$$\mathbf{T} = \begin{bmatrix} a & b & p \\ c & d & q \\ l & m & s \end{bmatrix}$$

where the terms a,b,c,d are the same as before but we have added l,m,s,p,q. Just as before, the most effective approach is to explore how these constants affect a transformation. Since we have already studied the effects of a,b,c,d, we will now consider only the effects of l,m,s,p,q, and the easiest way to do this is to fix a=d=1 and b=c=0 which was the unit transformation (no effect).

Consider first the situation when p=q=0 and s=1 so that only l and m will be varied. The transformation is then:

$$[x \quad y \quad 1]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ l & m & 1\end{bmatrix} = [x+l \quad y+m \quad 1] \Rightarrow [x+l \quad y+m]$$

It is clear that the l and m terms in the 3rd row of T are translation components in the x and y directions. Let's next examine the effect of the s term by considering the following:

$$[x \quad y \quad 1]\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & s\end{bmatrix} = [x \quad y \quad s] \Rightarrow [x/s \quad y/s \quad 1] \Rightarrow [x/s \quad y/s]$$

This represents an isotropic scale change that is proportional to 1/s. The effect of the p and q terms is much less clear as shown below:

$$[x \quad y \quad 1]\begin{bmatrix}1 & 0 & p\\ 0 & 1 & q\\ 0 & 0 & 1\end{bmatrix} = [x \quad y \quad px+qy+1]$$

$$\Rightarrow \left[\frac{x}{px+qy+1} \quad \frac{y}{px+qy+1} \quad 1\right] \Rightarrow \left[\frac{x}{px+qy+1} \quad \frac{y}{px+qy+1}\right]$$

This is a much more confusing situation and it is not a particularly useful result until we consider 3D Euclidean space with projections into 4D homogeneous space in a later section.

As a final remark, it should be noted that the 2D homogeneous transformation with p=q=0 defines what is called an "affine" transformation because it includes the basic linear transformation plus translation. We will see how this can be useful in the next section.

Rotation About an Arbitrary Point

It has been noted already that T(θ) produces a rotation about the origin. In many cases in geometric modeling, it is necessary to make a rotation about a particular point (x_0,y_0) instead of the origin. This can easily be accomplished by first applying translations to move the point (x_0,y_0) to the origin, then making the necessary rotation, and finally, translating the point (x_0,y_0) back to its original location. The process is illustrated in Fig. 3 and can be accomplished mathematically by concatenating each of the transformations as follows.

$$T_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -x_0 & -y_0 & 1\end{bmatrix} \; and \;\; T_\theta = \begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} and \; T_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ +x_0 & +y_0 & 1\end{bmatrix}$$

where T_1 translates the point at the origin, T_θ makes the rotation about the origin, and T_2 returns the point to its original position. Concatenating these transformations yields:

$$\mathbf{T} = \mathbf{T}_1 * \mathbf{T}_\theta * \mathbf{T}_2$$

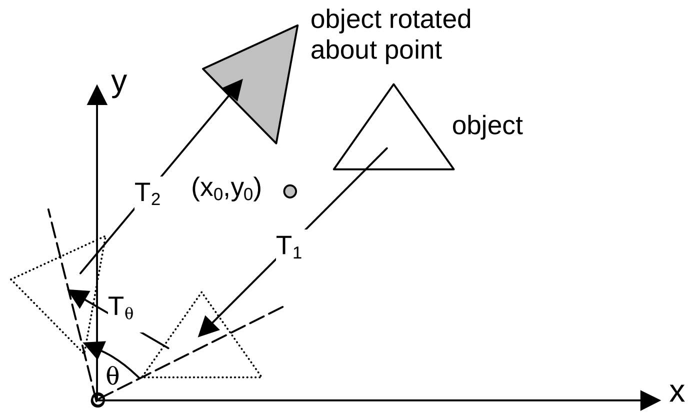

Figure 3. Rotation of Object About a Point (x_0,y_0)

Reflection Across a Centerline

A similar approach can be used to reflect an object across a given centerline but in this case it will be necessary to apply transformations until the centerline lies along either the x or y axes and then make the reflection across one of these axes. Then the centerline is returned to its initial position. Figure 4 illustrates the process geometrically.

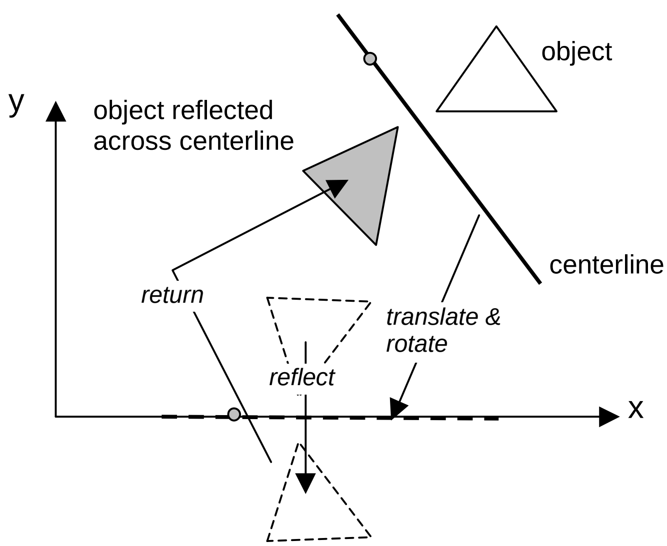

Figure 4. Reflection of Object Across a Centerline

Three Dimensional Transformations

It is a relatively simple matter to extend the previous 2D development to a full 3D representation. The point vector must be increased by one column to accommodate a z coordinate so that we can write:

$$\mathbf{p}^* = \begin{bmatrix} xh & yh & zh & h \end{bmatrix} = \begin{bmatrix} x & y & z & 1 \end{bmatrix}$$

and similarly, the general transformation matrix now becomes a 4x4 matrix:

$$\mathbf{T} = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & h & i & r \\ l & m & n & s \end{bmatrix}$$

Elements (a… i) in **T** play the same role as elements (a… d) in the 2D transformation. Similarly, elements in the 4th row and the 4th column play the same roles as those in the 3rd row and column from the 2D transformation. The example below illustrates this.

General Affine Transformation

A 3D affine transformation is defined when (p,q,r) are all zero. This yields the following:

$$\begin{aligned} \mathbf{p}^* = \mathbf{pT} &= \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ l & m & n & s \end{bmatrix} \\ &= \begin{bmatrix} ax+dy+gz+l & bx+ey+hz+m & cx+fy+iz+n & s \end{bmatrix} \\ &= \begin{bmatrix} (ax+dy+gz+l)/s & (bx+ey+hz+m)/s & (cx+fy+iz+n)/s & 1 \end{bmatrix} \end{aligned}$$

It is clear that the terms in T contribute to a linear scaling and translation of the original point.

Rotations about x, y and z Axes

It is also easy to generalize the previous 2D results for a rotation about the origin to the 3D case. In this case, one can visualize the 2D rotation as simply a special case of a 3D rotation about the z axis. This recognition leads to the following results:

Rotation θ about z axis:

The appropriate transformation matrix is given by:

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to see(e.g., simply multiply [x y z 1] times **T**) that this transformation has no effect on the z coordinate and only changes the x and y coordinates in exactly the same way as the 2D rotation.

Rotation ϕ about y axis:

This is simply a permutation of the previous result as follows:

$$\mathbf{T} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It should be noted that the sign of the sinϕ terms have been reversed across the diagonal of the matrix. This is due to the relative locations of the x and z axes for a rotation about y as compared to the previous rotation about the z axis.

Rotation ξ about the x axis:
Again, the result is another permutation of the previous results:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\xi & \sin\xi & 0 \\ 0 & -\sin\xi & \cos\xi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is nearly identical in structure to the rotation about the z axis.

It should be pointed out at this point that the finite rotations represented by the above transformations are not commutative. That is, the final result of successive rotations about different axes depends on the order in which the rotations are applied.. This can easily be illustrated by taking a familiar and readily oriented object (such as a pair of reading glasses) and subjecting it to successive 90° rotations about the x, y and z axes. Remember the final orientation. Now repeat this process starting from the same initial orientation, but this time make the 90° rotations about the x, z and y axes in that order. The result will be a different orientation of the object! As a final issue, it is simply noted without proof that while finite rotations are not commutative, infinitesimal rotations are commutative.

Rotation About an Arbitrary Point

The previous cases create rotations about the origin. We can consider an axis rotation about an arbitrary point (x_0,y_0,z_0) in a similar manner to that done for the 2D case. The procedure involves a translation to locate the point on the axis of rotation (the x, y or z axes). The rotation is next applied, and the reverse transformation is applied to return the object to its original location. Consider a rotation about the z axis at a point (a,b,c). The needed transformation is:

$$\mathbf{p}^* = \mathbf{p}\mathbf{T}_{-a}\mathbf{T}_{-b}\mathbf{T}_{\theta}\mathbf{T}_{a}\mathbf{T}_{b}$$

where $\mathbf{T}_a$ and $\mathbf{T}_b$ are the translations in the x and y directions, respectively. Note that $\mathbf{T}_a$ and $\mathbf{T}_b$ could be expressed as a single transformation, $\mathbf{T}_{ab}$ by entering appropriate values for l and m in the **T** matrix.

Rotation About an Arbitrary Axis

It is usually not sufficient to consider only rotations about the given coordinate axes since this is generally too inflexible. A more useful result would be a transformation matrix for a rotation of α about an arbitrary axis defined by a vector, **n**. This can easily be created by following a similar approach to that developed earlier to implement mirroring across an arbitrary axis.

Consider the situation as shown in Figure 5 below where the axis of rotation is specified as the vector, **n**, located at the origin of the coordinate system and the angle of rotation is α.

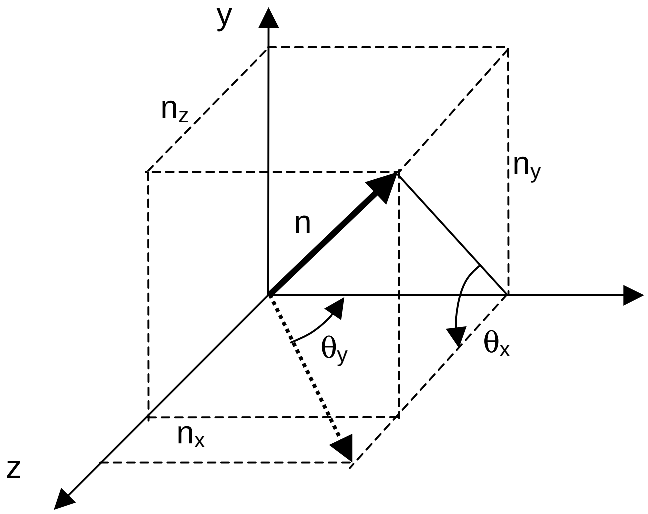

Figure 5. Rotation About Arbitrary Axis

The objective is to transform the problem such that the **n** vector lies along one of the coordinate axes, then apply the specified rotation about **n**, and finally, return the axis (and object) to the original location. To this end, it is necessary to break the rotation into simple steps as follows. First, consider that the vector, **n**, has been normalized such that |**n**|=1 so that the terms of **n** are the direction cosines. The first rotation, θ_x, is used to rotate the vector, n, about the x axis so that it lies in the xz plane. With reference to Fig. 5, the sine and cosine terms for this rotation are:

$$\cos\theta_x = n_z/n_1 \quad \text{and} \quad \sin\theta_x = n_y/n_1$$

where

$$n_1 = \sqrt{n_y^{\;2} + n_z^{\;2}}$$

The next rotation is, θ_y, about the y axis and this lines the n vector up with the x axis. Again, with reference to Fig. 5, the sine and cosine terms for this rotation are:

$$\cos\theta_y = n_x/n = n_x \qquad \text{and} \qquad \sin\theta_y = n_1/n = n_1$$

So finally, the needed rotations are the following:

Rotate θ_x about the x axis:
$$\mathbf{T}_{\theta_x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x & 0 \\ 0 & -\sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate θ_y about the y axis:
$$\mathbf{T}_{\theta_y} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotate α about the x axis: $$\mathbf{T}_{\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The final rotation is simply the concatenation of these rotations:

$$\mathbf{T} = \mathbf{T}_{\theta_x} \mathbf{T}_{\theta_t} \mathbf{T}_{\alpha} \mathbf{T}_{-\theta_x} \mathbf{T}_{-\theta_y}$$

where it should be noted that the reverse rotation matrices are simply the transposes of the initial rotation matrices. This final result can easily be evaluated numerically, given appropriate values for the angles.

Projections into a Plane

A projection from N dimensional space involves displaying the point (or objects) in a space of N-1 dimensions by simply ignoring one of the coordinates. In a rectilinear, orthogonal coordinate system, this produces what is called an orthographic projection since the effect is to project the objects using projection lines that are perpendicular to the plane of projection. Such projections can readily be accomplished from 3D into 2D (or from 2D into 1D) simply by zeroing one (or two) columns in the transformation matrix. This has the effect of zeroing the transformed coordinates corresponding to that column of **T**. For example, the following transformation matrix will create a projection into the z=0 plane:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Mirroring Across A Plane

The 3D equivalent of the 2D mirroring across a given line discussed in a previous section is to mirror across a given plane. The process for determining the needed transformations is similar to what was done earlier. The steps will be outlined below but the actual computations will be left as a student exercise. The resulting transformations are shown in ().

1. Translate until the mirroring plane passes through the origin ($\mathbf{T}_1$).
2. Rotate until the mirroring plane is aligned with either the x=0, the y=0 or the z=0 planes ($\mathbf{T}_2$).
3. Perform the mirror across the given plane ($\mathbf{T}_3$).
4. Reverse the transformations in steps #2 and #1.

The final concatenated transformation is:

$$\mathbf{T} = \mathbf{T}_1\, \mathbf{T}_2\, \mathbf{T}_3\, \mathbf{T}_2^{-1}\, \mathbf{T}_1^{-1}$$

where it should be noted that the inverse transformations are computed using the opposite signs for the indicated rotations and translations.

Axonometric Projections

Axonometric projections are orthographic projections in which the object has no particular orientation with respect to the axis system. This is in contrast to what are called multiview orthographic projections in which the principal object axes are aligned with the coordinate axes and projections are made into the x=0, y=0 and z=0 planes. Dashed lines are used to represent hidden lines, and the particular hidden lines depend on whether the object is in front of or behind the projection plane. US custom is to place the projection plane in front of the object while European convention is to place the plane behind the object. These two approaches are sometimes referred to at "third angle" and "first angle" projections, respectively.

There are three different kinds of axonometric projections and they are identified by the particular orientation of the object with respect to the projection plane. But rather than specify the orientation in terms of angles of rotation, the usual approach is to instead specify the change in length of the axis when it is projected into the projection plane. When an axis (or a line) is parallel to the projection plane, there is no change in length for the projection. On the other hand for any other orientation there will be some degree of shortening and the precise amount depends on the cosine of the angle between the axis and the projection plane. In the special case when the axis is perpendicular to the projection plane, the projection reduces to a single point.

To consider these cases, we will assume that an object (like a unit cube) has been arbitrarily oriented with respect to the coordinate system. You should recall from dynamics that this will require a minimum of two rotations about independent axes. Consider an arbitrary rotation ϕ about the y axis followed by a rotation of θ about the x axis:

$$\mathbf{T} = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & -\sin\phi\cos\theta & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ \sin\phi & -\cos\phi\sin\theta & \cos\phi\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Trimetric Projection

This is the most general case and no particular orientation is specified. In this case the original coordinate axes will each project into new axes on the plane of projection with different degrees of shortening. As a result, no particular values for ϕ or θ need be specified.

Dimetric Projection

For this case, two of the three axes project into the projection plane with equal shortening. That is, two axes make equal angles with the projection plane. This can be defined mathematically in terms of the above arbitrary 3D rotation by requiring two axes have the same projected lengths. We can use a point vector, [1 0 0 1], to represent a unit vector in the x direction (and similar forms for the other 2 axes). The projections are then:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 1 \\ 0 & \cos\theta & 0 & 1 \\ \sin\phi & -\cos\phi\sin\theta & 1 & 1 \end{bmatrix}$$

The projected lengths are simple the row norms for the first 3 columns as follows:

x vector length: $\left|\vec{i}\right| = \left(\cos^2\phi + \sin^2\phi\sin^2\theta\right)^{1/2}$

y vector length: $\left|\vec{j}\right| = \left(\cos^2\theta\right)^{1/2}$

z vector length: $\left|\vec{k}\right| = \left(\sin^2\phi + \cos^2\phi\sin^2\theta\right)^{1/2}$

We need only equate 2 of the 3 and choosing the first two yields the following relationship (after a bit of trigonometric manipulation):

$$\sin^2\phi = \frac{\sin^2\theta}{1-\sin^2\theta}$$

This specifies a relationship between the two angles so we are free to choose one and the other will be determined. A common approach is to add another constraint that the third axis should experience a specified shortening. For example, assume that we specify the z axis is shortened by a factor of ½. This means that we must require for the z vector length:

$$\left(\frac{1}{2}\right. = \left(\sin^2\phi + \cos^2\phi\sin^2\theta\right)^{1/2}$$

Solving the above two equations for both ϕ and θ can be done by eliminating ϕ from the equations to yield a single equation for θ as follows:

$$8\sin^4\theta - 9\sin^2\theta + 1 = 0$$

which can be solved to yield, $\sin^2\theta = 1/8$. This in turn allows solution for ϕ from one of the remaining equations. The final result is:

θ = 20.7°

ϕ = 22.2°

It should be noted that there are many other dimetric projections, but none of them are of any particular practical use. Yet, many classical engineering graphics texts still include a discussion of this projection and how to develop it graphically.

Isometric Projection

The third projection requires that all three axes are projected to equal lengths on the projection plane. This can be specified by requiring all the vector lengths to be equal and this yields two equations. To illustrate this we will equate the x and y vector lengths and the x and z vector lengths to yield (after some algebra) the two equations for ϕ and θ as follows:

Equate x & y vectors: $\sin^2\phi = \dfrac{\sin^2\theta}{1-\sin^2\theta}$

Equate x & z vectors: $\sin^2\phi = \dfrac{(1-2\sin^2\theta)}{\left(1-\sin^2\theta\right)}$

or solving:

$$\sin^2\theta = 1/3$$

$$\sin^2\phi = 1/2$$

These can be solved for ϕ and θ to yield:

ϕ = 45° about the y axis

θ = 35.26° about the x axis

The result is the familiar isometric projection in which all three axes projected equally onto the projection plane. The resulting axis system defines the projected axes to be oriented at 120° to each other as shown for the unit cube in Fig. 6 below. The 45° rotation is obviously needed but the meaning of the 35.26° rotation deserves more explanation. As shown in Fig. 6, the diagonal of the unit cube is oriented perpendicular to the projection plane (and shown as a point). The diagonals are in a plane rotated 45° about the y axis as noted but the appropriate angle for θ must be determined by analyzing the plane in which the diagonal lies. In this case it should be obvious that the diagonal is rotated in the plane by an angle whose tangent is $1/\sqrt{2}$ and this is the angle of 35.26° given above!

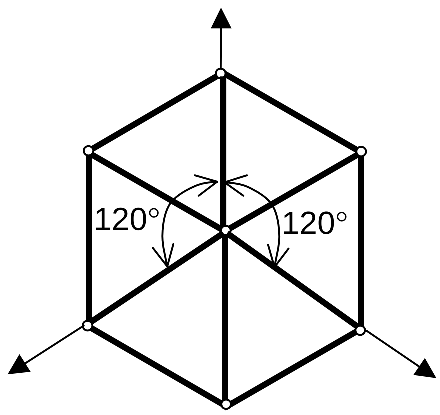

Figure 6. Isometric Projection

The isometric projection is very useful in engineering drawings because the same scale can be applied to all three axes, thus allowing information to be taken directly from the drawing using a ruler. By backsubstituting the ϕ and θ angles, we find that the shortening for each axis is 0.82 and this factor must be taken into account if dimensions are scaled off the drawing. Another approach is to increase the scale factor for the isometric view by the reciprocal (1.22) so that the dimensions are now at full scale. This allows dimensions to be taken off the drawing without any scaling but the object will appear to the eye to be about 22% larger.

Perspective Transformations

The previous transformations have all been affine transformations since they do not involve nonzero values for the p,q,r terms in the 4^{th} column. The role of p,q were discussed briefly in discussions of the 2D transformations but further analysis of these cases were postponed to the 3D treatment. Consider now the transformation with only p,q,r present. In particular, we will first consider r as the only nonzero term.

Perspective Projection

A perspective projection is usually formed by projecting a point (x,y,z) into the z=0 plane using a line of projection that passes through a point on the z axis called the center of projection. When the projection of multiple points and lines defining a geometric object is made onto the z=0 plane, the resulting image looks very much like a photographic image. In this case parallel lines on the object appear to come together in the projected image at points that are called vanishing points. A familiar example is the projection of the parallel rails of a railroad track which appear to come together in the projected image at a vanishing point.

We can create the same kind of projection by applying a transformation with diagonal terms equal to 1 and r as the only other nonzero term. We will follow this with a projection into the z=0 plane and this can be accomplished by zeroing any terms in the 3rd or z column of the transformation matrix as noted earlier. To illustrate this consider the following transformation:

$$\mathbf{p}^* = \mathbf{p}\mathbf{T} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x & y & 0 & rz+1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{rz+1} & \frac{y}{rz+1} & 0 & 1 \end{bmatrix} = \begin{bmatrix} x^* & y^* & 0 & 1 \end{bmatrix}$$

The geometric interpretation of this result is somewhat easier to make if only a single term in considered at a time. For example, the above equation indicates that the transformed x value, x*, is given as:

$$x^* = \frac{x}{rz+1}$$

This is somewhat easier to explain geometrically if we consider the y=0 plane as shown in Fig. 7 below. It can easily be shown that this result defines the particular geometry of the similar triangles shown in the figure. Using the geometry in Fig. 7, the ratio of the x and z sides of the two triangles can be expressed as:

$$\frac{x}{z+1/r} = \frac{x^*}{1/r}$$

This can be solved for x* to yield:

$$x^* = \frac{x}{rz+1}$$

which is exactly the result developed above from the transformation operation.

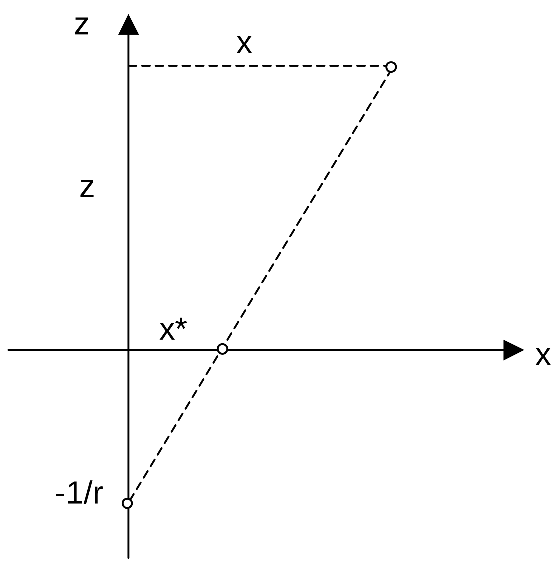

Figure 7. Perspective Projection as Viewed in the y=0 Plane

A similar geometric interpretation can be made for the y coordinates (by plotting the geometry in the x=0 plane). These results define what is called a perspective projection of the original (x,y,z) coordinates into new coordinates (x*,y*) in the z=0 projection plane.. Specifically, points are projected into the z=0 plane using projection lines that all pass through the point at 1/r on the negative z axis (which is called the center of projection). Several interesting features of this projection are worth noting.

1. Points with positive x coordinates lying above the center of projection all project to points on the projection plane (z=0) with positive x values as well.
2. Points with positive x coordinates lying below the center of projection all project to points on the projection plane with negative x values. This produces an image reversal that is similar to what happens in an optical lens system.
3. Points with the same x (or y) coordinate value but different z coordinates will project to points on the projection plane with different x coordinates. In other words, points farther away in the z direction will project closer and closer to the origin in the projection plane. In fact all points at z=∞ project to the point (0,0) on the projection plane!

These results are illustrated in Fig. 8 where the perspective projection into the z=0 plane is shown for a vertical line, AB lying above the z=0 plane (the figure is just a 3D extension of Fig. 7). It is interesting to consider what happens as point A is extended vertically. Geometrically, the projected point, A*, moves towards the origin and will converge there when A reaches z=∞. Similarly, the bottom point projection, B*, will move towards the point, B_0, in the z=0 plane as endpoint B moves downward towards the z=0 plane.

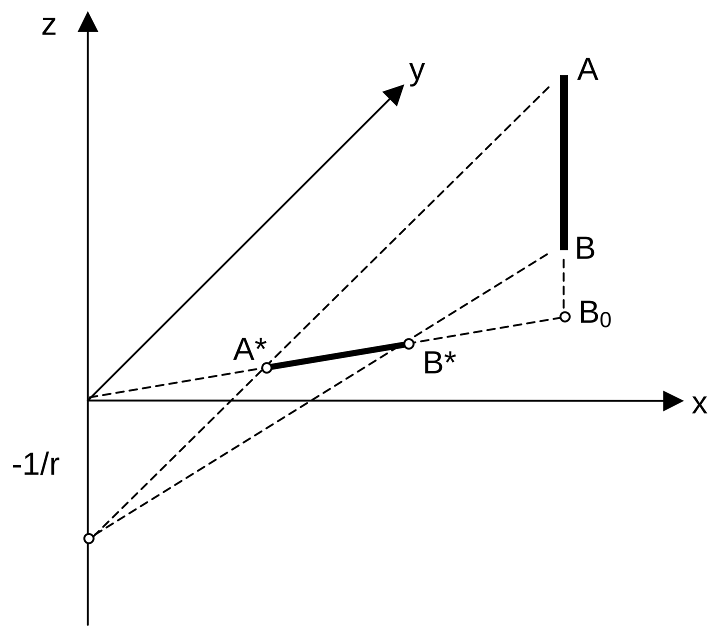

Figure 8. Perspective Projection of a Vertical Line AB

This behavior is also shown in the transformation of the points defining A and B as follows:

$$\begin{bmatrix} x_A & y_A & z_A & 1 \\ x_B & y_B & z_B & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_A & y_A & z_A & rz_A + 1 \\ x_B & y_B & z_B & rz_B + 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{x_A}{rz_A + 1} & \dfrac{y_A}{rz_A + 1} & \dfrac{z_A}{rz_A + 1} & 1 \\ \dfrac{x_B}{rz_B + 1} & \dfrac{y_B}{rz_B + 1} & \dfrac{z_B}{rz_B + 1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_A{}^* & y_A{}^* & z_A{}^* & 1 \\ x_B{}^* & y_B{}^* & z_B{}^* & 1 \end{bmatrix}$$

For a given value of r defining the center of projection, it is obvious that $x_A^* < x_B^*$ because $z_A > z_B$ so the denominator for x_A^* is larger than the denominator for x_B^*. Furthermore, as $z_A \to \infty$ it follows that $x_A^* \to 0$ and $y_A^* \to 0$ as illustrated geometrically in Fig. 7.

Perspective Projection

As a final point, Fig. 9 below shows the situation when the perspective projection is applied without a projection into the z=0 plane. In this case, the 3rd column in **T** is not zeroed out so that a z* coordinate will be computed. This is a perspective transformation and not a perspective projection. When the new z_A^* and z_B^* values are included it is apparent that the transformed points, A* and B*, are located on a line passing through a point located on the z axis at +1/r. This point is the true 3D vanishing point where both A* and B* will end up if their initial z coordinates are allowed to increase to ∞. (We saw in Fig. 8 that this point projects into the origin in the z=0 plane of projection for a perspective projection.)

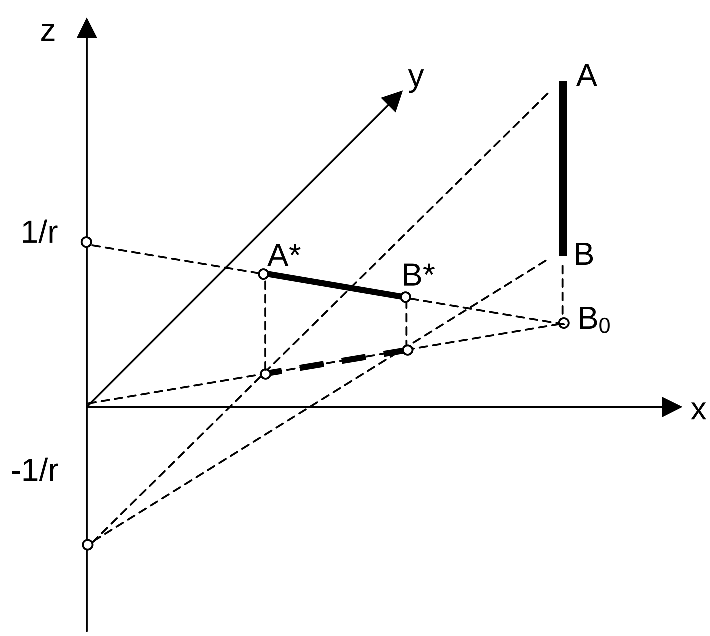

Figure 9. Full 3D Perspective Projection of Line AB

Multiple Vanishing Points

In the previous illustrations there is only a single vanishing point located at the origin. In general for a 3D object there may be one, two or three vanishing points in the transformed x*, y* and z* directions. To see this, we will consider applying a perspective projection into the z=0 plane after having first applied a rotation about the y axis ($\mathbf{T}_\theta$) and a general translation ($\mathbf{T}_t$). In this case we will choose the center of projection at a point z=k on the positive z axis. The resulting concatenated transformation is:

$$\mathbf{T} = \mathbf{T}_\theta \, \mathbf{T}_t \, \mathbf{T}_{pz} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{1}{k} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & 0 & 0 & \dfrac{\sin\theta}{k} \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & 0 & -\dfrac{\cos\theta}{k} \\ 0 & 0 & 0 & 1-\dfrac{n}{k} \end{bmatrix}$$

It is immediately apparent that the rotation and translation has produced an additional nonzero term in the 4th column of the resulting transformation matrix, and this leads to the appearance of two vanishing points. Note also that the (4,4) term which controls isotropic scaling is no longer unity which implies a scale change. This is the characteristic property of perspective projections to make objects closer to the center of projection seem larger and objects farther from the center of projection seem smaller. To determine the location of the vanishing points, we must project points located at ∞ along the x, y and z axes. Points at ∞ can be created as follows. Start with a point located on the x axis and consider what happens as the homogeneous coordinate, H, becomes vanishingly small:

$$\begin{bmatrix} x & 0 & 0 & H \end{bmatrix} = \begin{bmatrix} x/H & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x^* & 0 & 0 & 1 \end{bmatrix}$$

As H is reduced towards zero, the transformed coordinate, x*, approaches ∞. We conclude from this that points with H=0 correspond to points located at infinity. In particular we note that:

[1 0 0 0] → positive infinity along the x axis
[0 1 0 0] → positive infinity along the y axis
[0 0 1 0] → positive infinity along the z axis

Therefore, we can determine the locations of the vanishing points for the above case by transforming these 3 points at ∞ as follows:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} \cos\theta & 0 & 0 & \dfrac{\sin\theta}{k} \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & 0 & -\dfrac{\cos\theta}{k} \\ 0 & 0 & 0 & 1-\dfrac{n}{k} \end{bmatrix}
=
\begin{bmatrix} \cos\theta & 0 & 0 & \dfrac{\sin\theta}{k} \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & 0 & -\dfrac{\cos\theta}{k} \end{bmatrix}
$$

$$
= \begin{bmatrix} \dfrac{k}{\tan\theta} & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -k\tan\theta & 0 & 0 & 1 \end{bmatrix}
$$

This result indicates that vanishing points will appear for the x and z axes but not the y axis (its vanishing point is still located at ∞).To illustrate this transformation, Fig. 10 shows the transformation of a unit cube for which θ=30 and m=-2 with k=3. From the above results, the vanishing points will appear at:

x axis: $x^* = k/\tan\theta = 5.20$ and $y^*=z^*=0$.

z axis: $x^* = -k \tan\theta = -1.73$ and $y^*=z^*=0$.

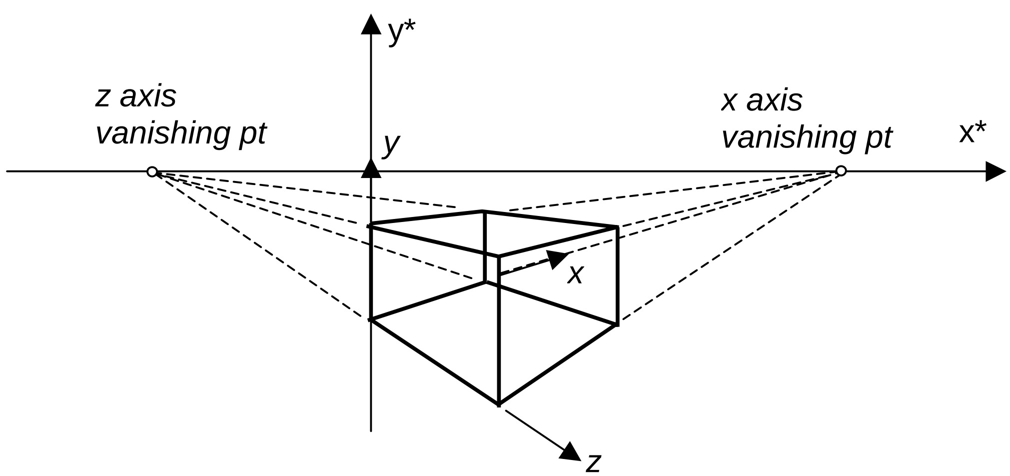

Figure 10. Projection of Unit Cube for θ=30, m=-2, r=-1/3.

Alternate Development of Homogeneous Coordinates

We have introduced homogeneous coordinates somewhat arbitrarily and from a purely mathematical point of view. However, we can also develop the same form simply by requiring that our linear transformation also include translation. Consider a general 3D linear transformation as follows:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} x^* & y^* & z^* \end{bmatrix}$$

In order to include translation given by components (x_0,y_0,z_0), we must modify this to the following:

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} = \begin{bmatrix} x^* & y^* & z^* \end{bmatrix}$$

Considering the application of this transformation to a 2-point polyline instead of a single point will reveal more details of the necessary process:

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix}^T \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix} = \begin{bmatrix} x_1^* & y_1^* & z_1^* \\ x_2^* & y_2^* & z_2^* \end{bmatrix}$$

Note that a column vector of 2 ones is needed to convert the translation component from a single row vector into a two-row vector with identical rows; this allows us to continue to represent the translation as a single row vector. We can express the above result in a more compact symbolic form as follows:

$$\mathbf{p}\,\mathbf{T} + \mathbf{1}^T \mathbf{p}_0 = \mathbf{p}^*$$

where $\mathbf{1}^{\mathrm{T}}$ is the special column vector noted above and $\mathbf{p}_0$ is the translation vector. This is still not a very useful format because it involves two separate terms on the left-hand side of the equation. In order to make this equation look more like a linear transformation, we can combine the pair of multiplications on the left-hand side as follows:

$$\begin{bmatrix} \mathbf{p} & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{p}_0 \end{bmatrix} = \mathbf{p}^*$$

This looks closer to what we want, and it is very interesting to note that the initial point vector, **p**, has now been augmented by a fourth column of 1's just like an homogeneous coordinate representation. But we will need to create the same representation for **p*** on the right-hand side and this can be done with a bit more matrix manipulation as follows:

$$\begin{bmatrix} \mathbf{p} & \mathbf{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{T} & \mathbf{0}^T \\ \mathbf{p}_0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{p}^* & \mathbf{1}^T \end{bmatrix}$$

The notation here is very tricky. We have added a new column to the right matrix on the left-hand side of the equation in order to produce the added column of 1's on the right-hand side result. In this case, we define the **0** vector as a row vector of 0's with the same number of 0's as used in the **1** vector above. This means that the term $\mathbf{0}^{\mathrm{T}}$ is a column of 0's. The final form for this result can be seen by expanding the terms in the above equation as:

$$\begin{bmatrix}\mathbf{p} & \mathbf{1}^T\end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{bmatrix}$$

$$\begin{bmatrix}\mathbf{T} & \mathbf{0}^T \\ \mathbf{p}_0 & 1\end{bmatrix} = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & h & i & 0 \\ x_0 & y_0 & z_0 & 1 \end{bmatrix}$$

$$\begin{bmatrix}\mathbf{p}* & \mathbf{1}^T\end{bmatrix} = \begin{bmatrix} x_1* & y_1* & z_1* & 1 \\ x_2* & y_2* & z_2* & 1 \end{bmatrix}$$

It should now be clear that the above result is really the same transformation matrix that we developed in homogeneous coordinates. In effect, we are required to add a 4^{th} column to our point vectors and set the value to 1. This allows us to expand the **T** matrix by adding a 4^{th} row containing the translation components and a 4^{th} column containing 0's (except for the **T**(4,4) element which must be 1 also).