Geometric Transformations and Projections

Prelude: Different Forms of the Transformation Matrix:

The transformation matrix can be expressed in two different forms:

- Post-multiplication form

  In this approach, a point is represented as a row array (or matrix). The new point is obtained by post-multiplying the old point in the following form, as illustrated using a 2D rotation and translation example.

$$p = [x \;\; y \;\; 1]$$

$$p_{new} = p_{old} \times [T]^{composite}_{post-multiply} = p_{old} \times [R]^{rotation\,about\,Z\,axis}_{post-multiply} \times [T]^{translation\,by\,(a,b)}_{post-multiply}$$

$$[R]^{rotation\,about\,Z\,axis}_{post-multiply} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; and$$

$$[T]^{translation\,by\,(a,b)}_{post-multiply} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & b & 1 \end{bmatrix}$$

  The post-multiplication was once popular and used in our old notes on the web. However, this approach is seldom used in more recent textbook and literatures.

- Pre-multiplication form

  In this approach, a point is represented as a column array (or matrix). The new point is obtained by pre-multiplying the old point in the following form, as illustrated using the same 2D rotation and translation example.

$$p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x & y & 1 \end{bmatrix}^T$$

$$p_{new} = [T]^{composite}_{pre-multiply} \times p_{old} = [T]^{translation\,by\,(a,b)}_{pre-multiply} \times [R]^{rotation\,about\,Z\,axis}_{pre-multiply} \times p_{old}$$

$$[R]^{rotation\,about\,Z\,axis}_{pre-multiply} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; and$$

$$[T]^{translation\,by\,(a,b)}_{pre-multiply} = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}$$

  The pre-multiplication approach is widely used today in textbooks and computer manuals. **We use this approach in the lecture and in additional notes (on the web and distributed in class).**

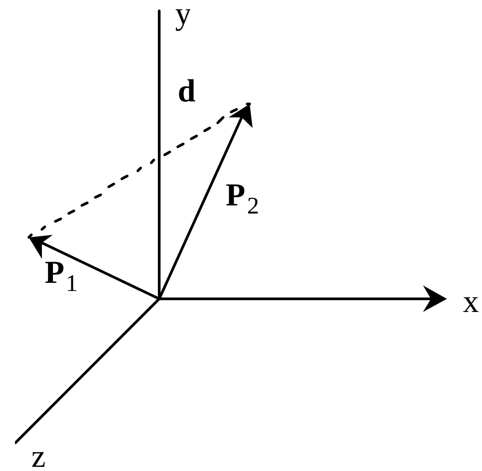
  The two approaches are equivalent. However, we do want to be consistent with current practice. We will use pre-multiplication in our assignments and exam.

1. Translations and Rotations

a) Translation

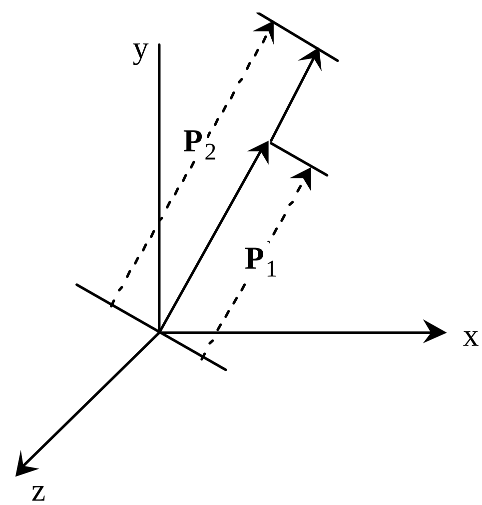
$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_d \\ y_d \\ z_d \end{bmatrix} = \begin{bmatrix} x_1 + x_d \\ y_1 + y_d \\ z_1 + z_d \end{bmatrix} = \mathbf{P}_1 + \mathbf{d}$$



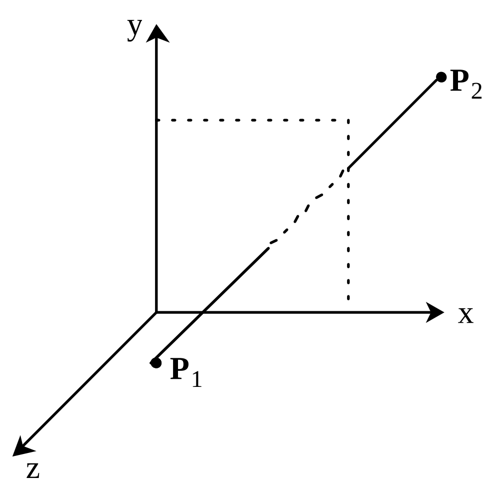
b) Scaling

$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} sx_1 \\ sy_1 \\ sz_1 \end{bmatrix} \quad \mathbf{P}_2 = s\mathbf{P}_1$$

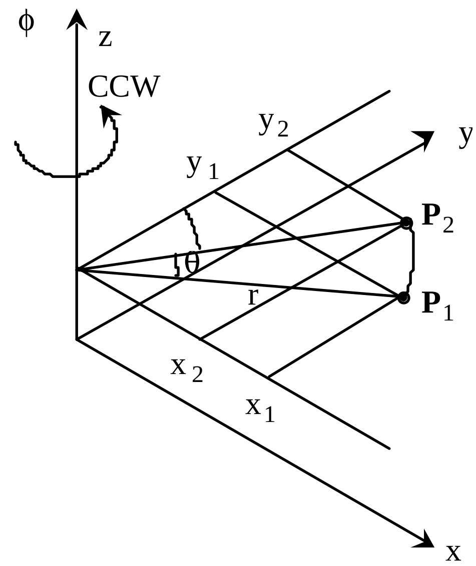


c) Reflection (about XOY Plane)

$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} x_1 \\ y_1 \\ -z_1 \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$



d) Rotation about z Axis



$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \mathrm{r}\cos\phi \\ \mathrm{r}\sin\phi \\ \mathrm{z} \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \mathrm{r}\cos(\phi+\theta) \\ \mathrm{r}\sin(\phi+\theta) \\ \mathrm{z} \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{r}\cos\phi\cos\theta - r\sin\phi\sin\theta \\ \mathrm{r}\cos\phi\sin\theta + r\sin\phi\cos\theta \\ \mathrm{z} \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta - y_1\sin\theta \\ x_1\sin\theta + y_1\cos\theta \\ z_1 \end{bmatrix}$$

2. Homogeneous Representation

The representation is introduced to express all geometric transformations in the form of matrix multiplication for the convenience of manipulation.

a) Translation

$$\mathbf{P}_2 = [\mathrm{D}]\ \mathbf{P}_1 \ \text{ or } \begin{bmatrix} x_2 \\ y_2 \\ z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + T_x \\ y_1 + T_y \\ z_1 + T_z \\ 1 \end{bmatrix}$$

b) Scaling

$$\mathbf{P}_2 = [\mathrm{S}]\ \mathbf{P}_1 \quad \text{and} \quad [\mathrm{S}] = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

c) Reflection (Mirroring)

$$\mathbf{P}_2 = [\mathrm{M}]\mathbf{P}_1 \quad \text{and} \quad [\mathrm{M}] = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ about } \begin{matrix} YOZ \\ XOZ \\ XOY \end{matrix} \text{ with “-“ sign.}$$

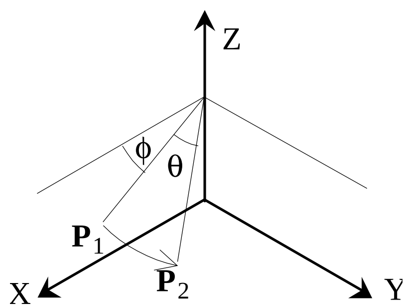
d) Rotation about the z Axis

$$\mathbf{P}_2 = [\mathrm{R_z}]\mathbf{P}_1 \quad \text{and} \quad [\mathrm{R_z}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly

$$[\mathrm{R_x}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad [\mathrm{R_y}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about Z Axis - CCW by θ



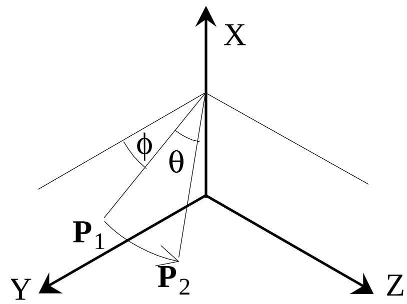
$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} r\cos\phi \\ r\sin\phi \\ z_1 \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \\ z_1 \end{bmatrix} = \begin{bmatrix} r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ r\cos\phi\cos\theta + r\sin\phi\cos\theta \\ z_1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta - y_1\sin\theta \\ x_1\sin\theta + y\cos\theta \\ z_1 \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = R_{[z]}\mathbf{P}_1$$

Rotation about X Axis - CCW by θ



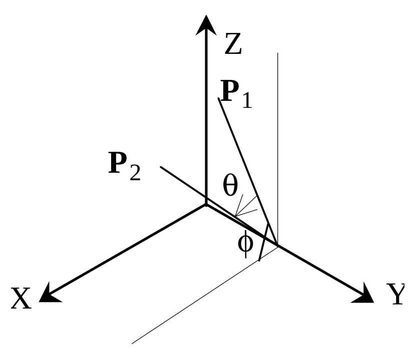
$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ r\cos\phi \\ r\sin\phi \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ r\cos(\phi+\theta) \\ r\sin(\phi+\theta) \end{bmatrix} = \begin{bmatrix} x_1 \\ r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ r\cos\phi\cos\theta + r\sin\phi\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ y_1\cos\theta - z_1\sin\theta \\ y_1\sin\theta + z_1\cos\theta \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = R_{[x]}\mathbf{P}_1$$

Rotation about Y Axis - CCW by θ



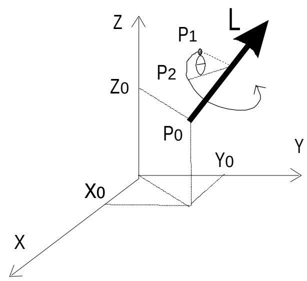
$$\mathbf{P}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} r\cos\phi \\ y_1 \\ r\sin\phi \end{bmatrix}$$

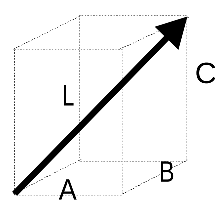
$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} r\cos(\phi-\theta) \\ y_1 \\ r\sin(\phi-\theta) \end{bmatrix} = \begin{bmatrix} r\cos\phi\cos\theta + r\sin\phi\sin\theta \\ y_1 \\ r\sin\phi\cos\theta + r\cos\phi\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta + z_1\sin\theta \\ y_1 \\ -x_1\sin\theta + z_1\cos\theta \end{bmatrix}$$

$$\mathbf{P}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = R_{[y]}\mathbf{P}_1$$

Rotation about an Arbitrary Axis



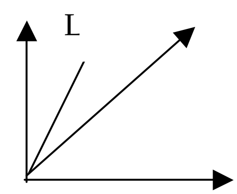


Parametric Rep. of the Axis:

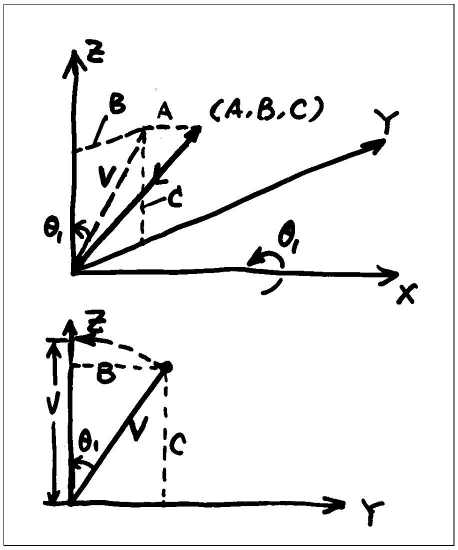
$$x = Au + x_0$$
$$y = Bu + y_0 \qquad 0 < u < 1$$
$$z = Cu + z_0$$
$$L = \sqrt{A^2 + B^2 + C^2}\,u$$

Step 1: Translate $\mathbf{P}_0$ to Origin

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Step 2: Rotate Vector $\vec{L}$ about X Axis to get $\vec{L}$ into the x - z plane



$$L = \sqrt{A^2 + B^2 + C^2}$$

$$V = \sqrt{B^2 + C^2}$$

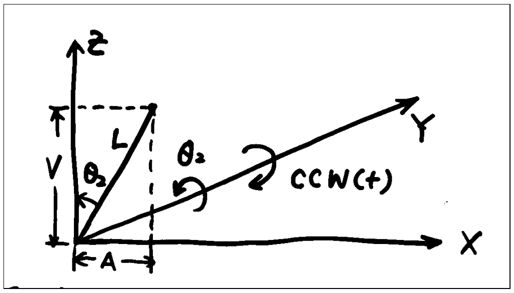
$$\sin\theta_1 = \frac{B}{V}$$

$$\cos\theta_1 = \frac{C}{V}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 & 0 \\ 0 & \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & -\frac{B}{V} & 0 \\ 0 & \frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Rotate $\vec{L}$ about the Y axis to get it in the z direction

Rotate a negative angle (CW)!

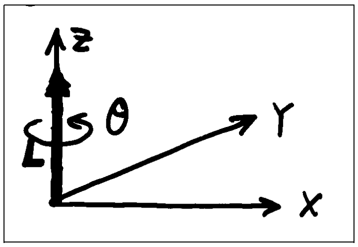


$$\sin\theta_2 = -\frac{A}{L}$$

$$\cos\theta_2 = \frac{V}{L}$$

$$[R_y] = \begin{bmatrix} \text{cox}\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{V}{L} & 0 & -\frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 4: Rotate angle θ about axis $\vec{L}$.



$$[R_z] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 5: Reverse the rotation about the Y axis

$$[R_y]^{-1} = \begin{bmatrix} \frac{V}{L} & 0 & \frac{A}{L} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{A}{L} & 0 & \frac{V}{L} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Rotation:

Replace	θ	by	– θ
	sin θ	by	– sin θ
	cos θ	by	– cos θ

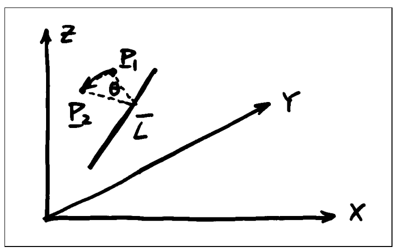
Step 6: Reverse rotation about the X axis

$$[R_x]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C}{V} & \frac{B}{V} & 0 \\ 0 & -\frac{B}{V} & \frac{C}{V} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 7: Reverse translation

$$[D]^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Overall Transformation



$$[T] = [D]^{-1}[R_x]^{-1}[R_y]^{-1}[R_z^{\theta}][R_y][R_x][D]$$
$$\mathbf{P}_2 = [T]\mathbf{P}_1$$

Assignment 4:

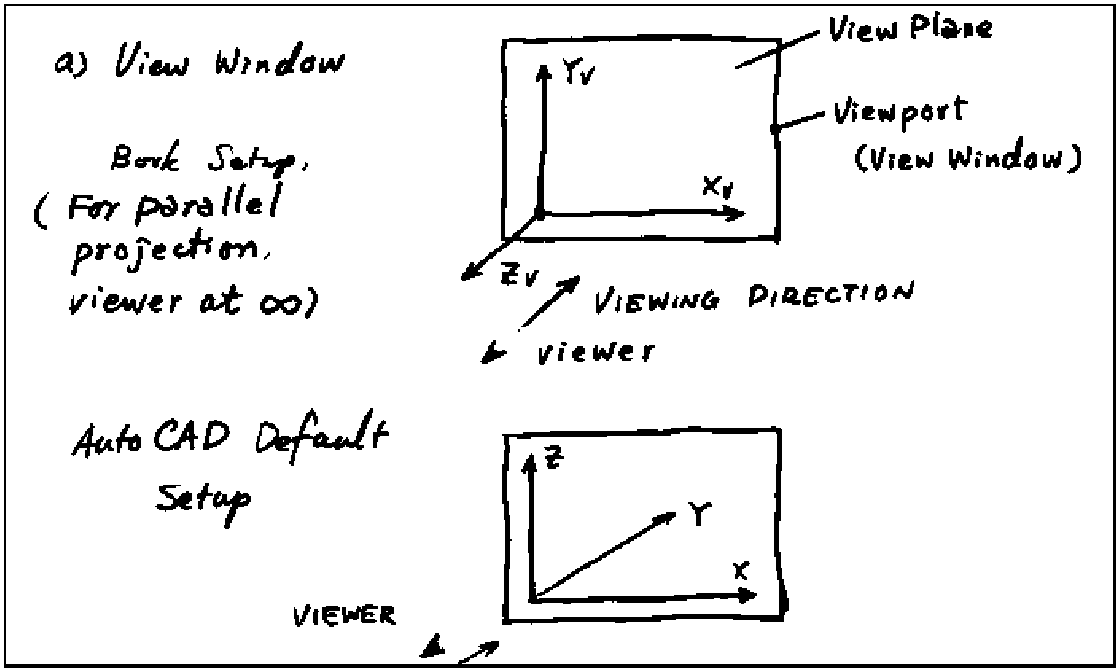
1. Given $\begin{bmatrix} x_0 & y_0 & z_0 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 & 4 & 1 \end{bmatrix}^T$, $\begin{bmatrix} A & B & C & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, and $\theta = 90^o$, find the composite transformation matrix [*T*] for **Rotation about an Arbitrary Axis**.

2. A line connects the point ***A*** at $[1,0,0]^T$ to the point ***B*** at $[1,0,1]^T$. A second line extends from ***C*** at $[1,0,2]^T$ to ***D*** at $[1,1,2]^T$. Rotate line ***AB*** about line ***CD*** using vector-matrix methods. The rotation should be 90^0 counter-clockwise as seen from the +*Y* axis.

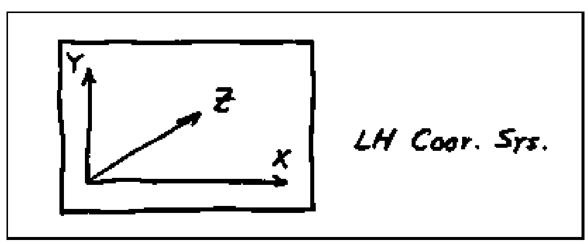
3. A plane surface intersects the coordinate axes at three points ***A*** = $[5,0,0]^T$, ***B*** = $[0,5,0]^T$ and ***C*** = $[0,0,l0]^T$. A given point ***P*** is on the plane. Find the matrix of geometric transformation that move the point ***P*** five units down on the plane to ***P'***. (Line ***PP'*** is perpendicular to edge **AB** and is on plane ***ABC***. ***PP'*** = 5.)

Viewing Coordinate System

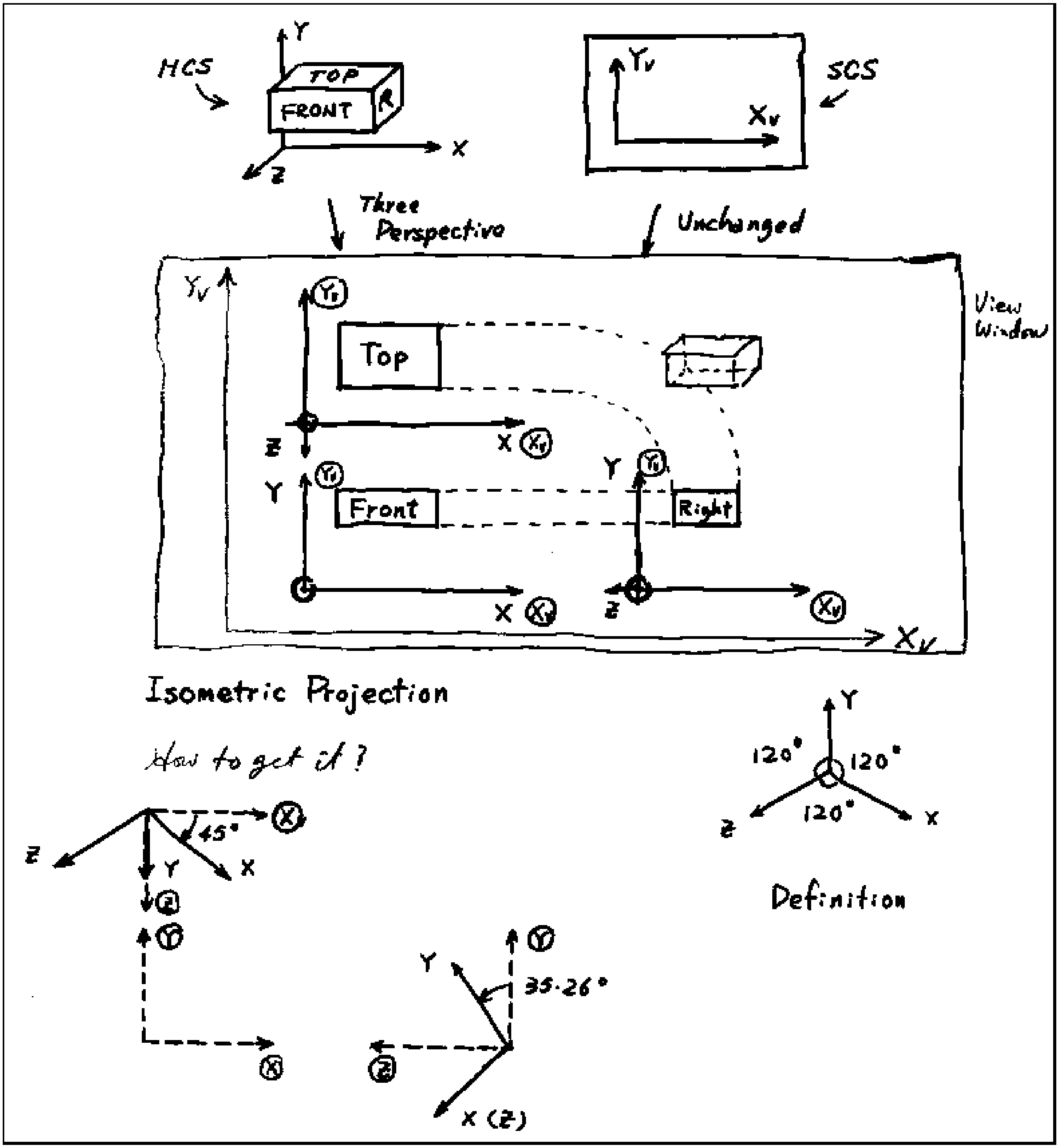
a) View Window



- Right hand coordinate system
- 2D coordinate system on the view plane.
- Some "old" graphics systems use a left-hand coordinate system. A different geometric transformation matrix must be used.



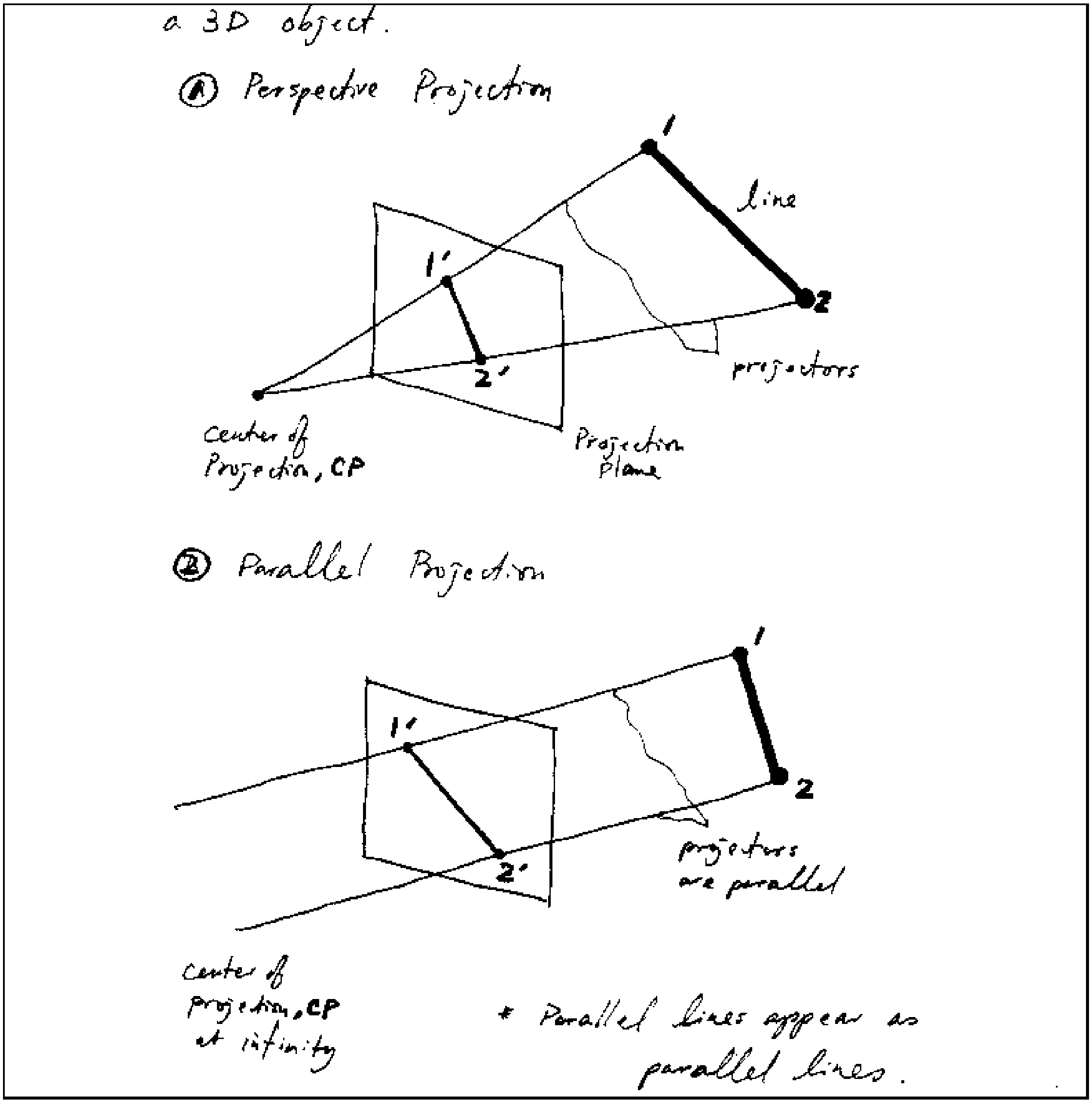
b) Orthographic View versus World Coordinate System



c) Perspective and Parallel Projections

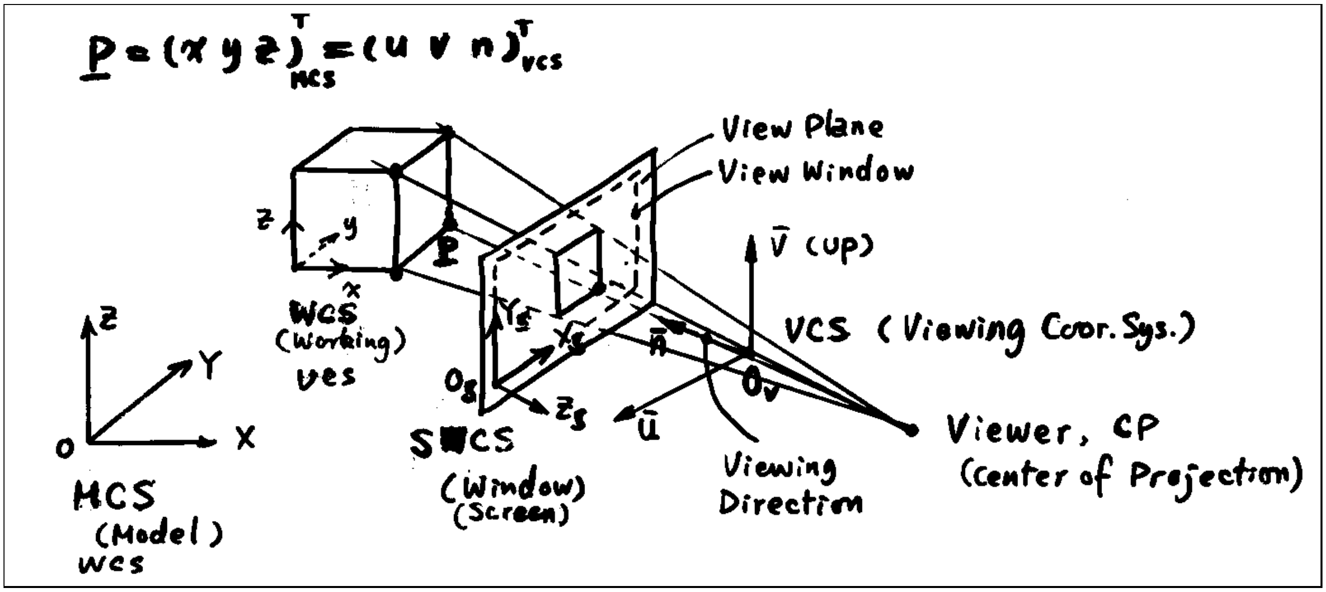
We want to produce a 2D image (projection) of a 3D object.

- Perspective Projection
- Parallel Projection



Definition of a General Viewing Coordinate System

a) What are involved?



b) How to Set Up the Viewing Coordinate System (VCS)!

i)Define the view reference point

$$\mathbf{P} = \left(P_x,\ P_y,\ P_z\right)^T$$

ii) Define the line of the sight vector $\vec{n}$ (normalized)

$$\vec{n} = \left(N_x,\ N_y,\ N_z\right)^T \text{ and } N_x^2 + N_y^2 + N_z^2 = 1$$

iii) Define the "up" direction

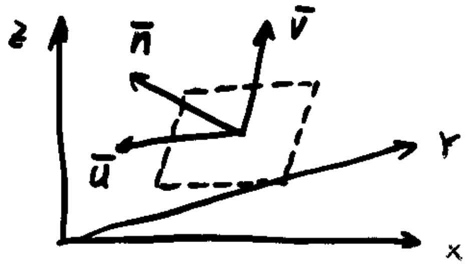
$$\vec{V} = \left(V_x,\ V_y,\ V_z\right)^T \perp \vec{n}, \qquad \vec{V} \cdot \vec{n} = 0$$

This also defines an orthogonal vector, $\bar{u}$

$$\vec{u} = \vec{V} \times \vec{n}$$

$\left(\vec{u},\ \vec{V},\ \vec{n}\right)$ forms the viewing coordinates

iv) Define the View Window in $\vec{U} - \vec{V} - \vec{W}$ coordinates



c) Parallel Projection

First transform coordinates of objects into the UVn coordinates (VCS), then drop the n component. (n – depth)

Overlapping x - y - z and U - V - n

i) Translate O_v to O.

$$[D] = \begin{bmatrix} 1 & 0 & 0 & -0_{vx} \\ 0 & 1 & 0 & -0_{vy} \\ 0 & 0 & 1 & -0_{vz} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii) Align the $\vec{n}$ axis with the Z axis.

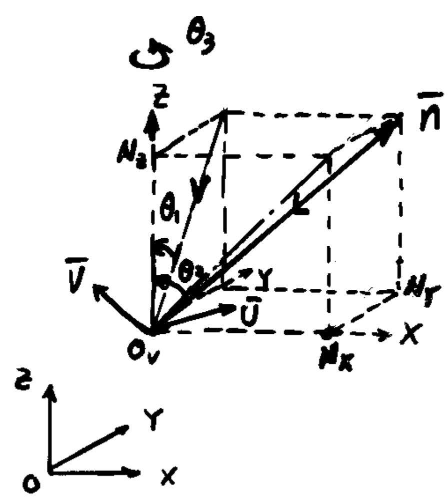
$$A = N_x, \quad B = N_y, \quad C = N_z$$

$$L = \sqrt{N_x^2 + N_y^2 + N_z^2}$$

$$V = \sqrt{N_y^2 + N_z^2}$$

The procedure is identical to that given in 5.2.

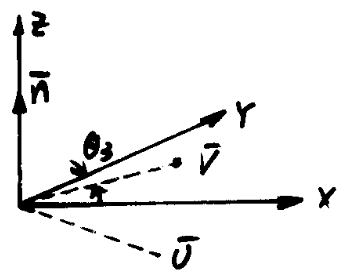
• rotate θ_1 about x: $[R_x]$



• rotate θ_2 about y: $[R_y]$

iii) Rotate θ_3 about the Z axis to align $\overline{U}$ with x and/or $\overline{V}$ with y. At this point, $\overline{V}$ is given by $\left(V'_x,\ V'_y,\ 0\right)^T$ where

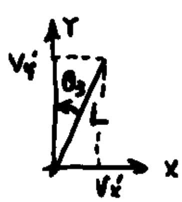
$$\begin{pmatrix} V'_x \\ V'_y \\ 0 \\ 1 \end{pmatrix} = \left[R_y\right]\left[R_x\right]\left[Do_V,o\right]\begin{pmatrix} V_x \\ V_y \\ V_z \\ 1 \end{pmatrix}$$

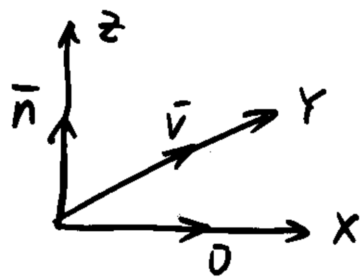


We need to rotate by an angle θ_3 about the Z axis

$$L = \sqrt{V'^2_x + V'^2_y}, \quad \sin\theta_3 = \frac{V'_x}{L}, \quad \cos\theta_3 = \frac{V'_y}{L}$$

$$\left[R_z\right] = \begin{bmatrix} V'_r/L & -V'_x/L & 0 & 0 \\ V'_x/L & V'_y/L & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





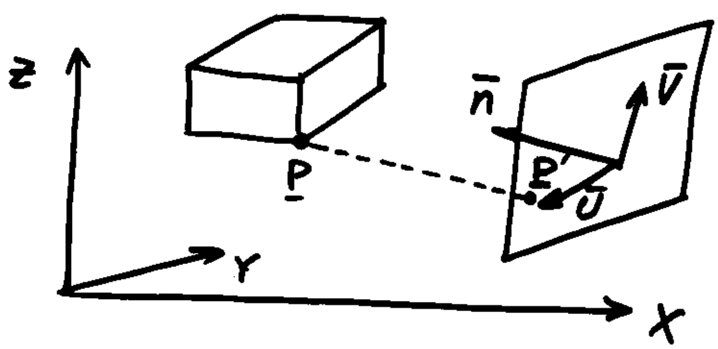
iv) Drop the **n** coordinate

$$[D_n] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{pmatrix} u \\ V \\ 0 \\ 1 \end{pmatrix} = [D_n] \begin{pmatrix} u \\ V \\ n \\ 1 \end{pmatrix}$$

In summary, to project a view of an object on the UV plane, one needs to transform each point on the object by:

$$[T] = [D_n]\,[R_z]\,[R_y]\,[R_x]\,[Do_v, o]$$

$$\mathbf{P}' = \begin{pmatrix} u \\ V \\ 0 \\ 1 \end{pmatrix} = [T]\mathbf{P} = [T] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



Note: The inverse transforms are not needed! We don't want to go back to x - y - z coordinates.