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A POSTERIOR REGION FOR PARALLEL PROFILE DIFFERENTIALS*

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A simple Bayesian solution for the joint estimation of several parallel profile differentials for multivariate normal populations is obtained. This generalizes a previous result for a single profile differential.

1. Introduction

The problem, in profile analysis, of testing whether or not constant differences exist among several multivariate normal populations has been discussed in detail by Greenhouse and Geisser, [1959]. In addition, Geisser, [1965] has given a Bayesian solution for the problem of estimating the constant difference between two parallel profiles. The purpose of this paper is to present a Bayesian solution for the more general problem of producing an estimate and a posterior region for the set of constant differences between k multivariate normal populations. That is to say, let $X_1^{(i)}, \cdots, X_{N_i}^{(i)}$ be a set of p-vectorial observations from a population which is $N(\eta + \mu_j e, \Sigma)$, for $j = 1, \cdots, k$. Here η is an unknown mean vector, $\mu_1 = 0, \mu_2, \cdots, \mu_k$ are unknown scalars, e is the vector whose elements are all unity and Σ is the common, unknown covariance matrix. We seek an estimate and a posterior region for $\mu' = (\mu_2, \cdots, \mu_k)$, the vector of profile differentials, on the assumption of the above model that the profiles are parallel.

2. A Posterior Density for μ

The joint likelihood function of η, μ, Σ^{-1} is

$$L(\eta, \mu, \Sigma^{-1}) \propto |\Sigma^{-1}|^{N/2}$$
$$\exp\cdot\left\{-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}\left[A + \sum_{j=1}^{k} N_j(\bar{X}^{(j)} - \eta - \mu_j e)\cdot(\bar{X}^{(j)} - \eta - \mu_j e)'\right]\right\}$$

where

$$N = \sum_{j=1}^{k} N_j, \quad \bar{X}^{(j)} = N_j^{-1} \sum_{i=1}^{N_j} X_i^{(j)}, \quad \text{for} \quad j = 1, \cdots, k,$$

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and

$$A = \sum_{j=1}^{k} \sum_{i=1}^{N_j} (X_i^{(j)} - \bar{X}^{(j)})(X_i^{(j)} - \bar{X}^{(j)})'.$$

In the absence of prior information, we assign as the joint prior density of μ, η, and Σ^{-1}

$$g(\mu, \eta, \Sigma^{-1}) \propto |\Sigma|^{(p+1)/2}.$$

Then the joint posterior density of μ, η, and Σ^{-1} is

$$(1) \qquad P(\mu, \eta, \Sigma^{-1}) \propto |\Sigma^{-1}|^{(N-p-1)/2} \cdot \exp \cdot \left\{ -\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} \left[A + \sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \eta - \mu_j e)(\bar{X}^{(j)} - \eta - \mu_j e)' \right] \right\}.$$

Using the fact that

$$\sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \eta - \mu_j e)(\bar{X}^{(j)} - \eta - \mu_j e)' = \sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \mu_j e - R)(\bar{X}^{(j)} - \mu_j e - R)' + N(\eta - R)(\eta - R)',$$

where

$$R = N^{-1} \sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \mu_j e),$$

in (1) and integrating out η, we obtain for the joint posterior density of μ and Σ^{-1}

$$P(\mu, \Sigma^{-1}) \propto |\Sigma^{-1}|^{(N-1-p-1)/2} \exp \cdot \left\{ -\tfrac{1}{2} \operatorname{tr} \Sigma^{-1} \left[A + \sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \mu_j e - R)(\bar{X}^{(j)} - \mu_j e - R)' \right] \right\}.$$

Integration with respect to Σ^{-1} yields the posterior density of μ which is

$$(2) \qquad P(\mu) \propto \left| A + \sum_{j=1}^{k} N_j (\bar{X}^{(j)} - \mu_j e - R)(\bar{X}^{(j)} - \mu_j e - R)' \right|^{-(N-1)/2}$$

$$\propto \left| A + \sum_{j=2}^{k} N_j (Z^{(j)} - \mu_j e)(Z^{(j)} - \mu_j e)' - N^{-1}(N - N_1)^2 (\bar{Z} - \bar{\mu} e)(\bar{Z} - \bar{\mu} e)' \right|^{-(N-1)/2}$$

where

$$\bar{\mu} = (N - N_1)^{-1} \sum_{j=2}^{k} N_j \mu_j, \quad Z^{(j)} = \bar{X}^{(j)} - \bar{X}^{(1)}, \quad \text{for} \quad j = 2, \cdots, k,$$

and

$$\bar{Z} = (N - N_1)^{-1} \sum_{j=2}^{k} N_j Z^{(j)}.$$

Let $Z = (Z^{(2)}, \cdots, Z^{(k)})$ and

$$\mathfrak{n} = N^{-1} \begin{bmatrix} N_2(N - N_2) & -N_2N_3 & -N_2N_4 & \cdots & -N_2N_k \\ -N_2N_3 & N_3(N - N_3) & -N_3N_4 & \cdots & -N_3N_k \\ -N_2N_4 & -N_3N_4 & & & \\ \vdots & \vdots & & & \vdots \\ -N_2N_k & -N_3N_k & & \cdots & N_k(N - N_k) \end{bmatrix}.$$

Then (2) is equivalent to

$$\begin{aligned} P(\mu) &\propto |A + (e\mu - Z)\mathfrak{n}(e\mu' - Z)'|^{-(N-1)/2} \\ &\propto |\mathfrak{n}^{-1} + (\mu e' - Z')A^{-1}(\mu e' - Z')'|^{-(N-1)/2} \\ &\propto |\mathfrak{n}^{-1} + (e'A^{-1}e)\mu\mu' - \mu e'A^{-1}Z - (e'A^{-1}Z)'\mu' + Z'A^{-1}Z|^{-(N-1)/2} \\ &\propto |I + (e'A^{-1}e)[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e][\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]' \\ &\quad \cdot [\mathfrak{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1}(Z'A^{-1}ee'A^{-1}Z)]^{-1}|^{-(N-1)/2} \\ &\propto \{1 + [\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]'(e'A^{-1}e)[\mathfrak{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1} \\ &\quad \cdot (Z'A^{-1}ee'A^{-1}Z)]^{-1}[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]\}^{-(N-1)/2} \end{aligned}$$

Thus we see that a posteriori μ has the generalized "Student" distribution with mean and modal vector $(e'A^{-1}e)^{-1}Z'A^{-1}e$, covariance matrix $[(N - k - 2)(e'A^{-1}e)]^{-1}[\mathfrak{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1}(Z'A^{-1}ee'A^{-1}Z)]$, and $N - k$ degrees of freedom.

3. A Posterior Region on μ

From the work of Geisser and Cornfield, [1963], it is clear that

$$\begin{aligned} Q(\mu) = (k - 1)^{-1}(N - k)(e'A^{-1}e)[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]'[\mathfrak{n}^{-1} + Z'A^{-1}Z \\ - (e'A^{-1}e)^{-1}(Z'A^{-1}ee'A^{-1}Z)]^{-1}[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e] \end{aligned}$$

has the F distribution with $k - 1$ and $N - k$ degrees of freedom. Hence, a Bayesian posterior region on μ can be based on the probability statement

$$\Pr \{Q(\mu) \leq F_{1-\alpha}(k - 1, N - k)\} = 1 - \alpha.$$

The region is hyper-ellipsoidal and is centered at the mean and modal vector of the posterior distribution of μ.

4. A Special Case

We consider now the case where $\eta = 0$ and $\mu_1 \neq 0$. This is a generalization of the problem of correlated estimates which was discussed by Halperin, [1961] and Geisser, [1965].

As before, if we take the joint prior density of μ and Σ^{-1}, where $\mu' = (\mu_1, \cdots, \mu_k)$, to be

$$g(\mu, \Sigma^{-1}) \propto |\Sigma|^{(p+1)/2}$$

then the marginal posterior density of μ is

$$P(\mu) \propto \left| A + \sum_{j=1}^{k} N_j(\bar{X}^{(j)} - \mu_j e)(\bar{X}^{(j)} - \mu_j e)' \right|^{-N/2}.$$

N, N_j and $\bar{X}^{(j)}$, for $j = 1, \cdots, k$, and A are as defined previously.

Let $Z = (\bar{X}^{(1)}, \cdots, \bar{X}^{(k)})$ and $\mathbf{n} = \{n_{ij}\}$ be the diagonal matrix such that $n_{ii} = N_i$, for $i = 1, \cdots, k$. Then

$$\begin{aligned} P(\mu) &\propto |A + (e\mu' - Z)\mathbf{n}(e\mu' - Z)'|^{-N/2} \\ &\propto |\mathbf{n}^{-1} + (e\mu' - Z)'A^{-1}(e\mu' - Z)|^{-N/2} \\ &\propto |(e'A^{-1}e)^{-1}[\mathbf{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1}Z'A^{-1}ee'A^{-1}Z] \\ &\quad + [\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e][\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]'|^{-N/2} \\ &\propto \{1 + [\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]'e'A^{-1}e[\mathbf{n}^{-1} + Z'A^{-1}Z \\ &\quad - (e'A^{-1}e)^{-1}Z'A^{-1}ee'A^{-1}Z]^{-1}[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]\}^{-N/2}. \end{aligned}$$

A posterior region on μ can be produced by making use of the fact that the posterior distribution of the quantity

$$\begin{aligned} &k^{-1}(N - k)(e'A^{-1}e)[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e]' \\ &\qquad \cdot[\mathbf{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1}Z'A^{-1}ee'A^{-1}Z]^{-1}[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e] \end{aligned}$$

is F with k and $N - k$ degrees of freedom.

If interest is to be focused on a linear combination $a'\mu$ then, [Geisser & Cornfield, 1963] the posterior distribution of

$$\begin{aligned} &a'[\mu - (e'A^{-1}e)^{-1}Z'A^{-1}e][e'A^{-1}e(N - k)]^{1/2}/\{a'[\mathbf{n}^{-1} + Z'A^{-1}Z \\ &\qquad\qquad - (e'A^{-1}e)^{-1}Z'A^{-1}ee'A^{-1}Z]a\}^{1/2} \end{aligned}$$

which is t with $N - k$ degrees of freedom can be utilized to provide an interval on $a'\mu$.

5. Remarks

In the paper by Geisser, [1965], "non informative" and natural conjugate densities were employed. Natural conjugate prior densities may also be utilized here. However, the algebra is similar, though more tedious, to the "non informative" case and hence is omitted.

An obvious application of this work is to "repeated measurement designs" in which there is "no interaction." In particular consider several groups of subjects which are measured on the same variable at p time points. The assumption of no group by time interaction is equivalent to the hypothesis of parallel group profiles [Greenhouse and Geisser, 1959]. When this hypothesis is tenable and the joint estimation of the profile differentials are of interest, the method described here is pertinent.

We may also obtain an estimate of $\mathbf{n}$ by computing its posterior mean. This is accomplished by first conditioning on μ yielding $E(\mathbf{n} \mid \mu) = R$ which is a function of μ. Hence the unconditional mean of $\mathbf{n}$ is $N^{-1} \sum_{i=1}^{k} N_i(\bar{X}^{(i)} - \hat{\mu}_i e)$ where $\hat{\mu}_j$ is the jth element of $(e'A^{-1}e)^{-1}Z'A^{-1}e$, $j = 2, \cdots, k$ and $\hat{\mu}_1 = 0$. Unfortunately, the posterior distribution of $\mathbf{n}$ is rather complicated and does not readily lend itself to the calculation of an estimating region for η.

6. Worked Illustration

Suppose we have three groups; G_1 a control group, and say G_2 and G_3 are experimental groups. A random sample of 11 subjects from each group is selected and a response for each of the 33 subjects is recorded at the same four time points. Over the given time span it is assumed that the group means have the same unknown functional form except for a difference in level. It is the intent to jointly estimate the two differentials, μ_2, μ_3, *i.e.*, the constant differences between the control group and the two experimental groups. Calculation leads to the following table of sample group means for the four time points.

Time points	G_1 $N_1 = 11$	G_2 $N_2 = 11$	G_3 $N_3 = 11$
t_1	16.4	18.7	22.6
t_2	13.2	15.3	19.6
t_3	15.7	18.5	21.5
t_4	12.1	14.8	18.0

For a graph of the group profiles see Figure 1.

The sample matrix of the sum of squared deviations is

$$A = \begin{bmatrix} 337.85 & 282.31 & 214.62 & 101.52 \\ 282.31 & 406.12 & 221.61 & 76.63 \\ 214.62 & 221.61 & 347.36 & 78.52 \\ 101.52 & 76.63 & 78.52 & 174.25 \end{bmatrix}$$

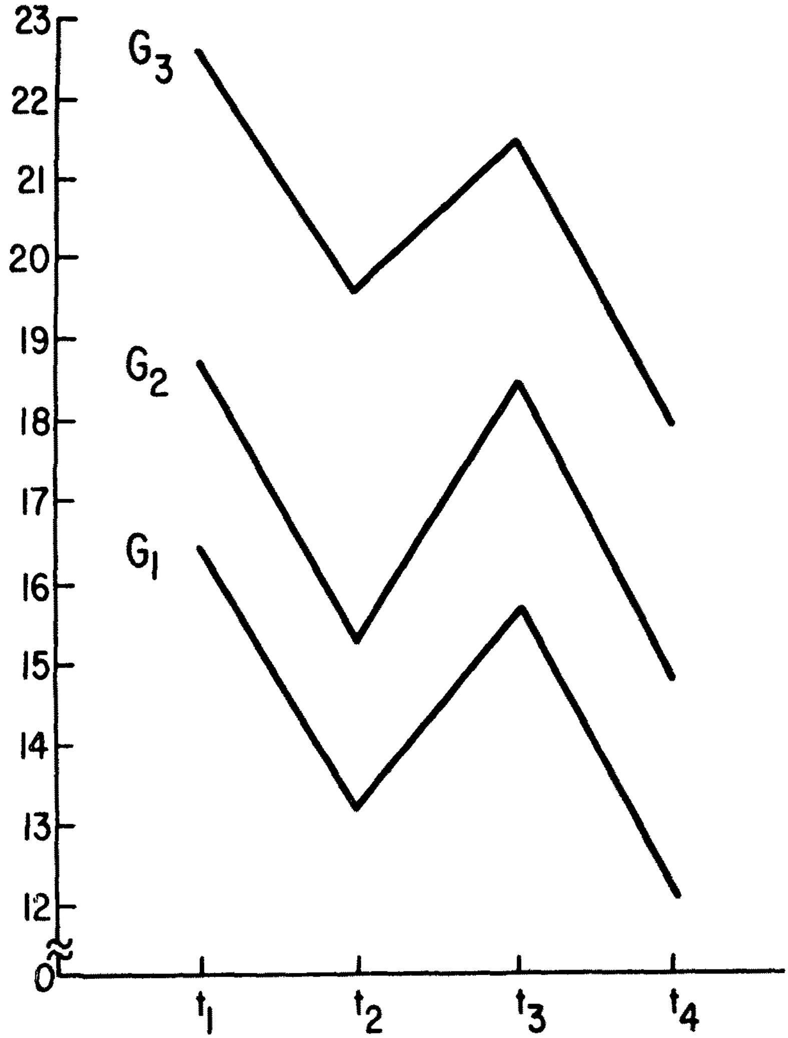

FIGURE 1
Graphical display of group profile averages

$$A^{-1} = \begin{bmatrix} .008635 & -.004526 & -.001960 & -.002157 \\ -.004526 & .006215 & -.001277 & .000479 \\ -.001960 & -.001277 & .005032 & -.000564 \\ -.002157 & .000479 & -.000564 & .007039 \end{bmatrix}$$

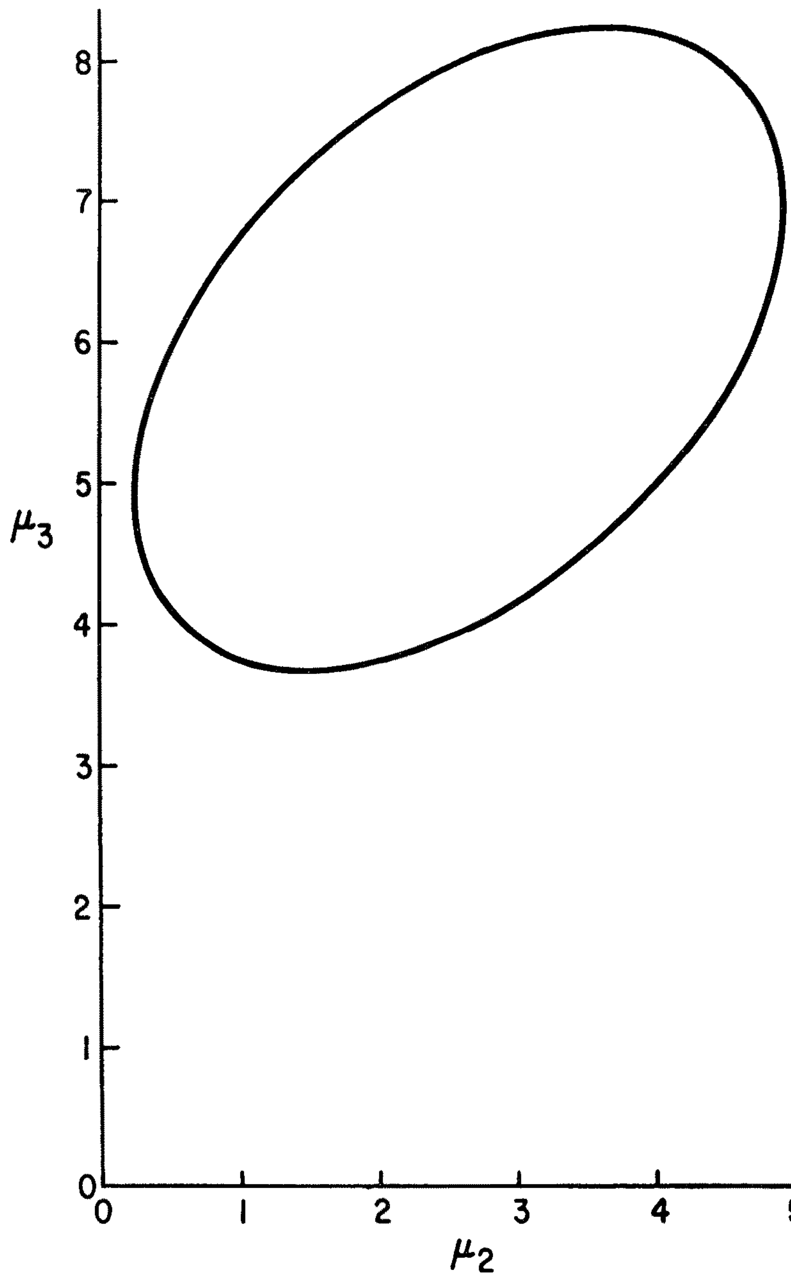

FIGURE 2
.95 joint region for μ_2 and μ_3

The sample matrix of mean differences Z, $\mathfrak{n}$ and $\mathfrak{n}^{-1}$ are

$$Z = \begin{bmatrix} 2.3 & 6.2 \\ 2.1 & 6.4 \\ 2.8 & 5.8 \\ 2.7 & 5.9 \end{bmatrix}, \quad \mathfrak{n} = \frac{11}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathfrak{n}^{-1} = \frac{1}{11}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The point estimate which is the center of the posterior distribution is

$$\hat{\mu} = \begin{pmatrix} \hat{\mu}_2 \\ \hat{\mu}_3 \end{pmatrix} = (e'A^{-1}e)^{-1}Z'A^{-1}e = \begin{pmatrix} 2.62 \\ 5.96 \end{pmatrix}.$$

Further we compute

$$\underline{\Delta} = \mathfrak{n}^{-1} + Z'A^{-1}Z - (e'A^{-1}e)^{-1}Z'A^{-1}ee'A^{-1}Z = \begin{pmatrix} .181534 & .083697 \\ .083697 & .167758 \end{pmatrix}$$

and

$$\Delta^{-1} = \begin{pmatrix} 7.1540 & -3.5694 \\ -3.5694 & 7.7418 \end{pmatrix}$$

Now we may compute a 95% joint region for μ_2 and μ_3, noting that $k = 3$ and $N = 33$ so that for the formula of section 3 all μ_2 and μ_3 satisfying

$$Q(\mu) \leq F_{.95}(2, 30) = 3.32$$

compose the 95% probability region for μ_2 and μ_3. Hence

$$Q(\mu) = \tfrac{1}{2}30[e'A^{-1}e](\mu_2 - \hat{\mu}_2, \mu_3 - \hat{\mu}_3)\, \Delta^{-1}\begin{pmatrix} \mu_2 - \hat{\mu}_2 \\ \mu_3 - \hat{\mu}_3 \end{pmatrix} \leq 3.32$$

and is equivalent to

$$.2363(\mu_2 - 2.62)^2 - .2371(\mu_2 - 2.62)(\mu_3 - 5.96) + .2557(\mu_3 - 5.96)^2 \leq 1,$$
which is sketched in Figure 2.

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