SIAM J. Appl. Math.
Vol. 65, No. 2, pp. 702–719

PROJECTIVE GEOMETRY OF HUMAN MOTION, WITH AN APPLICATION TO INJURY RISK*

H. LAURIE† AND R. PENNE‡

Abstract. We give an exposition of Plücker vectors for a system of joint axes in projective 3-space. We use Plücker vectors to analyze dependencies among joint axes and in particular to show that two rotational joints rigidly joined by a bar and each with 3 degrees of freedom always form a 5-dimensional system. We introduce the concept of reduced redundancy in a dependent set of projective Lines and argue that reduced redundancy in the axes of a body position increases injury risk. We apply this to a simple two-joint model of bowling in cricket and show by analysis of some experimental data that reduced redundancy around ball release is observed in some cases.

Key words. Plücker coordinates, human motion, reduced redundancy, injury risk

AMS subject classifications. 51N15, 51505, 70E17, 70E60, 92C10, 92C50

DOI. 10.1137/S0036139903429658

1. Introduction.

A variety of techniques exist for the mathematical analysis of human motion, including techniques that are also used in robotics [15, 2]. However, to our knowledge, nobody has yet employed the formalisms and insights of projective geometry, well known in robotics [18, 3].

We are motivated by the analysis of certain complicated athletic effects achieved by throwlike motions, such as a topspin serve in tennis and an away-swinger in cricket. It is clear that the brief interval ending in the release of the ball is crucial: after release, the ball is in free fall, except for some aerodynamic and gyroscopic effects. Thus the athlete must release the ball in a particular state of motion (translational as well as rotational).

Several questions arise: By what movements of the joints does a given athlete achieve a given effect? Is there more than one way to achieve a given effect? Do some effects require motions that are inherently more risky than others, and if so, can this risk be characterized analytically?

Many of these questions can be illuminated by using techniques from the mathematics of robotics, in which the following are possible: a simple description which unifies all aspects of the motion of the athlete and the ball, a representation in which rotation and translation are easily combined, and a level of generality at which all cases of reduced mobility can be found (and explicitly calculated).

In this report, we analyze what appears to us be a simple, interesting case: the motion of a cricketer's arm (much simplified) near the moment of delivery. We regard the hips as fixed, the torso as rigid, and the waist and shoulder as joints, each of which provides 3 degrees of freedom. Alert readers will notice that we ignore the elbow, wrist, and fingers, as well as any contact motion of the ball in the hand prior to release. For our purposes, it is assumed that the requirements of a given delivery have prescribed the motion at the center of the wrist. However, we will see that in our

*Received by the editors June 11, 2003; accepted for publication (in revised form) June 11, 2004; published electronically January 27, 2005.

http://www.siam.org/journals/siap/65-2/42965.html

†Department of Mathematics and Applied Mathematics, UCT, Rondebosch, 7701, South Africa (henri@pc001a.mth.uct.ac.za).

‡Industrial Sciences and Technology, Karel de Grote-Hogeschool, Antwerpen, Belgium (rudi.penne@kdg.be).

model in all positions there are only 5 degrees of freedom for the motion of the ball. Consequently, our system of 6 axes possesses intrinsic redundancy by design! It can be regarded as the solution of nature to the human desire to accomplish complicated motions. Indeed, kinematic redundancy offers an opportunity to distribute stress over many joints (see also [15]). Our main result is to prove the existence of special positions where *reduced redundancy* occurs: that is, for a given motion the amount of rotation about one or more joint axes is fixed, while there is some freedom in distributing motion about the remaining joint axes. These positions should not be confused with standard kinematic singularities, as no decrease of mobility is involved; 5 degrees of freedom are always maintained. We interpret reduced redundancy as a source of injury risk.

The paper is organized as follows: First, we describe the use of Plücker coordinates as a unified framework for computations concerning rotations and translations in many linked joints. Second, we describe a simplified model for the motion of an arm in the act of releasing a legal cricket delivery, a motion associated with risk of overuse injury [12, 14, 11]. Third, we analyze the degrees of freedom for the motion of the ball in this model, for the general case as well as all special cases; this includes a full analysis of reduced redundancy. Fourth, we discuss the analysis from two points of view: the prevention of injury and the forbidden motions of the wrist. Finally, we show that reduced redundancies indeed occur in real bowling actions by analyzing data from two bowlers with an injury history.

2. Plücker coordinates for human motion. Human motion is the result of rotations around joint axes, at least, infinitesimally in the first approximation (that is, neglecting the play in the joints and the deformation of bone, cartilage, and ligament). However, the desired motion of the end effector (in our case, the cricket ball) will in general have components of both rotation and translation. In projective geometry, translation can be rendered as rotation about an axis at infinity. In this view, all motions are rotations, and Plücker coordinates are merely a convenient way of describing them.

2.1. Projective points. In projective geometry, we identify all points on a line through the origin in $\mathbb{R}^4$ with a projective Point in the corresponding projective space $\mathbb{P}^3$. Thus the vector $\mathbf{x} = (\lambda a, \lambda b, \lambda c, \lambda d) \in \mathbb{R}^4$ corresponds to $\mathbf{p} \in \mathbb{P}^3$ for all $\lambda \neq 0$. Such a 4-vector is referred to as a set of homogeneous coordinates for $\mathbf{p}$. By the usual convention, the hyperplane $H{:}x_4 = 1$ in $\mathbb{R}^4$ is considered as (a copy of) affine 3-space. All Points of $\mathbb{P}^3$ which correspond to lines intersecting this hyperplane are called *finite points*, and these are identified with the affine point of H where they intersect. So for finite points,

$$(a, b, c, d) \sim (a/d, b/d, c/d, 1) \sim (a/d, b/d, c/d).$$

Notice that some lines through the origin in $\mathbb{R}^4$ do not intersect the hyperplane H, and therefore some projective Points are not finite. They are said to lie *at infinity*, and they are represented by homogeneous coordinates with 0 as the fourth coordinate: $(a, b, c, 0)$.

Similarly, planes through the origin of $\mathbb{R}^4$ correspond to Lines in $\mathbb{P}^3$. If such a plane is parallel to the hyperplane H, then it represents a Line at infinity. Finally, each 3-dimensional subspace of $\mathbb{R}^4$ is associated with a Plane in $\mathbb{P}^3$. The Plane corresponding to $x_4 = 0$ is the Plane at infinity of $\mathbb{P}^3$, and it contains all Points at infinity.

2.2. Plücker coordinates. From an algebraic point of view, a chosen set of homogeneous coordinates for a Point $\mathbf{p} \in \mathbb{P}^3$ represents a vector in the vector space $V = \mathbb{R}^4$. Now we can consider the *exterior algebra* built on V:

$$\wedge V = V^{(0)} \oplus V^{(1)} \oplus V^{(2)} \oplus V^{(3)} \oplus V^{(4)},$$

which enables us to make computations with scalars ($\mathbb{R} = V^{(0)}$), vectors ($V = V^{(1)}$), but also with more complicated objects called *antisymmetric tensors*, and this in the same framework. The *exterior product* $\wedge$ is a bilinear, antisymmetric operation on $\wedge V$, such that for $\mathbf{A} \in V^{(i)}$ and $\mathbf{B} \in V^{(j)}$ we get $\mathbf{A} \wedge \mathbf{B} \in V^{(i+j)}$ if $i + j \leq 4$ or $\mathbf{A} \wedge \mathbf{B} = \mathbf{0}$ otherwise (also in the case $i + j \leq 4$ it can happen that $\mathbf{A} \wedge \mathbf{B} = \mathbf{0}$ in $V^{(i+j)}$).

Example 1. The elements in $V^{(2)}$ (the so-called 2-tensors) are products $\mathbf{p} \wedge \mathbf{q}$ of vectors $\mathbf{p}$ and $\mathbf{q}$ in V, or linear combinations of these. Notice that $\mathbf{p} \wedge \mathbf{q} = \mathbf{0}$ in $V^{(2)}$ if $\mathbf{p}$ and $\mathbf{q}$ represent the same projective point, due to the antisymmetry.

For the reader who is not familiar with the exterior algebra it suffices to know for our purposes that each tensor can be regarded as just some vector, and $V^{(i)}$ as a real vector space of dimension $\binom{4}{i}$. For example, $V^{(2)} \cong \mathbb{R}^6$. Furthermore, using the standard basis of V, there is a canonical way to construct a basis for $V^{(i)}$. The corresponding coordinates arising in this manner for tensors are called *Plücker coordinates*.

Let us be more specific in the case of 2-tensors, because they will be needed most in this article. If $\mathbf{L} \in V^{(2)}$ then we have 6 Plücker coordinates for $\mathbf{L}$, by convention labeled by double-indices:

$$\mathbf{L} = (L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}).$$

In the special case that $\mathbf{L} = \mathbf{p} \wedge \mathbf{q}$ with $\mathbf{p} = (p_1, p_2, p_3, p_4)$ and $\mathbf{q} = (q_1, q_2, q_3, q_4)$, there is an easy rule to obtain the coordinates for $\mathbf{L}$: $L_{ij} = p_i q_j - p_j q_i$.

$$\mathbf{p} \wedge \mathbf{q} = \begin{pmatrix} p_1q_2 - p_2q_1 \\ p_1q_3 - p_3q_1 \\ p_1q_4 - p_4q_1 \\ p_2q_3 - p_3q_2 \\ p_2q_4 - p_4q_2 \\ p_3q_4 - p_4q_3 \end{pmatrix},$$

the elements of which are the 2×2 minors of the matrix $\left(\begin{smallmatrix} p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \end{smallmatrix}\right)$ in lexicographic order. If $\mathbf{p}$ and $\mathbf{q}$ represent different projective Points, then $\mathbf{L} = \mathbf{p} \wedge \mathbf{q}$ represents the projective Line through these two Points. Of course, many other 2-tensors represent the same Line in $\mathbb{P}^3$. Indeed, we can use a multiple of $\mathbf{p}$ or $\mathbf{q}$ without changing the involved projective Points, or we can even choose another pair of Points on the same Line. Fortunately, the new 2-tensor $\mathbf{L}' = \mathbf{p}' \wedge \mathbf{q}'$ will always be a multiple of $\mathbf{L}$: $\mathbf{L}' = \alpha \mathbf{L}$. We conclude that the Plücker coordinates of $\mathbf{L}$ can be considered as a 6-tuple of homogeneous coordinates for the projective Line represented by $\mathbf{L}$. Notice that Lines at infinity are characterized by having Plücker coordinates with $L_{14} = L_{24} = L_{34} = 0$.

Because not every 2-tensor in $V^{(2)}$ can be written as the exterior product of two vectors in V, not every 6-tuple of Plücker coordinates represents a Line in $\mathbb{P}^3$. More precisely, one can prove that $(L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34})$ corresponds to a projective line if and only if it differs from zero and the *Grassmann–Plücker relation* is satisfied:

$$L_{14}L_{23} - L_{24}L_{13} + L_{34}L_{12} = 0. \tag{GP}$$

Thus, “most” 6-tuples in $\mathbb{R}^6$ are not the Plücker coordinates of a projective Line. However, there is an interesting theorem, *Poinsot's central axis theorem*, which says that each 2-tensor $\mathbf{A}$ not obeying (GP) can be expressed as a sum $\mathbf{A} = \mathbf{L} + \mathbf{M}$ such that

1. $\mathbf{L}$ corresponds to a finite Line (not at infinity),
2. $\mathbf{M}$ corresponds to a Line at infinity,
3. Every affine Plane through $\mathbf{M}$ is perpendicular to $\mathbf{L}$.

Let us give one more illustration of Plücker coordinates. Given is a 2-tensor $\mathbf{A} = (A_{12}, A_{13}, A_{14}, A_{23}, A_{24}, A_{34})$ and a vector $\mathbf{p} =(p_1, p_2, p_3, p_4)$. Then $\mathbf{P} = \mathbf{A} \wedge \mathbf{p}$ belongs to the 4-dimensional space $V^{(3)}$: $\mathbf{P} = (P_{123}, P_{124}, P_{134}, P_{234})$, with

$$\begin{aligned}
P_{123} &= A_{12}p_3 - A_{13}p_2 + A_{23}p_1,\\
P_{124} &= A_{12}p_4 - A_{14}p_2 + A_{24}p_1,\\
P_{134} &= A_{13}p_4 - A_{14}p_3 + A_{34}p_1,\\
P_{234} &= A_{23}p_4 - A_{24}p_3 + A_{34}p_2.
\end{aligned}$$

In particular, if $\mathbf{A}$ represents a projective Line, which moreover does not contain the projective Point represented by $\mathbf{p}$, then $\mathbf{P}$ represents the projective Plane determined by this Line and this Point. If the Point lies on the Line, then $\mathbf{P} = \mathbf{0}$. In any case, if a 3-tensor differs from zero, it will represent a Plane in $\mathbb{P}^3$. Furthermore, it is the Plane at infinity if and only if $P_{124} = P_{134} = P_{234} = 0$. On the other hand, if $\mathbf{P} \in V^{(3)}$ represents a finite Plane, the vector $(P_{234}, -P_{134}, P_{124}) \in \mathbb{R}^3$ is perpendicular to the associated affine plane.

For a good introduction to Plücker coordinates and antisymmetric tensors, including the formal definitions, we refer to [18].

2.3. Dependencies among lines. A set of Lines in $\mathbb{P}^3$, finite or at infinity, is called independent (resp., dependent) if the corresponding 2-tensors are linearly independent (resp., dependent) in $V^{(2)}$ or, equivalently, if the corresponding Plücker coordinates are linearly independent (resp., dependent) in $\mathbb{R}^6$. These concepts are defined in algebraic terms; nevertheless, the possible dependencies among projective Lines have a transparent geometric characterization. We refer to [5] for a complete description of this. We quote only those situations that will be relevant for our analysis.

- Two Lines can only be dependent when they coincide.
- Three Lines are dependent if and only if they lie in the same Plane *and* go through the same Point.
- Four Lines are dependent if and only if at least one of the following cases occur:
  1. Three of the four Lines are dependent.
  2. The four Lines lie in the same Plane.
  3. The four Lines go through the same Point.
  4. Two of the Lines lie in a Plane α, intersecting in Point p, and the remaining two Lines lie in a Plane β, intersecting in Point q, such that the Planes α and β meet in the line pq.
  5. The four Lines belong to the same system of rulers on a quadratic surface.

In particular, if we are given two parallel Lines (intersecting at infinity), then a linear combination of their Plücker coordinates will always represent a Line in the unique Plane through the given Lines, and which either lies at infinity, or which is parallel to the given Lines. Four parallel Lines in 3-space are always dependent.

Example 2. As an illustration, let us consider a situation of 4 Lines, with $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ concurrent (but not coplanar) through Point $\mathbf{p}$, and $\mathbf{L}_4$ not containing $\mathbf{p}$. Clearly, these 4 Lines are independent. By taking linear combinations of the 3 concurrent Lines we can generate any Line $\mathbf{L}$ through $\mathbf{p}$. Furthermore, if $\mathbf{L}$ happens to intersect $\mathbf{L}_4$ (in $\mathbf{q}$, say), then linear combinations of $\mathbf{L}$ and $\mathbf{L}_4$ generate Lines through $\mathbf{q}$, lying in the Plane determined by $\mathbf{L}$ and $\mathbf{L}_4$. If $\mathbf{L}$ and $\mathbf{L}_4$ do not intersect each other, then we cannot obtain new Lines by combining them (a violation of (GP)). We conclude that the Lines which depend on $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \mathbf{L}_4$, are exactly those that contain $\mathbf{p}$ or that lie in the Plane through $\mathbf{L}_4$ and $\mathbf{p}$.

Next, we will elaborate on a special case which will be important for the applications in this paper.

THEOREM 2.1. *Let* $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ *and* $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$ *be two triples of concurrent Lines in* $\mathbb{P}^3$ *through different Points* $\mathbf{w}$ *and* $\mathbf{s}$. *Assume moreover that* $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ *are not coplanar, neither are* $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$. *Then the set* $\{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ *always has rank* 5. *Or more explicitly, these* 6 *Lines are always dependent but always contain a subset of* 5 *Lines which is independent.*

Proof. Choose a Plücker vector $\mathbf{P}$ to represent the Line $\mathbf{sw}$. By abuse of notation, we let $\mathbf{W}_1, \dots, \mathbf{S}_3$ stand for the Plücker vectors of the corresponding lines as well. Then there exist linear combinations

$$\begin{aligned} \mathbf{P} &= \alpha_1 \mathbf{W}_1 + \alpha_2 \mathbf{W}_2 + \alpha_3 \mathbf{W}_3, \\ \mathbf{P} &= \beta_1 \mathbf{S}_1 + \beta_2 \mathbf{S}_2 + \beta_3 \mathbf{S}_3, \end{aligned}$$

which gives rise to the claimed dependency:

$$\alpha_1 \mathbf{W}_1 + \alpha_2 \mathbf{W}_2 + \alpha_3 \mathbf{W}_3 - \beta_1 \mathbf{S}_1 - \beta_2 \mathbf{S}_2 - \beta_3 \mathbf{S}_3 = \mathbf{0}.$$

Next, we observe that at least one of $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ does not pass through $\mathbf{w}$, say $\mathbf{S}_1$. From the example above we learn that $\{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{S}_1\}$ is a set of independent Lines, and moreover, the only Lines which are dependent on these 4 Lines are Lines through $\mathbf{w}$, or Lines in the Plane determined by $\mathbf{S}_1$ and $\mathbf{w}$. Because the triple $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3\}$ is assumed to be nonplanar, it is impossible that both $\mathbf{S}_2$ and $\mathbf{S}_3$ depend on $\{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{S}_1\}$, which completes the proof. $\square$

2.4. Describing kinematics by Plücker coordinates. For a more extended exposition of the material presented in this paragraph, we refer to [18] and [4]. Consider a motion of a rigid body B in 3-space. Then, every point p of B traces a path, $p = p(t)$. If the motion is sufficiently *smooth* from a mathematical point of view, we can compute the derivative at a certain time t_0, giving us the infinitesimal motion of B at $t = t_0$. This results in a velocity vector $v_p = \dot{p}(t_0)$ for every point p of B. The rigidity of B can be translated into the statement that for every pair of its points $\{p, q\}$ the distance between these points must remain constant during the motion,

$$||p(t) - q(t)||^2 = \text{constant},$$

or infinitesimally (*preserved distance property*),

$$\text{(PDP)} \qquad (v_p - v_q) \cdot (p - q) = 0.$$

From now on, when we use the term “motion,” we always mean an infinitesimal rigid motion: the assignment of a velocity vector to every point of B such that (PDP) is satisfied. Thus, we associate a vector v_p to every point p of B, taking (PDP) into account.

One important example of such a motion is a spatial rotation about the origin. Here, there is always a line A involved, the so-called axis of rotation, containing the origin. Points on A remain fixed (zero velocity vector), but for other points p the velocity v_p is perpendicular to the plane determined by A and p. As a matter of fact, the rotation is specified by a vector ω along A, such that $v_p = \omega \times p$ (vector cross product). The length of ω is called the angular velocity, and together with the distance of p from the axis A, it determines the length of v_p.

Another fundamental motion is a translation along a given vector v. Here, we have a constant velocity: for every point p we put $v_p = v$.

A crucial theorem says that every rigid motion is the composition of rotations and translations, or infinitesimally, the velocity vectors can be written as the sum of rotation velocities and/or translation velocities.

Consider a rotation about some axis A, not necessarily containing the origin. If we embed affine 3-space into $\mathbb{P}^3$, as described in section 2.1, then we can associate with A a projective line $\mathbf{A}$, and hence a Plücker vector $\mathbf{P}_A$. For each point p in $\mathbb{R}^3$ we choose the standard homogeneous coordinates for the associated projective point $\mathbf{p}$ (having $p_4 = 1$). Now we can define the "motion of $\mathbf{p}$" as the following 3-tensor:

$$M(\mathbf{p}) = \mathbf{M} = \mathbf{P}_A \wedge \mathbf{p} \in V^{(3)}.$$

To see that this makes sense, consider a vector $\mathbf{M} = (M_{123}, M_{124}, M_{134}, M_{234})$ of Plücker coordinates. This determines the vector $v_p = (M_{234}, -M_{134}, M_{124}) \in \mathbb{R}^3$, which is zero if $p \in A$, or else it is perpendicular to the plane determined by p and A. And indeed, as one can prove that (PDP) holds for these vectors, they represent a rotation about axis A. The unused coordinate M_{123} in $M(\mathbf{p})$ is determined by the fact that this 3-tensor corresponds to a plane through $\mathbf{p}$. Of course, the magnitude of the vectors v_p depends on the chosen Plücker coordinates $\mathbf{P}_A$ for $\mathbf{A}$, but then again there are an infinite number of possible rotations about axis A in $\mathbb{R}^3$. One can say that the magnitude of the chosen Plücker vector accounts for the involved angular velocity. We conclude that the 2-tensor $\mathbf{P}_A$ encodes both the rotation axis A and the angular velocity. Therefore, it is called the *center* of the motion. Taking a multiple of $\mathbf{P}_A$ does not change the axis, only the angular velocity. If you are interested in the velocity of a specific point p under this motion, just perform the exterior product $\mathbf{P}_A \wedge \mathbf{p}$, using standard homogeneous coordinates for $\mathbf{p}$.

Now that we have put spatial rotations in the setting of projective geometry, we can extend the notion of rotation axis. Indeed, we can take $\mathbf{A}$ to be a line at infinity, so if $\mathbf{P} = \mathbf{P}_A$, then $\mathbf{P}_{14} = \mathbf{P}_{24} = \mathbf{P}_{34} = \mathbf{0}$. If we copy the previous computations for some point p, we observe, surprisingly, that the last three Plücker coordinates of $M(\mathbf{p})$ do not depend on p. Therefore, we see that v_p is a constant vector if we perform a rotation about an axis at infinity, which must be a translation! More precisely, $v_p = (\mathbf{P}_{23}, -\mathbf{P}_{13}, \mathbf{P}_{12})$, a vector which is perpendicular to any plane in $\mathbb{R}^3$ whose projective extension contains the given axis at infinity $\mathbf{A}$. For the sake of uniformity, we will again call the 2-tensor $\mathbf{P}_A$ the center of the motion, and the 3-tensor $M(\mathbf{p})$ the motion itself of the point p.

Our arguments will directly take place in $\mathbb{P}^3$ or $\wedge\mathbb{R}^4$, but readers who like to switch to affine space now and then should remember

$$\begin{aligned} p = (p_1, p_2, p_3) &\longrightarrow \mathbf{p} = (p_1, p_2, p_3, 1), \\ v_p = (M_{234}, -M_{134}, M_{124}) &\longleftarrow M(\mathbf{p}) = (M_{123}, M_{124}, M_{134}, M_{234}). \end{aligned}$$

In this setting, the zero-tensor in $V^{(3)}$ corresponds to the zero velocity.

As mentioned before, composing two motions comes down to adding the velocity vectors in each point p. Let the corresponding centers of motion be denoted by $\mathbf{C}_1$ and $\mathbf{C}_2$, Plücker vectors in $\mathbb{R}^6$. Then the resulting motion of p equals

$$\mathbf{C}_1 \wedge \mathbf{p} + \mathbf{C}_2 \wedge \mathbf{p} = (\mathbf{C}_1 + \mathbf{C}_2) \wedge \mathbf{p}$$

due to a basic property of the exterior product. Now we can consider $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ to be the center of the composite motion. This means that every Plücker vector $\mathbf{P}$ in $\mathbb{R}^6$ can play the part of a center of some motion. More precisely, if $\mathbf{P}$ represents a projective Line (satisfying (GP)), then it gives rise to a rotation (finite line) or a translation (line at infinity); otherwise it is the center of a composition of rotations and translations. As a consequence of Poinsot's central axis theorem (section 2.2) we can be even more specific in the latter case. To this end, we define a *screw motion* as the composition of a rotation (infinitesimal, of course) about some axis, and a translation (ditto) along the same axis. **If a motion is not a pure translation or rotation, then it is a screw motion.**

From now on, Plücker coordinates of 2-tensors (the space $\mathbb{R}^6$) are interpreted as centers of infinitesimal rigid motions.

3. A simple model for bowling a cricket delivery. Biomechanical models for cricket motions are not that rare, but few exist for bowling [1, 11]. For our purposes, a model simpler than either of these will suffice. We make the following assumptions about motion just prior to delivery:

1. There is no rotation in the elbow (as is required in a legal delivery).
2. There is no rotation in the wrist, and the state of motion of the ball upon release prescribes the motion of the so-called tool center (a term from robotics), which we take to be the wrist.
3. The spine is taken as rigid but free to rotate as if its base is attached to the pelvis in a ball joint (i.e., we ignore deformation of the torso), and the shoulder is rigidly joined to the spine.
4. The joint axes of both joints pass through the center of the joint.

For greater realism, one might add more joints; for example, it is known that the shoulder does rotate relative to the torso [8], and the ball might leave the hand in a contact motion. This is not conceptually difficult but is computationally and experimentally challenging. The same applies to relaxing assumption 4, to allow noncoincident joint axes. Still more challenging would be the direct modeling of muscle groups (as in [6, 16]), as this would increase the number of axes of rotation substantially, and one might be hard put to identify an axis of rotation for every muscle group, particularly those with attachments over more than one joint.

3.1. Introducing the joint axes of our model. With assumptions 1 to 4, the system reduces to two joints, which we call the waist (w) and the shoulder (s). Although in general w may be in motion, there is no loss of generality if we place w at the fixed origin and identify its joint axes with axes of a reference coordinate system XYZ. They are interpreted as follows: for a person standing, X points horizontally forward, Y horizontally points to the left, and Z points vertically along the spine, in our case upwards.

We choose units of length so that the right shoulder joint s is at $(0, -1, 1)$ in the system; since the torso is rigid, it stays there. The three joint axes through s follow the usual convention: We choose S_1 as the axis that passes through shoulder and

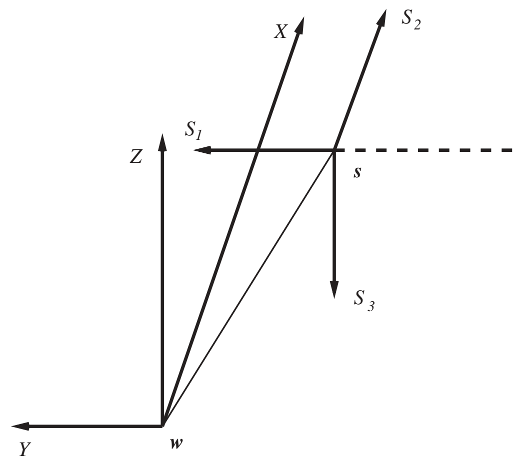


FIG. 3.1. *Configuration of joint axes in our simplified model of a right-handed cricket bowler, facing away from the reader.* **w** *is the waist joint and* **s** *is the shoulder joint. Distances are normalized so that the shoulder is at* $(0, -1, 1)$ *in the waist joint axes. The dashed line corresponds to the arm in standard position: horizontal, palm down. Note that the shoulder axes move with the arm so that* S_3 *is always pointing in the same direction as the palm.*

elbow, in the direction of the shoulder, and S_2, S_3 perpendicular to each other and to S_1 so that when the arm is extended sideways horizontally wrist down, S_2 points forwards and S_3 points downwards. This means that the $S_1S_2S_3$ system moves with the arm, and in particular that S_3 is always perpendicular to the palm. The general configuration is illustrated in Figure 3.1.

We note that some of our results below depend on the choice of shoulder axes. In particular, we find a case where rotation about the s_3 axis plays a significant role in predicting injury risk. This would be indefensible if all we knew of the shoulder joint was that it had 3 degrees of rotational freedom, because our result would disappear under many other apparently equivalent choice of axes.

However, we do know more about how the shoulder moves and about the motion of the arm of a fast bowler near the point of release. In that context, the $s_1s_2s_3$ system as described above is preferred and has intrinsic interpretation for two reasons. First, s_1 is an anatomically intrinsic axis in all rotations of the shoulder, because of the role of the rotator cuff, which are the only muscles that cause rotation around s_1. Second, during the final phase of the delivery of a fast ball in cricket, the bowler's arm moves in a plane. Near the moment of release, the direction of s_3 is tangential to this motion (since these bowlers aim for high speed deliveries), so the plane of motion is the s_1s_3 plane, and in that plane the motion is a pure rotation around the s_2 axis. By orthogonality to both the s_1 and s_2 axes, the s_3 axis is also intrinsic.

3.2. Plücker coordinates of the 6 joint axes. The positions of the waist and the shoulder are given by

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s = \begin{pmatrix} 0 & -1 & 1 & 1 \end{pmatrix}.$$

The directions of the joint axes are

$$\begin{aligned} W_1 &= \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}, \\ W_2 &= \begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}, \\ W_3 &= \begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}, \\ S_1 &= \begin{pmatrix}a_1 & b_1 & c_1 & 0\end{pmatrix}, \\ S_2 &= \begin{pmatrix}a_2 & b_2 & c_2 & 0\end{pmatrix}, \\ S_3 &= \begin{pmatrix}a_3 & b_3 & c_3 & 0\end{pmatrix}. \end{aligned}$$

The six centers of rotation are then

$$\begin{aligned} P_1 = W_1 \wedge w &= \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}, \\ P_2 = W_2 \wedge w &= \begin{pmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}, \\ P_3 = W_3 \wedge w &= \begin{pmatrix}0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}, \\ P_4 = S_1 \wedge s &= \begin{pmatrix}a_1 & b_1 & c_1 & 0 \\ 0 & -1 & 1 & 1\end{pmatrix} = \begin{pmatrix}-a_1 & a_1 & a_1 & b_1 + c_1 & b_1 & c_1\end{pmatrix}, \\ P_5 = S_2 \wedge s &= \begin{pmatrix}a_2 & b_2 & c_2 & 0 \\ 0 & -1 & 1 & 1\end{pmatrix} = \begin{pmatrix}-a_2 & a_2 & a_2 & b_2 + c_2 & b_2 & c_2\end{pmatrix}, \\ P_6 = S_3 \wedge s &= \begin{pmatrix}a_3 & b_3 & c_3 & 0 \\ 0 & -1 & 1 & 1\end{pmatrix} = \begin{pmatrix}-a_3 & a_3 & a_3 & b_3 + c_3 & b_3 & c_3\end{pmatrix}. \end{aligned}$$

We collect these in the columns of the motion matrix M:

$$M = \begin{pmatrix} 0 & 0 & 0 & -a_1 & -a_2 & -a_3 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \\ 1 & 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \\ 0 & 1 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & 1 & c_1 & c_2 & c_3 \end{pmatrix}.$$

All the information regarding configurations of the joints and possible motions can be found by analyzing M. More precisely, infinitesimally, the motion of the wrist is a composition of a rotation about w and a rotation about s (in our model). Thus, the center of this motion is a linear combination of the six Plücker coordinates which we assigned to the six given axes. This motivates us to define the column space of the matrix M to be the *motion space* (of the wrist in the given position of the human body), $\mathbb{MS}$. Recall from Theorem 2.1 that the matrix M always has rank equal to 5, implying a constant dimension of 5 for the motion space. Notice that we never obtain the full $\mathbb{R}^6$ as motion space in our model; this would require including further rotations in our model, such as about the elbow or the wrist.

3.3. Possible motions under the model. Suppose the human body (in particular, the torso and the bowling arm) is in a certain position. If one intends to propel the ball in some specific way, then this is accomplished by performing an infinitesimal motion with the hand. In our model, the only way to realize a hand motion is by means of rotations about the waist (3 joint axes) and/or about the shoulder (3 joint axes). Every (infinitesimal) rotation about one of these 6 axes is given by an appropriate multiple of the corresponding Plücker vector. We conclude that the motion of the ball is controlled by a 2-tensor which is a linear combination of the 6 Plücker vectors of our model; that is, it belongs to the column space of the matrix M ($\mathbb{MS}$). In particular, a linear combination which gives the zero 2-tensor corresponds to not moving at all (the zero center of motion).

Clearly, the first two rows of M are equal in magnitude but opposite in sign. This implies that every possible motion is represented by a 2-tensor with opposite Plücker coordinates in the first two places,

$$\mathbf{B} = (-a, a, b, c, d, e),$$

or equivalently, a possible motion is a point of $\mathbb{R}^6$ in the hyperplane $\mathbb{H} : p_{12} = -p_{13}$, so $\mathbb{MS} \subset \mathbb{H}$. Furthermore, since both spaces have dimension 5, we can state that $\mathbb{MS} = \mathbb{H}$.

Example 3. Try to perform a *pure* translation with your hand along the Z-axis (the direction of the spine) by only using the waist joint and the shoulder joint. You will not succeed! The algebraic proof for this goes as follows. Each translation along Z is represented by a set of Plücker coordinates of the line at infinity of the XY-plane. This means that it is a multiple of

$$(1, 0, 0, 0) \wedge (0, 1, 0, 0) = (1, 0, 0, 0, 0, 0),$$

which is not a possible motion, because it does not belong to $\mathbb{H}$.

Example 4. In an analogous fashion we see that a *pure* translation along the Y-axis is not possible. Indeed, such a translation is always represented by a multiple of $(0, 1, 0, 0, 0, 0)$, the Plücker vector for the line at infinity of the XZ-plane.

Example 5. However, a translation along the X-axis appears to be possible (this is the direction perpendicular to the plane of the torso; fortunately for cricketers, this direction is the one they want the ball to go). Indeed, the corresponding 2-tensor is a multiple of $(0, 0, 1, 0, 0, 0)$, the line at infinity of YZ; hence it belongs to $\mathbb{MS}$. But how can this be accomplished in practice? Let L be the line through s and parallel to Y. Because the Plücker vector of L is a linear combination of the Plücker vectors of S_1, S_2, and S_3, any rotation about L can be realized. Notice that L lies in the YZ-plane, as does the shoulder joint s in our model. Because Y and L intersect at infinity, an appropriate linear combination of their Plücker vectors yields the line at infinity of YZ. We conclude that a pure translation along X can be realized by composing a rotation about Y and a rotation about L.

Forbidden motions are interesting for two reasons. First, they are a simple way of describing what is possible. Second, they have an associated injury risk: attempting forbidden motion will introduce extremely large stresses, and coming close to forbidden motion (in the sense of a path through the motion space) may also require large stresses, a well-known phenomenon in robotics [3].

3.4. Critical positions of the human body. In our model, the possible motions are supported by six joint axes, each with a natural physical interpretation. Giving the spatial positions of these six axes determines what we will call the "position of the human body." In the previous sections we explained that our analysis of cricket bowling comes down to exploring the linear relations between the Plücker coordinates of these six axes. To simplify computations we are entitled to make the fixed coordinate frame coincide with the three waist axes, and that is what we did in section 3.2, yielding the simple structure of matrix M. So, in fact, by describing a body position we will mean the specification of the relative position of the shoulder axes with respect to the waist axes. It needs 3 parameters to be specified in order to fix the orientation of $S_1S_2S_3$ relative to $W_1W_2W_3$ $(= XYZ)$, for example the 3 Euler angles. Thus the very limited positions of the human body relevant to this paper can be regarded as points in a 3-dimensional space **Pos**.

3.5. Redundancy and supports. As a consequence of Theorem 2.1 we know that, in each position of the body, our six joint axes span a 5-dimensional space ($\mathbb{MS}$). We say that our kinematic system has a *generic redundancy.* Further, still in each position, basic linear algebra teaches us that we have a 1-dimensional space of linear dependencies between our six 2-tensors $(6 - 5 = 1)$. Redundancy in a model for human motion is also treated in [15], where the emphasis is also on the potential for fatigue management but the operational definition and mathematical treatment are different.

DEFINITION 3.1. *The* support *of a linear dependency among a set A of vectors is the subset of A consisting of exactly those vectors with nonzero coefficient in this dependency.*

In a given position of the body, each (nontrivial) linear dependency of the six joint axes is a multiple of every other one. Thus, we can define merely the "support of a body position" without specifying the linear dependency. Notice that, whatever position we are in, we always use the same notation for our six joints axes; hence the support can always be considered as a subset of $J = \{X, Y, Z, S_1, S_2, S_3\}$. This can be mathematically encoded in a map:

$$\mathbf{supp} : \mathbf{Pos} \to 2^J : p \mapsto \mathbf{supp}(p).$$

Before proceeding, let us explain the relevance of the this concept. Suppose the body is in some position p. Let $\mathbf{M} = (M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34})$ be the motion in $\mathbb{MS}$ that we want to perform. This is achieved by finding appropriate coefficients (angular velocities):

$$\mathbf{M} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma\mathbf{Z} + \sigma_1\mathbf{S}_1 + \sigma_2\mathbf{S}_2 + \sigma_3\mathbf{S}_3,$$

where the bold font reminds us of the fact that we switched to Plücker vectors (or 2-tensors). Now suppose that $\mathbf{supp}(p) = \{Y, Z, S_3\}$, corresponding to the following relation:

$$\lambda\mathbf{Y} + \mu\mathbf{Z} + \nu\mathbf{S}_3 = \mathbf{0}$$

with nonzero coefficients λ, μ, ν. Then we can realize the same motion $\mathbf{M}$ as

$$\mathbf{M} = \alpha\mathbf{X} + (\beta + k\lambda)\mathbf{Y} + (\gamma + k\mu)\mathbf{Z} + \sigma_1\mathbf{S}_1 + \sigma_2\mathbf{S}_2 + (\sigma_3 + k\nu)\mathbf{S}_3$$

with k an arbitrary constant. This means that the efforts done by Y, Z, and S_3 can be traded among each other, while the contributions by X, S_1, and S_2 are given by fixed coefficients with no chance for compensation. From this we learn two important things:

1. The concept of redundancy of joint axes is inherent to the human body. It is the solution supplied by nature to distribute the necessary efforts among the several joints for achieving a certain motion.
2. Positions in which the human body has abundant support are less strenuous than positions with limited support.

3.6. Critical positions. Now we arrive at the core of this paper. We will classify the possible supports in our model. A position of the human body is called *critical* if the support is smaller than expected, that is, smaller than in generic positions. We say that a critical position suffers from *redundancy with reduced support* or shortly, *reduced redundancy*. Our first observation says that the required work for joint axis X can never be compensated by one of the other five axes.

THEOREM 3.2. *For each position* $p \in \mathbf{Pos}$ *we have that* $X \notin \mathbf{supp}(p)$.

Proof. Since the shoulder joint s is assumed to lie in the YZ-plane, the Line $\mathbf{L} = sw$ is a linear combination of $\mathbf{Y}$ and $\mathbf{Z}$. And of course, $\mathbf{L}$ is a linear combination of $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$; hence the set $\{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{Y}, \mathbf{Z}\}$ is dependent. Because the motion space $\mathbb{MS}$ has dimension 5 in every position, $\mathbf{X}$ cannot be a linear combination of $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{Y}, \mathbf{Z}$; hence, it does not belong to the support. ∎

THEOREM 3.3. *Let* p *be a position of the human body. We distinguish three cases for the Line* $\mathbf{L} = sw$.

1. *The Line* $\mathbf{L}$ *is not contained in a plane determined by any two Lines of* $\{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}\}$. *In this case*

$$\mathbf{supp}(p) = \{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{Y}, \mathbf{Z}\}.$$

2. *The Line* $\mathbf{L}$ *does not coincide with a line of* $\{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}\}$*, but it lies in the plane generated by two of them (*$\mathbf{L} \in \mathbf{S_i S_j}$*). Then*

$$\mathbf{supp}(p) = \{\mathbf{S_i}, \mathbf{S_j}, \mathbf{Y}, \mathbf{Z}\}.$$

3. *The Line* $\mathbf{L}$ *coincides with one of* $\{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}\}$ *(i.e.,* $\mathbf{L} = \mathbf{S_i}$*). Then*

$$\mathbf{supp}(p) = \{\mathbf{S_i}, \mathbf{Y}, \mathbf{Z}\}.$$

Proof. The claims are an immediate consequence of what is said in section 2.3.

In case 3, if $\mathbf{L} = \mathbf{S_i}$, then the Lines $\mathbf{Y}, \mathbf{Z}, \mathbf{S_i}$ are concurrent and coPlanar, and so they are dependent. The support cannot be smaller, because this would mean that at least two of these lines coincide.

In case 2, either Lines $\mathbf{S_i}, \mathbf{S_j}, \mathbf{Y}, \mathbf{Z}$ are coPlanar or the pairs $\{\mathbf{Y}, \mathbf{Z}\}$ and $\{\mathbf{S_i}, \mathbf{S_j}\}$ determine two Planes that meet in the line **sw** through their intersections. In both cases, the four Lines are dependent. Furthermore, no three of them are concurrent, implying that the support is not smaller.

In case 1, we can rule out the five possibilities for the dependency of four lines (section 2.3). We refer to Theorem 3.2 for the claim that $\{\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{Y}, \mathbf{Z}\}$ is a dependent set. ∎

Remark. Cases 2 and 3 of the previous theorem correspond to the critical positions of our model.

4. Reduced redundancy as injury risk. It is known that high levels of fitness are attained in many cricketers [13]; nevertheless, injuries are fairly common [10] and fatigue may play a significant role [7]. This is not the place to review the mechanisms of overuse injury (the interested reader is referred to [17] as a starting point). We adopt the common perspective that overuse injuries start as microinjuries such as bruised bone and microtorn ligament. We suggest that overuse is more likely in situations of reduced redundancy. In such cases, no compensation that reduces the strain on a microinjured site is possible. The subject, in repeating the action, is condemned to repeating, at the same intensity, a motion that already caused a microinjury. By contrast, the ability to achieve a desired motion with a range of different joint rotations amounts to having the option of avoiding a motion that has caused a microinjury. The probability of overuse injury should decrease; hence, redundancy should correlate with reducing the risk of overuse injury. If so, then bowlers whose body position at ball release has more reduced redundancy than others should be at higher risk of injury, because such bowlers are less able to adapt. We also assume that microinjury is more likely in fatigued tissues, and hence adopt the view that reducing the probability of overuse injury is equivalent to reducing fatigue.

4.1. The role of the various joint axes. We interpret a joint axis that does not belong to the support in a given body position as a "necessary" axis of that position.

The joint axis X is through the "waist" joint and perpendicular to the pelvis; it is more or less parallel to the direction of the ball around the time of release. It is always a necessary axis, so for a particular desired motion, the amount of sideways bending of the spine is prescribed.[1] One implication of this is that injury risk due to this motion cannot be modified.

We noted above that a largely supported redundancy should help to reduce fatigue. Similarly, if an axis is necessary then no fatigue management can reduce the rate of tiring in structures involved in rotations around it. While the human body will have many more joint axes, our analysis suggests that bowlers will find it hard to compensate for fatigue related to rotation around the X-axis. Anecdotal evidence suggests that bowlers may attempt compensation by "falling over" as they tire. However, studies on changes in bowling action over long spells [9] have not reported rotation around this axis, so no scientific judgment is possible.

In critical positions we even suffer from reduced redundancy. The calculations for reduced redundancy depend on the choice of shoulder axes. We argued above that the s_1 axis is anatomically an intrinsic axis of rotation and that s_2 is dynamically an intrinsic axis of rotation for fast bowlers, because the motion of the arm is in the $s_1 s_3$ plane around the time of delivery of a fast ball.

In the bowling of a cricket fast ball, the worst-case scenario of reduced redundancy is that S_1 passes through the waist joint, which corresponds to case 3 above and implies that rotation about the other two shoulder axes are prescribed in all motions. Let us consider the simplest (and also most common) example: a straight arm. For such bowlers, the most risky action is one in which wrist, elbow, shoulder, and waist all lie on the same line very near or at the moment of delivery. Their ability to modify the amount of rotation will be limited to axes S_1, Y, and Z; thus one expects overuse injury related to such rotations to be less common. So for them tradeoffs are only possible among axial rotations of the arm, twisting of the spine, and bending forward at the waist. On the other hand, coaches need to be aware that changing the rotation in one of these axes will cause compensation in the other two axes.

We note that this situation is avoided by releasing the ball either behind or in front of the plane of the torso (more on this below) by a round-arm action, where S_1 is nearer to horizontal, and by a very upright action, where S_1 is nearer to vertical. Vertical action is usually encouraged by coaches but in some cases may tend to align the wrist with shoulder and waist and so increase injury risk.

Furthermore, Y and Z are never necessary, so that the amount of twisting and bending (backwards/forwards, that is) can be modified. Thus, in case of excessive

[1] Some care is needed here: In our model, bending of the spine is approximated by rotation of an inflexible spine in the "waist" joint. It may be that more than one pattern of rotations of vertebrae can achieve the desired rotation.

rotation in these directions at the waist, it should be possible to modify the bowler's action to reduce these, no matter what the configuration of their joints at the moment of delivery. For instance, excessive twisting around the Z-axis during the delivery stride is currently regarded as a major source of injury risk (the "mixed" action, which starts with hips and shoulders facing forwards; then the shoulders rapidly rotate and counterrotate—see [9, 12] and many others). Our study suggests that bowlers using a mixed action should be able to change action with relative ease.

Finally, is it possible to deliver a cricket ball with a maximally supported redundancy? Yes, but such actions are unusual and discouraged by coaches. The Line $\mathbf{L}$ in the analysis above corresponds to the line through waist and shoulder; it is required that this line be perpendicular to none of the shoulder joint axes. For instance, suppose that at delivery, the X and S_3 axes are parallel (certainly an aim in some deliveries by fast- and medium- pace bowlers). Then the wrist should not be in the plane formed by spine and shoulder (otherwise case 2 applies: $\mathbf{L}$ perpendicular to $\mathbf{S_2}$ or, equivalently, $\mathbf{L}$ is a linear combination of $\mathbf{S_1}$ and $\mathbf{S_3}$). So these bowlers should deliver such balls from behind or in front of the torso (the former seems to be common). The other axes have similar requirements. $\mathbf{L}$ perpendicular to $\mathbf{S}_1$ would be an excessively round-arm action and perhaps unlikely (though it could occur in the slinging action of some fast bowlers). $\mathbf{L}$ perpendicular to $\mathbf{S}_3$ is perhaps harder to avoid but should still be rare; for instance a round-arm action with the palm down at the moment of release, which might occur in some spin bowling actions.

4.2. An example. We give an analysis of the action of two medium-fast bowlers, both from the youth academy and hence at risk of injury, as potentially elite medium-fast or fast bowlers. The data were kindly provided by Janine Gray of Sports Science Institute of South Africa, who collected the data on these two bowlers as part of a larger study. Both subjects were 17 years old and free of injury at the time the data were collected. Bowler B had a long history of injuries, some of them from noncricket activities. In particular, he had suffered a stress injury to the lower back, which was seen as due to cricket. Bowler A had never been injured. Their historical workloads were different—Bowler B had played cricket from early boyhood, while Bowler A was a recent recruit to the game.

For each bowler, reflectors were attached to the body surface. Under stroboscopic lighting (frequency 120 Hz), video cameras recorded the positions of the reflectors at intervals (interval length about 8 milliseconds). The following reflectors were used in the calculation below: two on the wrist, one on the shoulder, and three on the waist. The three waist coordinates were assumed to lie at the vertices of a symmetric trapezium, and the center of its circumrectangle was calculated to give w, the center of the waist joint. In calculating s, the center of the shoulder joint, we assumed that the shoulder is fixed relative to the waist, so a simple correction allowed us to move from the position of the reflector on the acromion to s. The midpoint of the two reflectors on the wrist provided the position of r, the center of the wrist. In Figures 4.1 and 4.2 we depict aspects of the raw data: wrist position as a function of time for both bowlers.

Calculation of redundancy then proceeds as follows. The simple subtraction $w-s$ and normalization gave us the unit vector $\mathbf{l}$, which gives the direction of the Line $\mathbf{L}$ through waist and shoulder. Similarly, $s-r$ gives $\mathbf{s_1}$, the direction of S_1. Since the wrist reflectors lie in the S_1S_2-plane as does $\mathbf{s_1}$, simple orthogonalization gives $\mathbf{s_2}$, and $\mathbf{s_3}$ is then available as the cross-product of $\mathbf{s_1}$ and $\mathbf{s_2}$. The dot products of $\mathbf{l}$ with the $\mathbf{s_i}$ are then calculated, giving the direction cosines of $\mathbf{l}$ in the $S_1S_2S_3$ axes. When

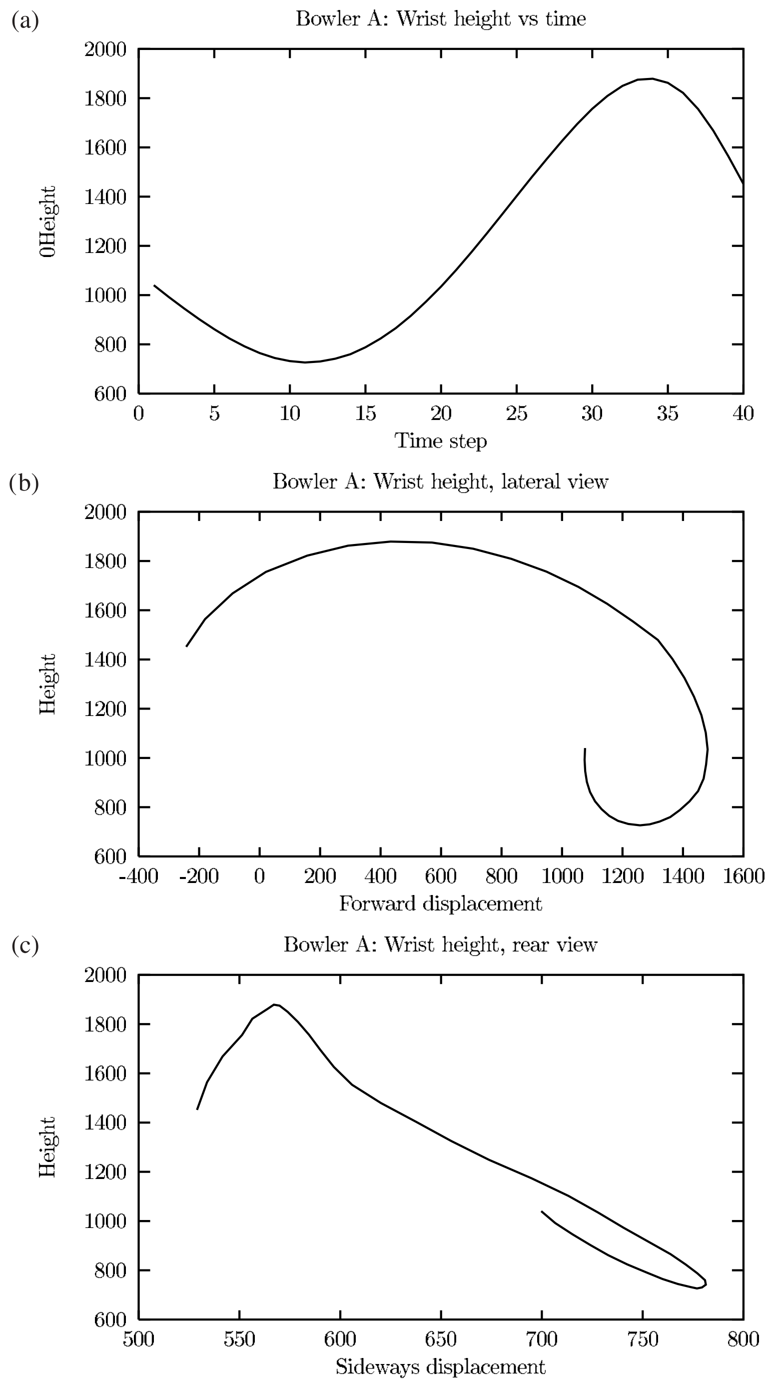


FIG. 4.1. *Wrist position as a function of time for Bowler A. The three diagrams give different aspects.* (a) *Wrist height as a function of time.* (b) *Wrist path in the YZ-plane (movement is leftward on diagram).* (c) *Wrist path in the XZ-plane. Height and displacement in millimeters; time steps* 0.83 *milliseconds apart.*

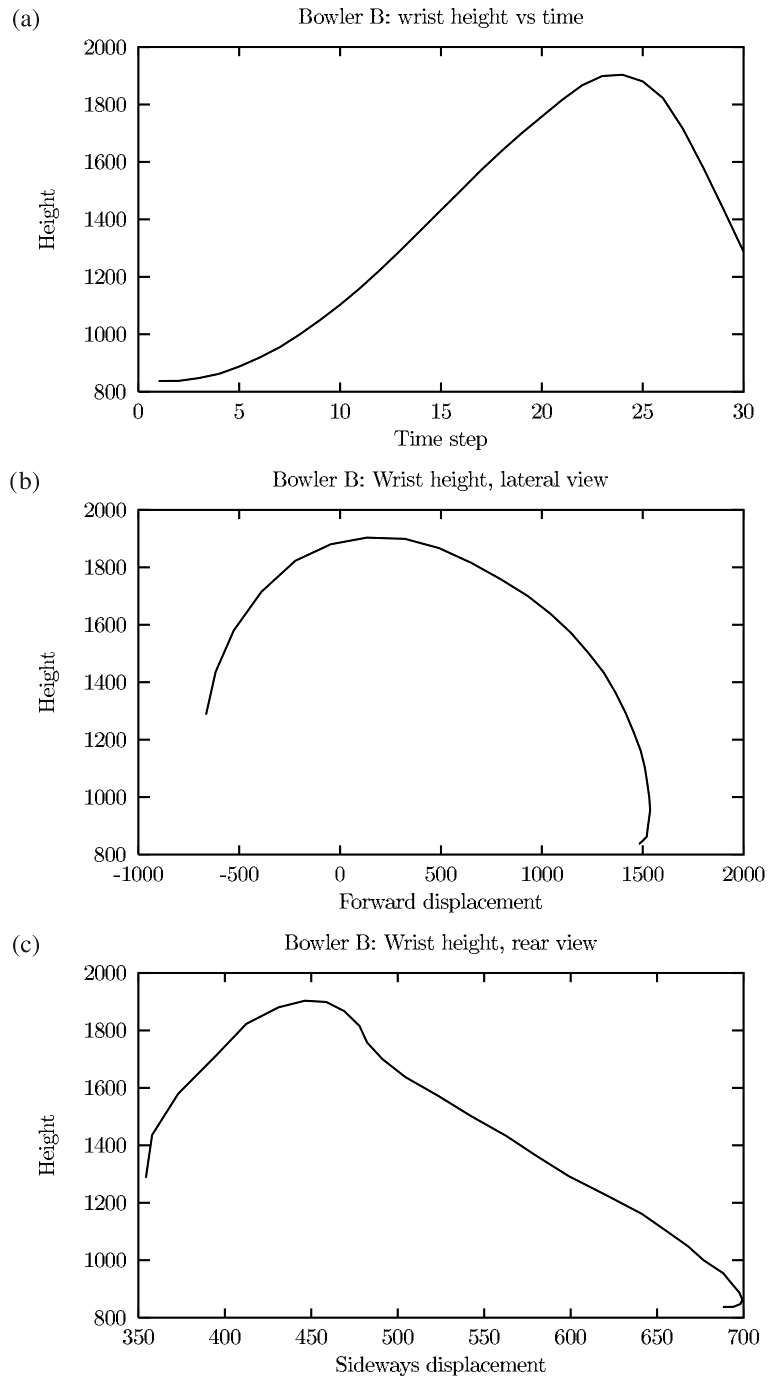
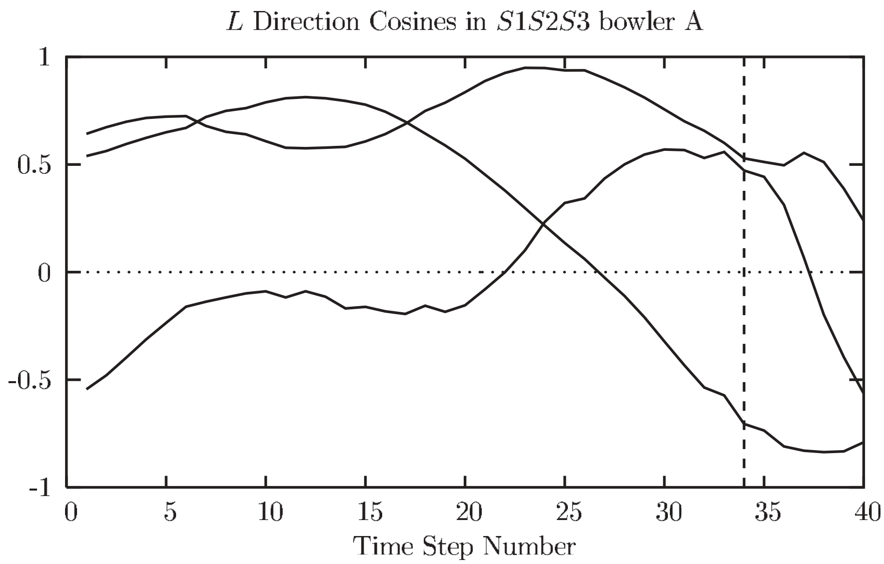


FIG. 4.2. *Wrist position as a function of time for Bowler B. The three diagrams give different aspects.* (a) *Wrist height as a function of time.* (b) *Wrist path in the YZ-plane.* (c) *Wrist path in the XZ-plane. Height and displacement in millimeters; time steps* 0.83 *milliseconds apart.*



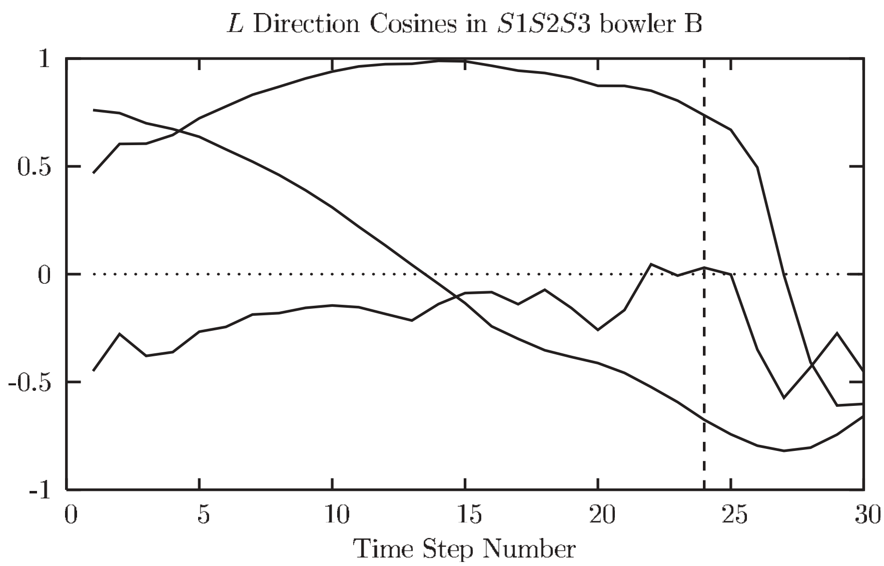


FIG. 4.3. *Reduced redundancy as revealed by the direction of the waist-shoulder line in shoulder coordinates. Curves give direction cosines of L, the line from waist to shoulder, in the axes $S1$, $S2$, and $S3$ of the shoulder; they are plotted against time steps (one unit of time is a few milliseconds). Vertical lines indicate the approximate points of release. Whenever one of the direction cosines goes to zero, reduction of redundancy occurs and the corresponding axis is absent from the support of the body position. Bowler A maintains full support until well after the moment of delivery, but Bowler B loses the S_3 axis from the support of the motion for about* 15 *milliseconds on either side of the moment of release.*

reduced redundancy occurs, then $\mathbf{l}$ is perpendicular to one or two of the $\mathbf{s_i}$—that is, one of the direction cosines is zero. This is easily spotted on a graph of direction cosines vs. time (see Figure 4.3).

The plots in Figure 4.3 cover approximately 0.4 s in time and forward motion of about 2 meters in space. Note that for Bowler A all three direction cosines stay well away from zero in the period prior to release, but that for Bowler B the S_3 axis goes to zero about 15 milliseconds before release, and stays there for about 30 milliseconds. With respect to our choice of axes, Bowler B operates with reduced redundancy around the time of release of the ball, but not Bowler A. This suggests that Bowler B may be less able to modify his action to cope with fatigue. This is consistent with their injury history, as Bowler B indeed has had more injuries than Bowler A. However, it is also true that Bowler B has had much more opportunity for overuse than Bowler A, due to a far longer playing career. We hope to track both subjects to test whether indeed Bowler A will remain relatively free from overuse injury, as we predict.

REFERENCES

[1] A. Burnett, C. Barrett, R. Marshall, B. Elliott, and R. Day, *Three-dimensional measurement of lumbar spine kinematics for fast bowlers in cricket*, Clin. Biomech., 13 (1998), pp. 574–593.

[2] L. Chèze, C. Gutierrez, R. San Marcelino, and J. Dimmet, *Biomechanics of upper limb using robotic techniques*, Human Movement Sci., 15 (1996), pp. 477–496.

[3] J. Craig, *Introduction to Robotics*, Addison-Wesley, 1989.

[4] H. Crapo and W. Whiteley, *Statics of frameworks and motions of panel structures: A projective geometric introduction*, Struct. Topol., 6 (1982), pp. 43–82.

[5] A. Dandurand, *The rigidity of compound spatial grids*, Struct. Topol., 10 (1984), pp. 41–56.

[6] P. Eberhard, T. Spägele, and A. Gollhofer, *Investigations for the dynamical analysis of human motion*, Multibody Sys. Dynam., 3 (1999), pp. 1–20.

[7] B. Elliot, *Back injuries and the fast bowler in cricket*, J. Sports Sci., 18 (2000), pp. 983–991.

[8] B. Elliott, R. Wallis, S. Sakurai, D. Lloyd, and T. Besier, *The measurement of shoulder alignment in cricket fast bowling*, J. Sports Sci., 20 (2002), pp. 407–510.

[9] P. Glazier, G. Paradisis, and S.-M. Cooper, *Anthropometric and kinematic influences on release speed in men's fast-medium bowling*, J. Sports Sci., 18 (2000), pp. 1013–1021.

[10] T. Leary and J. A. White, *Acute injury incidence in professional county club cricket players*, British J. Sports Med., 34 (2000), pp. 145–147.

[11] D. Lloyd, J. Alderson, and B. Elliott, *An upper limb kinematic model for the examination of cricket bowling: A case study of Mutiah Muralitheran*, J. Sports Sci., 18 (2000), pp. 975–982.

[12] A. McGrath and C. Finch, *Bowling Cricketing Injuries Over: A Review of the Literature*, Tech. Report 105, Monash University Accident Research Centre, 1996.

[13] T. Noakes and J. Durandt, *Physiological requirements of cricket*, J. Sports Sci., 18 (2000), pp. 919–929.

[14] M. Portus, P. Sinclair, S. Burke, D. Moore, and P. Farhart, *Cricket fast bowling performance and technique and the influence of selected physical factors during an* 8*-over spell*, J. Sports Sci., 18 (2000), pp. 999–1011.

[15] V. Potkonjak, S. Tzafestas, D. Kostic, and G. Djordjevic, *Human-like behaviour of robot arms: General considerations and the handwriting task—Part* I: *Mathematical description of human-like motion: distributed positioning and virtual fatigue*, Robotics and Computer Integrated Manufacturing, 17 (2001), pp. 305–315.

[16] A. van den Bogert and H. Schamhardt, *Multi-body modelling and simulation of animal locmotion*, Acta Anatomica, 146 (1993), pp. 95–102.

[17] A. van Mechelen, *Running injuries. A review of the epidemiological literature*, Sports Medicine, 14 (1992), pp. 320–335.

[18] N. White, *Grassmann-Cayley algebra and robotics*, J. Intelli. Robot. Sys., 11 (1994), pp. 91–107.