Frequency Spectra of Transient Electromagnetic Pulses in a Conducting Medium*

W. L. ANDERSON†, MEMBER, IRE, AND R. K. MOORE†, SENIOR MEMBER, IRE

Summary—The energy density spectra of transient electromagnetic fields generated by a pulsed ideal dipole source in an infinite conducting medium have been investigated for various distances from the source. A characteristic frequency ω_c, corresponding either to the peak of the spectrum or to its half-width, is defined and shown to vary inversely as the square of distance at large distances. The behavior of ω_c with distance is a measure of the behavior of the pulse energy. Thus, at large distances it appears that the attenuation factor associated with ω_c, $\exp\{-r\sqrt{\omega_c\mu\sigma/2}\}$, is independent of r, due to the constancy of the product $r\sqrt{\omega_c}$. From this point of view, the transient fields do not decrease exponentially as r, but as inverse powers of r.

This should not be construed as meaning that the transient possesses an advantage over CW. The attenuation for monochromatic components of the pulse is the same as for continuous waves of the same frequency and at large distances the energy put into the high frequency components is wasted.

The phenomenon is illustrated by calculations that have been carried out for the case of pulses in sea water.

Introduction

It has been shown by Richards[1] that at large distances in an infinite conducting medium the variation of the peak transient electromagnetic fields generated by a pulsed dipole source takes place as r^{-5} or r^{-4}. This rather unexpected situation is not clarified by the work of other authors[2–6] on the subject. Consequently, this study was undertaken in order to relate this phenomenon to the exponential attenuation which is known to apply to the monochromatic components of the pulse and to determine whether the apparent advantage of transients over pulse-modulated carriers might, in fact, be illusory. The results show that a pulse spectrum with appreciable high-frequency content wastes energy.

In this development, the displacement current term of the wave equation has been dropped, since the main interest is in the fields at considerable distances. At these distances the practical aspect of the problem is chiefly concerned with that part of the frequency spectrum containing the major portion of the pulse energy, and for these frequencies displacement currents are negligible.

For purposes of investigation the source has been taken as a magnetic dipole and some numerical results are developed for the case where the medium is sea water. Using triangular and rectangular pulses, expressions are derived for the relationship between pulse width, distance from source, and frequency ω_c at which either the peak or the half-width of the energy density spectrum occurs. It is shown that at large distances an approximation can be made which results in a simple inverse relation between ω_c and r^2. The product $r\sqrt{\omega_c}$ then appears constant whenever it occurs, including the exponential term, and, in the sense described, exponential attenuation does not occur. Any given spectral component of the pulse is attenuated exponentially with distance, however, and it therefore follows that energy should not be wasted on high frequencies which will be effectively filtered out at long distances.

Development of Equations

The fields from an idealized magnetic dipole in a conducting medium can be derived from the Hertz vector

$$\mathbf{\Pi}^*(\omega) = \frac{e^{-r\sqrt{i\omega\mu\sigma}}}{4\pi r}\,\boldsymbol{m}(\omega), \tag{1}$$

where Fourier transform quantities are used exclusively. The displacement current term is not included. The parameters are defined as follows:

μ = permeability of the medium,
σ = conductivity of the medium,
r = distance from source,
$\boldsymbol{m}(\omega)$ = magnetic moment (transform) of the dipole.

When $\boldsymbol{m}(\omega)$ is assumed to be along the polar axis and the medium is assumed infinite, the fields derived from (1) are given by:

$$E_\phi(\omega) = \frac{-\mu\,|\,\boldsymbol{m}(\omega)\,|\,\sin\theta}{4\pi r^2}(1 + r\sqrt{i\omega\mu\sigma})i\omega e^{-\sqrt{i\omega\mu\sigma}\,r} \tag{2}$$

$$H_\theta(\omega) = \frac{|\,\boldsymbol{m}(\omega)\,|\,\sin\theta}{2\pi r^3}(1 + r\sqrt{i\omega\mu\sigma} + i\omega\mu\sigma r^2)e^{-\sqrt{i\omega\mu\sigma}\,r} \tag{3}$$

$$H_r(\omega) = \frac{|\,\boldsymbol{m}(\omega)\,|\,\cos\theta}{2\pi r^3}(1 + r\sqrt{i\omega\mu\sigma})e^{-\sqrt{i\omega\mu\sigma}\,r}. \tag{4}$$

* Received by the PGAP, February 29, 1960; revised manuscript received, June 25, 1960.

† Elec. Engrg. Dept., University of New Mexico, Albuquerque, N. M.

[1] P. I. Richards, "Transients in conducting media," IRE Trans. on Antennas and Propagation, vol. AP-6, pp. 178–182; April, 1958.

[2] J. R. Wait, "Transient electromagnetic propagation in a conducting medium," *Geophysics*, vol. 16, pp. 213–221; April, 1951.

[3] W. J. Yost, "The interpretation of electromagnetic reflection data in geophysical exploration—Part I, general theory," *Geophysics*, vol. 17, pp. 89–106; January, 1952.

[4] J. R. Wait, "A transient magnetic dipole source in a dissipative medium," *J. Appl. Phys.*, vol. 24, pp. 341–343; March, 1953.

[5] J. Keilson and R. V. Row, "Transfer of transient electromagnetic surface waves into a lossy medium," *J. Appl. Phys.*, vol. 30, pp. 1595–1598; October, 1959.

[6] A. Grumet, "Penetration of transient electromagnetic fields into a conductor," *J. Appl. Phys.*, vol. 30, pp. 682–686; May, 1959.

The variation of the transient current in the loop constituting the dipole is now taken to be as shown in Fig. 1. The Fourier transform of this current is

$$I(\omega) = \frac{-I_m}{W}\left[\frac{1}{\omega^2}(1 - 2e^{-i\omega W} + e^{-i2\omega W}\right] \tag{5}$$

and from this we obtain

$$\begin{aligned}|m(\omega)|^2 &= |I(\omega)|^2 dA^2 \\ &= \frac{4I_m{}^2}{W^2}\left[\frac{1}{\omega^4}(1 - \cos\omega W)^2 dA^2\right]\end{aligned} \tag{6}$$

where dA is the area of the loop.

Lumping together factors not involving frequency into the term L, and making the abbreviation $a = r\sqrt{2\sigma\mu}$, we find

$$|E_\phi(\omega)|^2 = L_1{}^2\frac{e^{-a\sqrt{\omega}}}{\omega^2}\left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2}\right)(1 - \cos\omega W)^2 \tag{7}$$

$$|H_\theta(\omega)|^2 = L_2{}^2\frac{e^{-a\sqrt{\omega}}}{\omega^4}\left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2} + \frac{a^3\omega^{3/2}}{2} + \frac{a^4\omega^2}{4}\right)(1 - \cos\omega W)^2 \tag{8}$$

$$|H_r(\omega)|^2 = L_3{}^2\frac{e^{-a\sqrt{\omega}}}{\omega^4}\left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2}\right)(1 - \cos\omega W)^2. \tag{9}$$

Differentiating with respect to ω and setting the results equal to zero gives the following expressions, wherein ω_p denotes a frequency to be associated with the peak of the spectral density, and the abbreviation $p = a\sqrt{\omega_p}$ is made:

For $|E_\phi(\omega)|^2$:

$$\frac{2}{\omega_p W}\tan\frac{\omega_p W}{2}(p^3 + 4p^2 + 8p + 8) = 8p^2 + 16p + 16. \tag{10}$$

For $|H_\theta(\omega)|^2$:

$$\begin{aligned}\frac{2}{\omega_p W}\tan\frac{\omega_p W}{2}&(p^5 + 6p^4 + 12p^3 + 16p^2 + 32p + 32) \\ &= 8p^4 + 16p^3 + 16p^2 + 32p + 32.\end{aligned} \tag{11}$$

For $|H_r(\omega)|^2$:

$$\begin{aligned}\frac{2}{\omega_p W}\tan\frac{\omega_p W}{2}&(p^3 + 8p^2 + 16p + 16) \\ &= 8p^2 + 16p + 16.\end{aligned} \tag{12}$$

For arbitrary W, it is obviously difficult to deal with these equations. However, if trial values are assumed, it can be ascertained that for $W \ll a^2$, the value of ω_p satisfying (10) and (11) will be sufficiently small so that the approximation

$$\tan\frac{\omega_p W}{2} \approx \frac{\omega_p W}{2} \tag{13}$$

can be made. Under this condition the equations become independent of W, and a solution for p is readily obtained in each case. Eq. (12) is not satisfied for $\omega_p W/2$ in the first quadrant except for $p = 0$. Hence, in this range of frequency the energy spectral density for $H_r(\omega)$ is monotonically decreasing from its value at $\omega = 0$, and some quantity other than peak value must be used as a measure of spectrum behavior. One possibility, denoted by $\omega_{1/2}$, is the value of ω for which $|H_r(\omega)|^2 = \frac{1}{2}|H_r(0)|^2$. It can be shown that this latter quantity is $\frac{1}{2}(L_3{}^2(W^4/4))$. Assuming distances sufficiently great so that $\omega_{1/2}W$ is small, we approximate $(1 - \cos\omega W)^2$ by $(\omega W)^4/4$. The relation to be satisfied becomes

$$e^{-p'}\left(1 + p' + \frac{p'^2}{2}\right) = 1/2 \tag{14}$$

where p' is defined as $a\sqrt{\omega_{1/2}}$.

Eqs. (10), (11), and (14) can now be solved for p or p' under the given assumptions. Table I presents the values obtained for p and p', together with the values

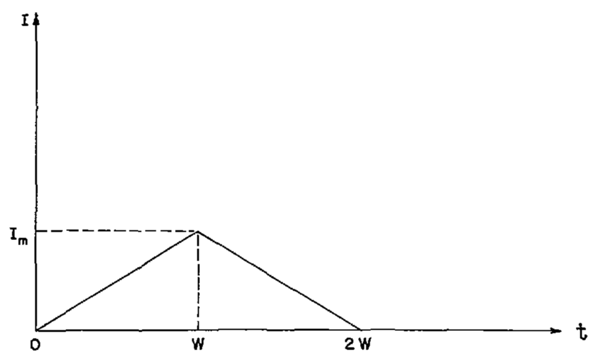

Fig. 1—Dipole current.

TABLE I

$\lvert E_\phi(\omega)\rvert^2$	$\lvert H_\theta(\omega)\rvert^2$	$\lvert H_r(\omega)\rvert^2$
$p = 5.662$	$p = 3.236$	$p' = 2.674$
$\omega_p = \frac{(5.662)^2}{a^2}$	$\omega_p = \frac{(3.236)^2}{a^2}$	$\omega_{1/2} = \frac{(2.674)^2}{a^2}$
$\omega_p(\text{sea water}) = \frac{3.19 \times 10^6}{r^2}$ rad/sec	$\omega_p(\text{sea water}) = \frac{1.04 \times 10^6}{r^2}$ rad/sec	$\omega_{1/2}(\text{sea water}) = \frac{7.11 \times 10^5}{r^2}$ rad/sec

of $\omega_c(\omega_p$ or $\omega_{1/2})$ both for an arbitrary medium and for sea water ($\sigma = 4$ mho/meter, $\mu = 4\pi \times 10^{-7}$ henry/meter).

To determine more specifically the limits within which the assumption (13) is valid, we may note that when $\omega_p W/2 = \frac{1}{2}$ radian, the tangent differs from its argument by about 10 per cent. Since $\omega_p = p^2/a^2$, we require

$$p^2 W \leq a^2 \tag{15}$$

$$a \geq p\sqrt{W} \tag{16}$$

for the assumption to be reasonably good. This condition, with p replaced by p', applies also in obtaining (14). For a particular field component, p or p' assumes its appropriate numerical value from Table I.

At lesser distances the trigonometric term in the spectral density equations plays a more important role. By assuming specific values of W, the problem can be attacked on a point-by-point basis, and the results plotted as in Fig. 2. Although only the electric field situation is shown, a similar analysis can be made for H_θ and H_r.

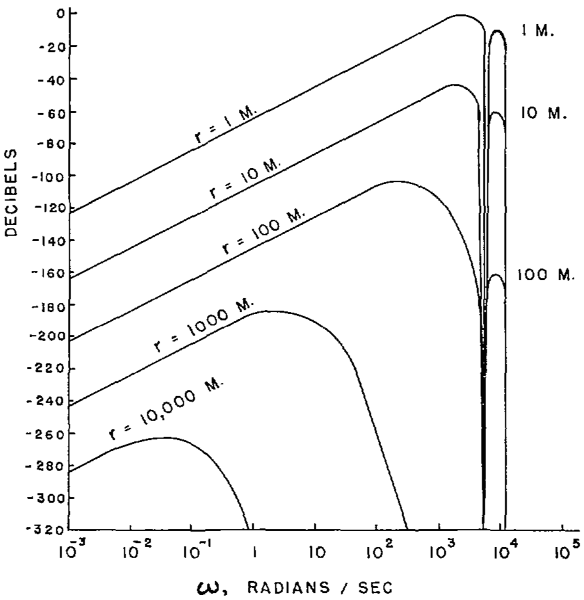

Fig. 2—Energy density spectrum for E_ϕ, triangular pulse, at various distances. ($\omega < 4000\pi$.)

The steeply sloping line of Fig. 2 represents the energy spectral peak behavior under the conditions defined by (15). It is therefore an absolute upper limit on the spectral peak regardless of the value of W. The gently sloping lines are the result of the point by point calculations, and show that the peak frequency decreases relatively slowly until the critical distance is reached. Beyond the critical distance, the product $a\sqrt{\omega_p}$ is constant; that is, ω_p varies as r^{-2}.

In order to determine the attenuation with distance of the main energy of the pulse, an evaluation can now be made of $|E_\phi(\omega)|$ and $|H_\theta(\omega)|$ at $\omega = \omega_p$. $|H_r(\omega)|$ at $\omega = \omega_{1/2}$ is probably of less interest and is therefore not calculated. Using the relations given by Table I, writing

$$(1 - \cos \omega_p W) \approx \frac{(\omega_p W)^2}{2}, \tag{17}$$

and reintroducing factors not involving frequency into (7) and (8), it is found that:

$$|E_\phi(\omega)|_{\omega_p} = \frac{0.358}{\sigma} \frac{W I_m dA \sin\theta}{r^4},$$

$$\text{(under condition } a \geq 5.662\sqrt{W}) \tag{18}$$

$$|H_\theta(\omega)|_{\omega_p} = 0.116 \frac{W I_m dA \sin\theta}{r^3}$$

$$\text{(under condition } a \geq 3.236\sqrt{W}). \tag{19}$$

For the pulse considered, these equations are applicable to any highly conducting medium ($\sigma \gg \omega\epsilon$), provided the restriction as to distance of (16) is satisfied. The electric field magnitude for sea water is obtained immediately by substituting $\sigma = 4$ in (18). The magnetic fields do not depend on the parameters of the medium. An interesting point is that varying widths of the transmitted pulse are converted into proportionally varying amplitudes at large distances.

It should be noted that a physical measurement of the field quantities expressed as transforms necessarily involves an integration over the bandwidth of the measuring device. This multiplies the units of (18) and (19) by (sec^{-1}).

The variation of electric field associated with the spectrum peak at distances for which a is less than $p\sqrt{W}$ seems to be rather difficult to formulate analytically. An approximate expression can be obtained by examining

$$|E_\phi(\omega)| = \frac{\mu I_m dA \sin\theta}{2\pi r^2 \omega W}(1 - \cos \omega W)e^{-a\sqrt{\omega}/2} \cdot \left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2}\right)^{1/2}. \tag{20}$$

It can easily be verified that the product

$$e^{-a\sqrt{\omega}/2}\left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2}\right)^{1/2}$$

does not differ greatly from unity until $a\sqrt{\omega}$ becomes larger than about 3. From Fig. 2 it can be seen that this condition holds for a major part of the small slope linear variation of $\sqrt{\omega_p}$. The product $1/\omega W$ $(1 - \cos \omega W)$ is also nearly constant when evaluated at ω_p in this region. Consequently, the electric field as defined by the above equation varies approximately as $1/r^2$ for $a \leq 3\sqrt{W}$ and as $1/r^4$ for $a \geq 5.662\sqrt{W}$. It is apparent from (2) that the transition from $1/r^2$ to $1/r^4$ behavior is due to the presence of the multiplicative factor ω_p

and to the fact that it begins to vary as $1/r^2$ after the critical distance has been reached. The absence of this ω factor in (3) and (4) assures us that a $1/r^3$ behavior will obtain for both "near" and "far" situations for the magnetic field.

The above considerations can be clarified by reference to Fig. 3, where the energy spectral density for E_ϕ of a pulse with $W=0.001$ second is plotted for various distances from the source in sea water. The spectrum is, of course, continuous from zero to infinity, with nulls in this case separated by $2000\,\pi$. The plot is carried out only up to the second null. The first spectral peaks for $r=1$ meter and $r=10$ meters differ by 40 db, whereas those for $r=100$, 1000, and 10,000 meters differ by 80 db.

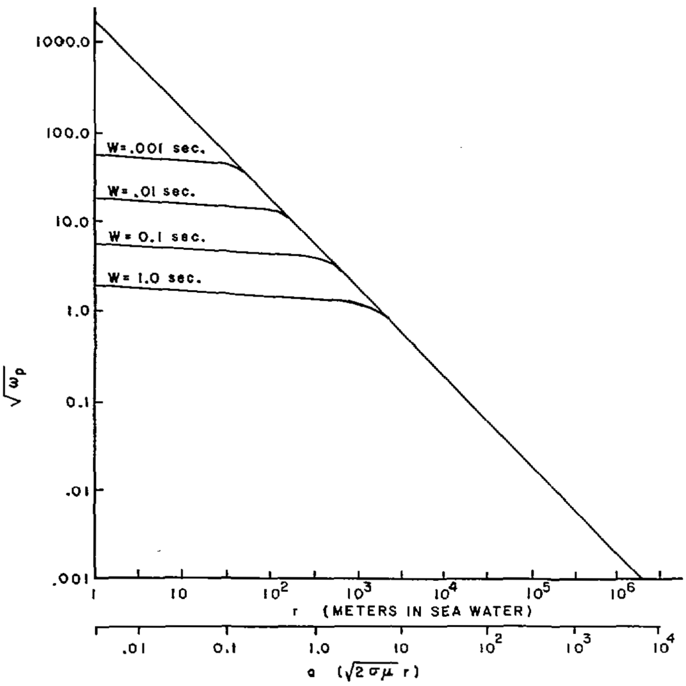

Fig. 3—Behavior of electric field energy spectral peak with distance.

Alternatively, the figure can be used to compare the attenuation of monochromatic waves of different frequencies. For example, if we wished to transmit a single frequency with minimum attenuation for a distance of 10,000 meters, entirely through water, we would be required to select a frequency of around 10^{-2} radian/second or less. If the maximum distance were only 300 meters, a frequency of around 30 radian/second could be used.

Likewise, if transient propagation with minimum attenuation is desired, the duration of the transient should be chosen so that the ω_c of its predominant energy spectral peak undergoes the minimum attenuation of 40 db per decade of distance over the range required.

Rectangular Pulse

To demonstrate that the spectrum for a different type of transient will behave in essentially the same way, a development for the rectangular pulse can be carried out along exactly the same lines. In this case the transform current is

$$I(\omega) = \frac{I_m}{i\omega}(1 - e^{-i\omega W}). \tag{21}$$

The energy spectral density for the electric field is as follows:

$$|E_\phi(\omega)|^2 = L_4^2 \frac{e^{-a\sqrt{\omega}}}{\omega^2}\left(1 + a\sqrt{\omega} + \frac{a^2\omega}{2}\right)(1 - \cos\omega W). \tag{22}$$

Since this density is monotonically decreasing for $\omega W<\pi$, as in the case of H_r for the triangular pulse, we find it necessary to investigate $\omega_{1/2}$. It can be shown that

$$|E_\phi(0)|^2 = L_4^2W^2/2, \tag{23}$$

so that for distances such that ωW is small,

$$L_4^2\,\frac{e^{-a\sqrt{\omega}}}{\omega^2}(1 + a\sqrt{\omega} + a^2\omega/2)\,\frac{(\omega W)^2}{2} = \frac{1}{2}L_4^2W^2/2 \tag{24}$$

is the determining equation for $\omega^{1/2}$. This gives

$$e^{-p'}(1 + p' + p'^2) = 1/2, \tag{25}$$

with the result $p'=2.674$ or

$$\omega_{1/2} = \frac{(2.674)^2}{a^2}. \tag{26}$$

Evidently the loss of high frequencies occurs in a manner similar to that of the triangular pulse; *i.e.*, either ω_p or $\omega_{1/2}$ decreases with distance as $1/r^2$.

Phase Velocity

In a highly conducting medium ($\sigma\gg\omega\epsilon$) the phase velocity for a monochromatic wave is known to be

$$v = \sqrt{\frac{2\omega}{\mu\sigma}}. \tag{27}$$

It has been shown that ω_c for a transient pulse at large distances varies continuously as r^{-2}. By substituting $\omega=\omega_p$ in the above equation, we define a velocity v_p which is associated with the peak of the energy spectrum. Taking ω_p for the electric field (triangular pulse), $\sigma=4$ mho per meter, and $\mu=4\pi\times10^{-7}$ henry per meter, we find:

$$v_p = \frac{1.143\times10^6}{r}\ \text{meters/second}. \tag{28}$$

At 1 kilometer in sea water, v_p is thus seen to be 1143 meters/sec. Since ω_p at 1 kilometer is 3.28 radians/sec ($W<0.1$ sec) and since a monochromatic wave of frequency 3.28 radians/sec propagates at a constant phase velocity of 1143 meters/sec, it is evident that the action of the medium on the spectrum of the pulse is simply that of low-pass filtering.

General Considerations

Although the foregoing calculations have been carried out for only two particular types of transient, it is obvious that the same procedure is applicable in any case. When computational difficulties arise, the information of interest can always be obtained from point-by-point graphical plots, numerical calculations, etc.

The most obvious and the most important factor in an analysis of this kind is the assumed linearity of the medium. After a transient has traveled an arbitrary distance, its spectrum can be considered to consist of a superposition of monochromatic components, each of which has been acted upon as if it alone existed. In any particular situation, knowledge of the general attenuation vs the frequency-distance characteristics of the medium enables one to make an educated guess as to how a spectrum will appear at various distances from the source. Guesswork on such a basis should tell us that the generation of high-frequency components is quite undesirable for communication over appreciable distances. It is this point that has been demonstrated by a specific analysis for triangular and rectangular pulses.

Propagation in a conducting medium is analogous to propagation along a transmission line under two different situations. This analogy can be seen by inspection of the general equation for voltage on a line with distributed parameters R, L (in series) and C, G (in shunt):

$$\frac{\partial^2 v}{\partial x^2} = RGv + (RC + LG)\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \cdot \qquad (29)$$

Setting L and G *or* R and C equal to zero gives an equation corresponding to that obtained by omission of the displacement current term from the wave equation:

$$\nabla^2 \boldsymbol{E} = \sigma\mu\frac{\partial \boldsymbol{E}}{\partial t} + \mu\epsilon\frac{\partial^2 \boldsymbol{E}}{\sigma t^2} \cdot \qquad (30)$$

The result is the diffusion equation in each case. Physically, it is somewhat more satisfactory to use the analog obtained when R and C are zero, since the units then correspond more directly.

This well-known analogy is mentioned here because the RC transmission line was originally investigated more than 100 years ago,[7] and, as would be expected, some important conclusions as to its performance have long been known.[8] These results are immediately applicable to the situation that is the topic of this paper. Some of the major conclusions are as follows:

1) Signal velocity is inversely dependent on distance.
2) The pulse shape is "smeared out" in time.
3) Consequently, there is a limitation on signaling rate.

Conclusions[9]

After a fairly small distance of travel in an infinite conducting medium, most of the energy in a transient pulse is contained in a frequency range around its lowest spectral peak. An analysis of the behavior of this peak or its half width, therefore, gives a good description of what is happening to the major portion of the energy in the pulse. Under the conditions assumed, it has been shown in the case of triangular and rectangular pulses generated by a magnetic dipole that the energy spectrum displays the following characteristics:

1) As distance increases up to a critical value which can be determined in terms of pulse width, there is a slow shift downward in the frequency ω_c at which the spectrum peak or its half width occurs. In this range, the electric field associated with ω_c decreases approximately as r^{-2}, while magnetic fields decrease as r^{-3}.

2) After the critical distance is reached, ω_c shifts downward proportionally as r^{-2}. Then the electric field associated with ω_c decreases as r^{-4}. In this "far" range an r^{-3} behavior continues to obtain for the magnetic fields.

3) The same considerations apply to propagation of transients as to propagation of continuous waves; namely, the range of transmission sets an upper limit to the frequencies which can be sent with minimum attenuation. Energy put into frequencies above this limit is essentially wasted.

[7] W. Thompson (Lord Kelvin), "On the theory of the electric telegraph," *Proc. Roy. Soc.*, vol. 7, pp. 382 ff; 1856.

[8] The authors of some recent papers on electromagnetic transients in a lossy medium have apparently been unaware of these transmission line investigations. A very good discussion of the RC transmission line is contained in the book by E. Weber, "Linear Transient Analysis," John Wiley and Sons, Inc., New York, N. Y., vol. II, ch. 7; 1956. Another briefer discussion of this subject is in R. K. Moore's "Traveling Wave Engineering," McGraw-Hill Book Co., Inc., New York, N. Y.; 1960.

[9] Since this paper was first submitted, a communication has appeared by S. H. Zisk, "Electromagnetic transients in a conducting medium," IRE Trans. on Antennas and Propagation, vol. AP-6, pp. 229–230; March, 1960. Zisk uses an integration, by means of the saddle-point method, to obtain a time-domain result from which he concludes, as in this paper, that Richards' results arise because of dispersion in the conducting medium.