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BOUNDARY LAYER ATTENUATION OF HIGHER ORDER MODES IN WAVEGUIDES

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The attenuation of higher order modes in rectangular and circular tubes is treated here by using results for the boundary layer admittance for the respective normal modes. Comparison with results available in the literature for propagating modes is given. Results for evanescent modes and at the cut-off frequencies are discussed. Finally, the well-known Kirchhoff theory is extended to obtain a test of validity for the proposed calculations.

INTRODUCTION

The boundary layer attenuation caused by viscous and thermal losses of lower modes in rigid walled small cavities has been calculated in previous work [1] by using the concept of boundary layer admittance [2]. On the other hand, the problem of extending the classical Kirchhoff result for the attenuation of plane waves in rigid walled tubes to the case of higher order propagating modes has been the subject of papers appearing in the 1950s. The substantial theoretical extension of the concept of boundary layer admittance for higher order modes in tubes was given by Beatty [3], but his work led to infinite attenuation at the cut-off frequency, which is not correct when losses are taken into account; in fact the theory is not valid near the cut-off frequency (of course there are no cut-off frequencies when dissipation occurs; we refer throughout this paper to "adiabatic" cut-off frequencies).

Some acoustical devices make use of cavities or ducts, the dimensions of which are of the same order of magnitude or greater than the acoustic wavelength, and the walls of which are perfectly rigid [4, 5]. In order to determine the acoustic response of such cavities or ducts, one needs to calculate the attenuation of the modes, be they evanescent, propagating or at their adiabatic cut-off frequency, by taking into account the attenuation due to viscous and thermal effects. The attenuation of higher order modes in rectangular and circular tubes is treated here by using the results for the boundary layer admittance for the respective normal modes. Comparison with some previous results is given.

Finally, the well-known Kirchoff theory, which provides a dispersion equation for $m = 0$ modes (in cylindrical ducts), is extended to obtain a test of the validity of the theory (in cylindrical ducts).

2. WAVENUMBERS IN RIGID WALLED WAVEGUIDES

The tubes considered here are either rectangular or circular. The waves are propagating in the x direction, along a rectangular tube of width l_y in the y direction and height l_z in the z direction, or along a circular tube of radius a. The transverse dimensions of the tubes are assumed to be greater than the boundary layer thickness, but small enough so

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that wall losses are preponderant. The effect of the thermal and shear modes on the boundary condition can then be treated by using the concept of boundary specific admittance [2, 6]:

$$\varepsilon = \sin^2\theta\varepsilon_v + \varepsilon_t, \tag{1}$$

where $\sin^2\theta$ is related to the second order derivative of the acoustic pressure p with respect to the normal to the side wall by the equation $\sin^2\theta = 1 + \partial^2_{\perp\perp}p/k^2p$: that is, for separable co-ordinates, $\sin^2\theta = 1 - k_\perp^2/k^2$ (no matter the kind of mode it is). Here

$$\begin{pmatrix}\varepsilon_v\\ \varepsilon_t\end{pmatrix} = \frac{1+\mathrm{i}}{\sqrt{2}}\sqrt{k}\begin{pmatrix} l_v^{1/2}\\ (\gamma-1)l_t^{1/2}\end{pmatrix},$$

$k = \omega/c$ is the wavenumber, l_v is the characteristic length $\mu/\rho c$, μ is the viscosity coefficient of the gas, ρ is the mass density, l_t is the characteristic length $\lambda M/\rho c C_p$, c is the speed of sound, λ is the coefficient of thermal conductivity, M is the molar mass, C_p is the specific heat coefficient at constant pressure, and γ is the specific heat ratio.

When the modes are propagating, θ can be interpreted as the local incidence angle with respect to the side wall. When the modes are evanescent, $\sin^2\theta$ is a negative quantity and therefore has no direct physical meaning.

For air at atmospheric pressure and room temperature, for $\theta = \pi/2$, $\mathrm{Re}(\varepsilon) = \mathrm{Im}(\varepsilon) \sim 3\times 10^{-5} f^{1/2}$ (where f is the frequency). Thus the results involving the effects of the wall admittances ε are only slightly different from those corresponding to Neumann boundary conditions. Consequently, for the mode considered, the factor $\sin^2\theta$ may be calculated by using the Neumann solutions. For the rectangular tube, one obtains

$$\sin^2\theta_y = 1 - (n_y\pi/l_y)^2/k^2 \qquad \text{and} \qquad \sin^2\theta_z = 1 - (n_z\pi/l_z)^2/k^2, \tag{2}$$

where n_y and n_z are quantum numbers for the mode considered ($n_y, n_z = 0, 1, 2, 3, \ldots$), and for the circular tube one obtains

$$\sin^2\theta = 1 - \frac{[(\gamma_{mn}/a)^2 - m^2/a^2]}{k^2}, \tag{3}$$

where m is the azimuthal quantum number, and γ_{mn} is the nth zero of the derivative of the cylindrical Bessel function of the first kind, $\mathrm{J}'_m(\gamma_{mn}) = 0$ ($n, m = 0, 1, 2, \ldots$).

3. THE WAVENUMBER IN THE AXIAL DIRECTION

The wavenumber k_x in the axial direction can be written as

$$k_x^2 = k^2 - k_\perp^2, \tag{4}$$

where $k_\perp$ is the radial eigenvalue of the mode considered.

For the modes (m, n) in a circular tube of radius a, the radial eigenvalues $k_\perp^2 = k_{mn}^2$ satisfy the boundary condition equation

$$(k_{mn}/k)(\mathrm{J}'_m(k_{mn}a)/\mathrm{J}_m(k_{mn}a)) = -\mathrm{i}\varepsilon_{mn}, \tag{5}$$

where ε_{mn} is the value of ε obtained by substituting in equation (1) the expression (3) for $\sin^2\theta$. Upon assuming that k_{mn} can be written as $k_{mn}a = \gamma_{mn} + \eta$, where η has the same order of magnitude as ε, and making use of the well-known Bessel equation, expression (5) yields

$$k_{mn} \simeq (\gamma_{mn}/a) + \mathrm{i}[k\gamma_{mn}/(\gamma_{mn}^2 - m^2)]\varepsilon_{mn}. \tag{6}$$

Substituting this value of $k_{mn} = k_\perp$ into equation (4) gives axial wavenumber,

$$k_x^2 \simeq k^2 - (\gamma_{mn}/a)^2 + \{(2k/a)/[1-(m/\gamma_{mn})^2]\}[\mathrm{Im}(\varepsilon_{mn}) - \mathrm{i}\,\mathrm{Re}(\varepsilon_{mn})]. \tag{7}$$

The same procedure [1] can be applied also to determine the axial wavenumber in a rectangular tube. One obtains

$$k_x^2 = k^2 - \left[\left(\frac{n_y \pi}{l_y}\right)^2 + \left(\frac{n_z \pi}{l_z}\right)^2\right] + 2k\left[(2-\delta_{n_y0})\frac{\operatorname{Im} \varepsilon_y}{l_y} + (2-\delta_{n_z0})\frac{\operatorname{Im} \varepsilon_z}{l_z} - \mathrm{i}\left((2-\delta_{n_y0})\frac{\operatorname{Re} \varepsilon_y}{l_y} + (2-\delta_{n_z0})\frac{\operatorname{Re} \varepsilon_z}{l_z}\right)\right] \tag{8}$$

where δ is the Kronecker delta index, $\varepsilon_y = \sin^2 \theta_y \varepsilon_v + \varepsilon_t$ and $\varepsilon_z = \sin^2 \theta_z \varepsilon_v + \varepsilon_t$.

In both cases, rectangular and circular tubes, the axial wavenumber can be written as

$$k_x^2 = A_\nu + (I_\nu - \mathrm{i} R_\nu), \tag{9}$$

where I_ν and R_ν are proportional to the specific admittances ε_{mn}, or ε_y and ε_z.

This last equation leads to the next expressions for the real part χ_ν and the imaginary part $(-\alpha_\nu)$ of k_x:

$$\chi_\nu = \pm\frac{1}{\sqrt{2}}[A_\nu + I_\nu + \sqrt{(A_\nu + I_\nu)^2 + R_\nu^2}]^{1/2}, \tag{10}$$

$$\alpha_\nu = \pm\frac{1}{\sqrt{2}}[-(A_\nu + I_\nu) + \sqrt{(A_\nu + I_\nu)^2 + R_\nu^2}]^{1/2}. \tag{11}$$

These results are valid for propagating and evanescent modes, and at the cut-off frequency.

The limiting expressions for χ_ν and α_ν, when the losses are neglected ($I_\nu = R_\nu = 0$), are well-known: above the cut-off frequency ($f > f_c$, propagating mode),

$$\chi_\nu = \pm(A_\nu)^{1/2} \quad \text{and} \quad \alpha_\nu = 0; \tag{12}$$

below the cut-off frequency ($f < f_c$, evanescent mode),

$$\chi_\nu = 0 \quad \text{and} \quad \alpha_\nu = \pm|A_\nu|^{1/2}.$$

Furthermore, for frequencies much greater than the cut-off frequency, A_ν is much greater than I_ν and R_ν, and looking for an approximate solution to equation (9) leads to the results given by Beatty [3]:

$$X_\nu \simeq \pm(A_\nu)^{1/2} = \pm k[1 - f_c/f)^2]^{1/2} \tag{13}$$

with $f_c/f = \gamma_{mn}/ka$ (circular tube) and $(f_c/f)^2 = (n_y\pi/kl_y)^2 + (n_z\pi^2/kl_z)^2$ (rectangular tube)

$$\alpha_\nu \simeq \pm R_\nu/2(A_\nu)^{1/2} = \pm[(1-(f_c/f)^2]^{-1/2}\begin{cases}[1-(m/\gamma_{mn})^2]^{-1}\operatorname{Re}(\varepsilon_{mn})/a & \text{(circular tube)}\\ [(2-\delta_{n_y0})\operatorname{Re}(\varepsilon_y)/l_y + (2-\delta_{n_z0})\operatorname{Re}(\varepsilon_z)/l_z] & \text{(rectangular tube)}\end{cases} \tag{14}$$

Note that formulas (3) and (7) in the paper by Beatty [3] would be the same as equation (14) but they are both incorrect because the terms $(2-\delta_{n0})$ are omitted. On the other hand, the major frequency dependence is due to the factor $[1-(f_c/f)^2]^{-1/2}$ (as Beatty pointed out), especially near the cut-off frequency, but in this last case the results (13) and (14) are not correct at all.

4. RESULTS AND DISCUSSION

The results mentioned above are summarized in Table 1. The attenuation ratio α/α_0 is plotted in Figures 1–6 for several modes and several tube dimensions (rectangular or

TABLE 1

Axial wavenumber $\chi_\nu - i\alpha_\nu$ for circular and rectangular tubes

Circular tube	Rectangular tube
$f_c/f = \gamma_{mn}/ka$	$(f_c/f)^2 = \left(\frac{n_y \pi}{kl_y}\right)^2 + \left(\frac{n_z \pi}{kl_z}\right)^2$
$A_\nu = k^2[1-(f_c/f)^2], \quad \nu \equiv (m, n)$	$A_\nu = k^2[1-(f_c/f)^2], \quad \nu \equiv (l_y, l_z)$
$R_\nu = 2k\dfrac{\mathrm{Re}\,(\varepsilon_{mn}/a)}{1-(m/\gamma_{mn})^2}$	$R_\nu = 2k[(2-\delta_{n_y 0})\,\mathrm{Re}\,(\varepsilon_y)/l_y + (2-\delta_{n_z 0})\,\mathrm{Re}\,(\varepsilon_z)/l_z]$
	$I_\nu = 2k[(2-\delta_{n_y 0})\,\mathrm{Im}\,(\varepsilon_y)/l_y + (2-\delta_{n_z 0})\,\mathrm{Im}\,(\varepsilon_z)/l_z]$
$I_\nu = 2k\dfrac{\mathrm{Im}\,(\varepsilon_{mn}/a)}{1-(m/\gamma_{mn})^2}$	$\varepsilon_y = \left[1 - \left(\frac{f_c}{f}\right)^2 \frac{u_n^2}{1+u_n^2}\right]\varepsilon_v + \varepsilon_t$
$\varepsilon_{mn} = \{1-(f_c/f)^2[1-(m/\gamma_{mn})^2]\}\varepsilon_v + \varepsilon_t$	$\varepsilon_z = \left[1 - \left(\frac{f_c}{f}\right)^2 \frac{1}{1+u_n^2}\right]\varepsilon_v + \varepsilon_t, \qquad u_n = \frac{n_y}{n_z}\frac{l_z}{l_y}$

Axial wave number, $k_x = \chi_\nu - i\alpha_\nu$

Without losses:

Propagating modes	$\chi_\nu = \pm A_\nu^{1/2}$,	$\alpha_\nu = 0$	$(A_\nu > 0)$
Evanescent modes	$\chi_\nu = 0$	$\alpha_\nu = \pm\lvert A_\nu\rvert^{1/2}$	$(A_\nu < 0)$

With losses, valid for propagating modes only, far from the cut-off frequency (corrected results from Beatty):

$$\chi_\nu = \pm A_\nu^{1/2}$$
$$\alpha_\nu = \pm R_\nu/(2A_\nu^{1/2})$$

Present paper (valid for all kinds of modes)

$$\chi_\nu = \pm\frac{1}{\sqrt{2}}\{A_\nu + I_\nu + [(A_\nu + I_\nu)^2 + R_\nu^2]^{1/2}\}^{1/2}$$

$$\alpha_\nu = \pm\frac{1}{\sqrt{2}}\{-(A_\nu + I_\nu) + [(A_\nu + I_\nu)^2 + R_\nu^2]^{1/2}\}^{1/2}$$

Plane wave attenuation

$\alpha_0 = \pm\mathrm{Re}(\varepsilon_{00})/a = \pm\dfrac{\mathrm{Re}\,(\varepsilon_v + \varepsilon_t)}{a}$ (well-known Kirchhoff results) $\qquad \alpha_0 = \left(\dfrac{1}{l_y} + \dfrac{1}{l_z}\right)\mathrm{Re}\,\varepsilon_v + \varepsilon_t)$

circular), with air under standard conditions ($c = 344\ m/s$, $\mathrm{Re}\,(\varepsilon_v) = \mathrm{Im}\,(\varepsilon_v) = 2{\cdot}03 \times 10^{-5} f^{1/2}$, $\mathrm{Re}\,(\varepsilon_t) = \mathrm{Im}\,(\varepsilon_t) = 0{\cdot}95 \times 10^{-5} f^{1/2}$). Curves (c) give the results for propagating modes, and curves (a) those for evanescent modes; detailed results near the cut-off frequencies can be seen from curves (b). Above the cut-off frequency, results for propagating modes are the same as those given by Beatty [3].

Similar curves for the real part χ_ν of the wavenumber are shown in Figures 7-12.

In describing the behavior of the complex wavenumber in the axial direction of waveguides, our formulas are in complete agreement with those from previous work on propagating modes, which are well explained in the paper by Beatty [3]. In addition, they remain valid for evanescent modes and at the cut-off frequency. The only limitation of the method appears when the frequency goes to zero, or when the transverse dimensions have the same order of magnitude as the boundary layer thickness (capillary tubes), or when losses in the whole volume must be taken into account (see details in section 5).

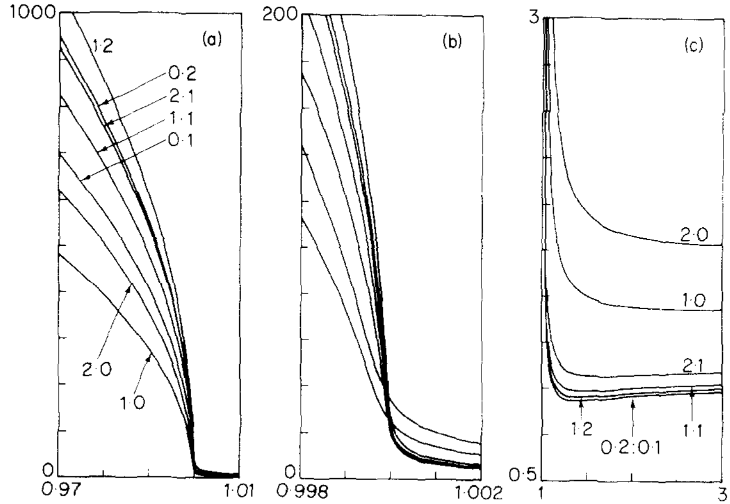

Figure 1. Attenuation ratio α/α_0 of (m, n) modes versus frequency parameter f/f_c for a circular tube ($a = 0{\cdot}1$ m).

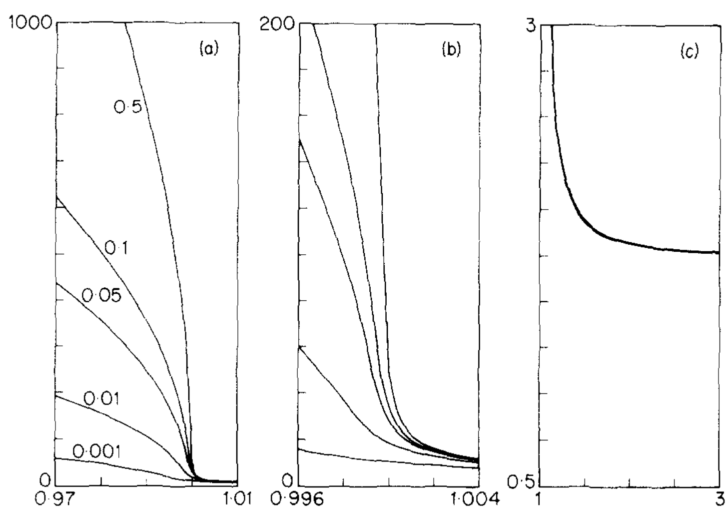

Figure 2. Attenuation ratio α/α_0 of (2, 0) mode versus frequency parameter f/f_c for several circular tubes (from $a = 0.5$ m to 0.001 m).

Finally, the method used here can be applied also to the calculation of characteristics of normal modes in rigid-walled cavities (rectangular, circular or spherical cavities).

5. EXTENDED KIRCHOFF THEORY FOR HIGHER MODES IN CIRCULAR TUBES

In this section the method of Kirchhoff is applied in order to obtain a test of the validity of equation (7) in circular tube. The well-known Kirchhoff theory [7] gives a dispersion equation only for modes $m = 0$ and only for plane wave. The aim of this last section is to find a new dispersion equation for all modes, propagating, evanescent, or at the cut-off frequency.

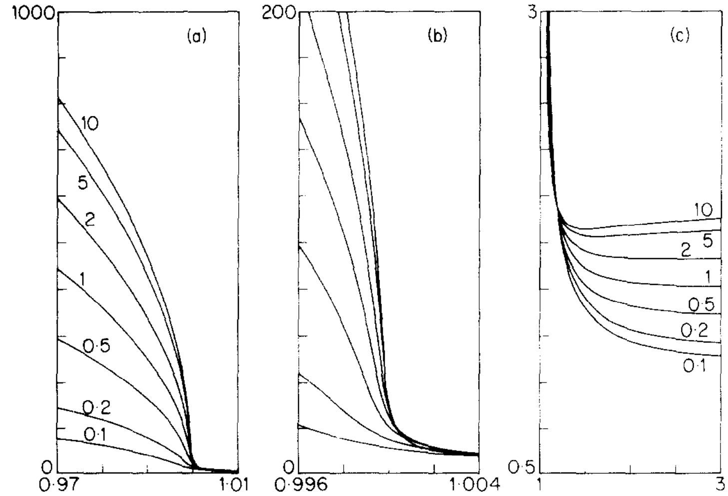

Figure 3. Attenuation ratio α/α_0 of $(n_y = 0,\ n_z = 1)$ mode versus frequency parameter f/f_c, for several rectangular tubes ($l_z = 0.2$ m, $l_y/l_z = 0.1$ to 10).

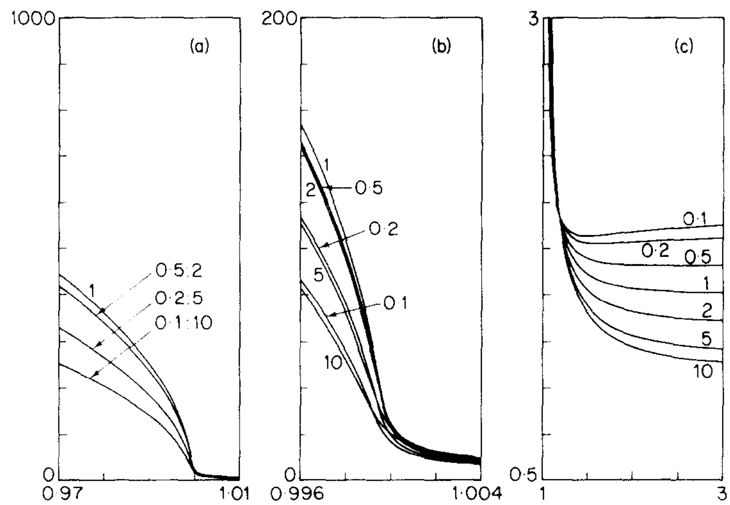

Figure 4. Attenuation ratio α/α_0 of $(n_y = 1,\ n_z = 0)$ mode versus frequency parameter f/f_c, for several rectangular tubes $l_z = 0{\cdot}2$ m, $l_y/l_z = 0{\cdot}1$ to 10).

The variables describing the dynamical and thermodynamical state of the fluid are as follows: p, the acoustic pressure; $\mathbf{v}$, the particle velocity, $\mathbf{v} = \mathbf{v}_l + \mathbf{v}_v + \mathbf{v}_t$; $\mathbf{v}_l$, the acoustic particle velocity; $\mathbf{v}_v$, the vorticity-mode velocity; $\mathbf{v}_t$, the entropy-mode velocity; τ, the temperature variation, $\tau = \tau_l + \tau_t$; τ_l, the acoustic temperature; τ_t, the entropic temperature.

Kirchhoff's theory of viscothermal effects in circular tubes starts with five differential equations, the Navier–Stokes equations, the conservation of mass equation, the Fourier equation for conduction of heat, and the equations expressing the entropy variations and the acoustic part of the density (both regarded as functions of the independent variables p and τ), considered as total differentials.

Any disturbance governed by this system of linear equations can be considered as a superposition of acoustic, vorticity and entropy modes. The corresponding acoustic pressure p, the acoustic temperature τ_l, the entropic temperature τ_t and the rotational

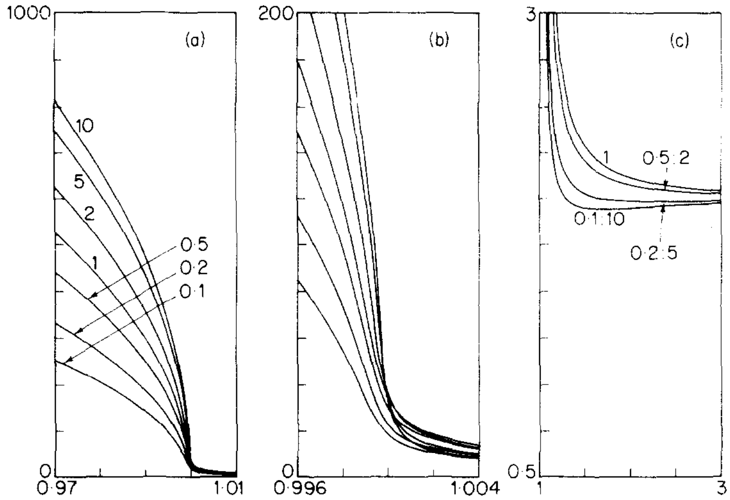

Figure 5. Attenuation ratio [redacted] of ($n_y = 1$, $n_z = 1$) mode versus frequency parameter f/f_c for several rectangular tubes $l_z = 2$ m, l_y/l_z [redacted] to 10).

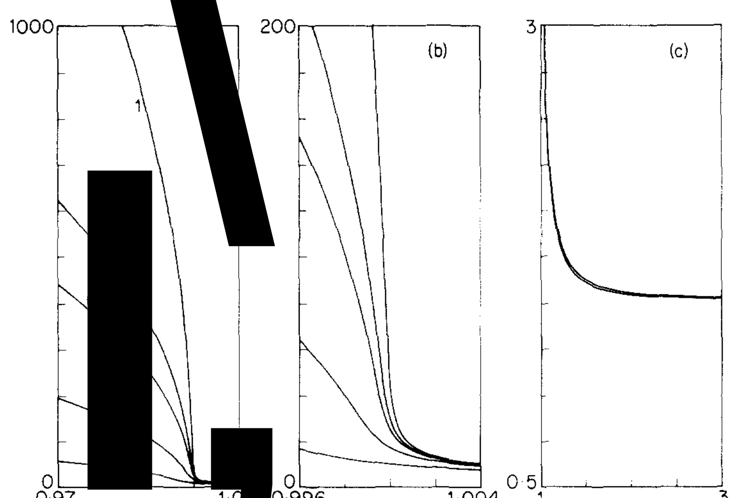

Figure 6. Attenuation ratio α/α_0 [redacted],0) mode versus frequency parameter f/f_c for several rectangular tubes ($l_y/l_z = 1$, $l_z = 0{\cdot}002$ m to 1 m).

velocity $\mathbf{v}_v$ satisfy respectively, for a simple harmonic motion, the equations

$$(\nabla^2 + k_1^2)p = 0, \qquad (\nabla^2 + k_1^2)\tau_1 = 0, \qquad (15\text{a}, 16\text{a})$$

$$(\nabla^2 + k_t^2)\tau_t = 0, \qquad (\nabla^2 + k_v^2)\mathbf{v}_v = 0 \quad \text{and} \quad \nabla \cdot \mathbf{v}_v = 0, \qquad (17\text{a}, 18\text{a})$$

where k_1 and k_t are respectively obtained from the smaller and larger roots y_1 and y_2 of the equation

$$\left[1 - y\left(1 + \mathrm{i}\frac{\omega}{c}(l'_v + (\gamma - 1)l_t)\right)\right]\left[1 - \mathrm{i}\frac{\omega}{c}l_t y\right] = (\gamma - 1)\left(\mathrm{i}\frac{\omega}{c}\right)^2 l_t(l_t - l'_v)y^2$$

as

$$k_1 = k y_1^{1/2} \quad \text{and} \quad k_t = k y_2^{1/2} \qquad (l_v = 4l'_v/3), \qquad (17\text{b})$$

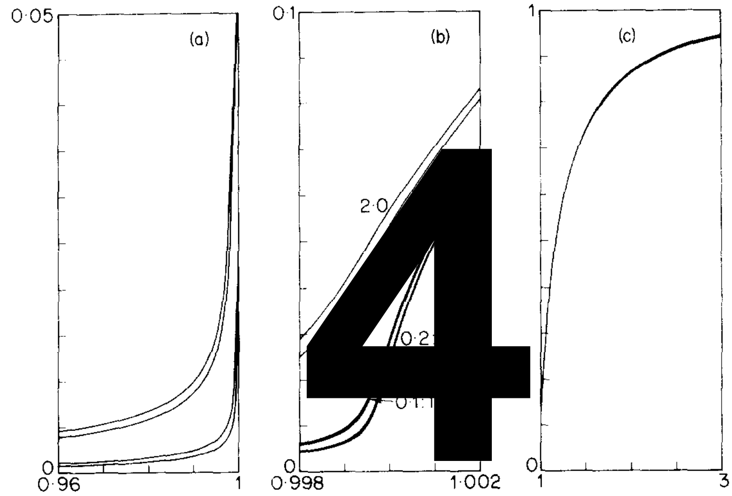

Figure 7. Normalized real part of the wavenumber χ/k of (m, n) mode versus frequency parameter f/f_c, for a circular tube ($a = 0{\cdot}01$ m).

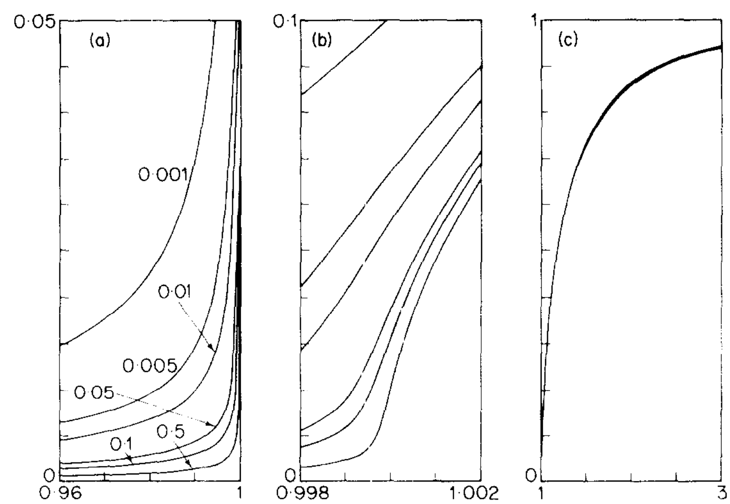

Figure 8. Normalized real part of the wavenumber χ/k of (2, 0) mode versus frequency parameter f/f_c, for several circular tubes ($a = 0{\cdot}5$ m to $0{\cdot}001$ m).

and

$$k_v^2 = -\mathrm{i}\omega/(cl_v). \tag{18b}$$

By using these equations, a simple calculation allows one to obtain (see, e.g., reference [6])

$$p = [\gamma\beta/(\gamma-1)](1 - k_1^2/k_t^2)A_1\tau_1, \tag{19}$$

$$\mathbf{v} = A_t\mathbf{v}_t + \frac{\gamma\beta}{\gamma-1}\frac{\mathrm{i}\omega}{\rho c^2}\left[\frac{1}{k_1^2}\left(1-\gamma\frac{k_1^2}{k_t^2}\right)A_1\nabla\tau_1 - \frac{\gamma-1}{k_t^2}A_h\nabla\tau_h\right], \tag{20}$$

where A_1, A_t and A_h are three arbitrary constants (β is the increase in pressure per unit increase in temperature at constant density).

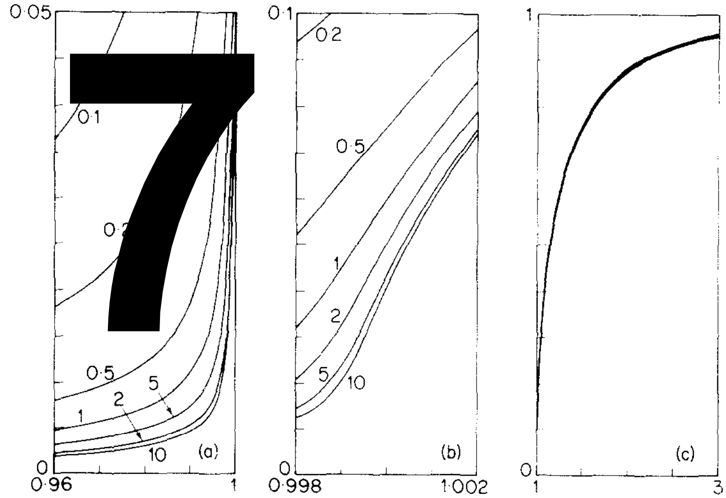

Figure 9. Normalized real part of the wavenumber χ/k of ($n_y = 0$, $n_z = 1$) mode versus frequency parameter f/f_c for several rectangular tubes ($l_z = 0{\cdot}02$ m, $l_y/l_z = 0.1$ to 10).

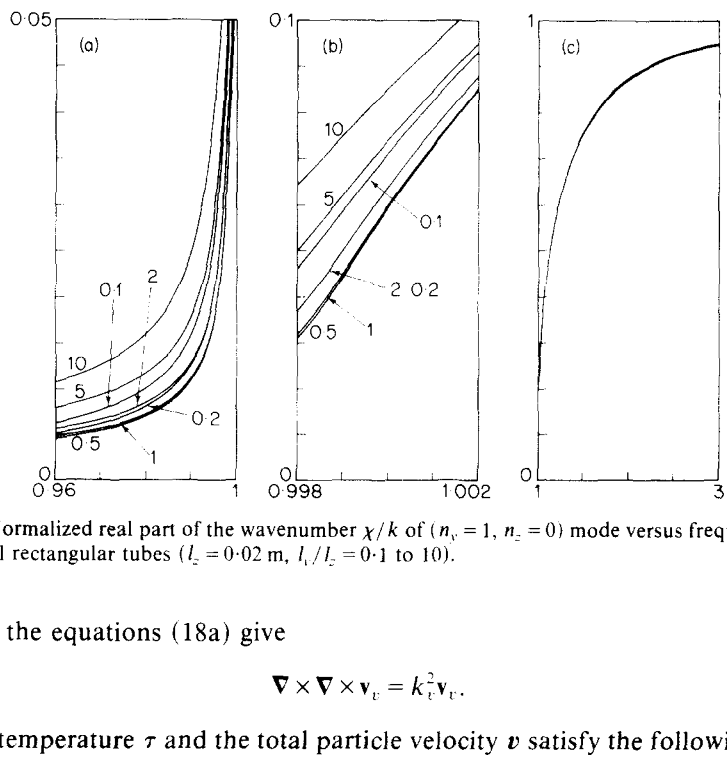

Figure 10. Normalized real part of the wavenumber χ/k of ($n_y = 1$, $n_z = 0$) mode versus frequency parameter f/f_c for several rectangular tubes ($l_z = 0{\cdot}02$ m, $l_y/l_z = 0{\cdot}1$ to 10).

Note that the equations (18a) give

$$\nabla \times \nabla \times \mathbf{v}_v = k_v^2 \mathbf{v}_v. \tag{21}$$

The total temperature τ and the total particle velocity $\boldsymbol{v}$ satisfy the following boundary conditions: the wall is highly conducting and has a high capacity for storing heat, so

$$A_1 \tau_1 + A_t \tau_t = 0 \qquad \text{or} \qquad A_1 \nabla_{\parallel} \tau_1 + A_t \nabla_{\parallel} \tau_t = 0. \tag{22}$$

where $\nabla_{\parallel}$ denotes the tangential "component" of the divergence; the no-slip condition requires $\mathbf{v}_{1\parallel} + \mathbf{v}_{t\parallel} + \mathbf{v}_{v\parallel} = \mathbf{0}$ or, upon substituting this into equations (20) and (22) for $\mathbf{v}$ and τ_t,

$$A_v \mathbf{v}_{v\parallel} = [\gamma\beta/(\gamma-1)](\mathrm{i}\omega/\rho c^2)(1/k_1^2)(1 - k_1^2/k_t^2) A_1 \nabla_{\parallel} \tau_1; \tag{23}$$

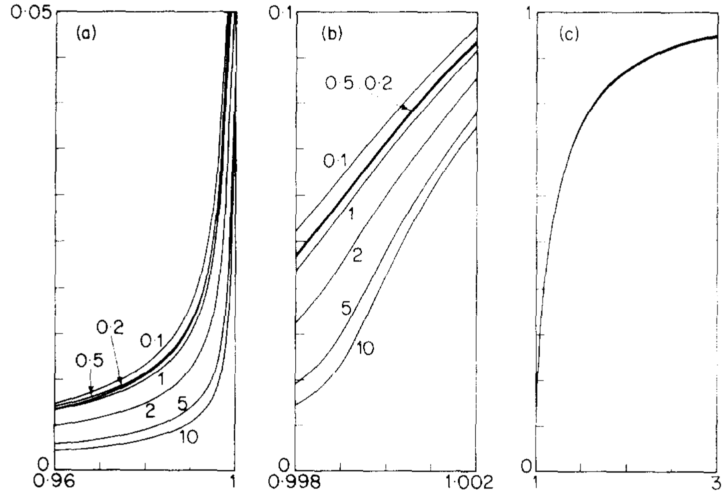

Figure 11. Normalized real part of the wavenumber χ/k of ($n_y=1$, $n_z=1$) mode versus frequency parameter f/f_c, for several rectangular tubes ($l_z=0{\cdot}02$ m, $l_y/l_z=0{\cdot}1$ to 10).

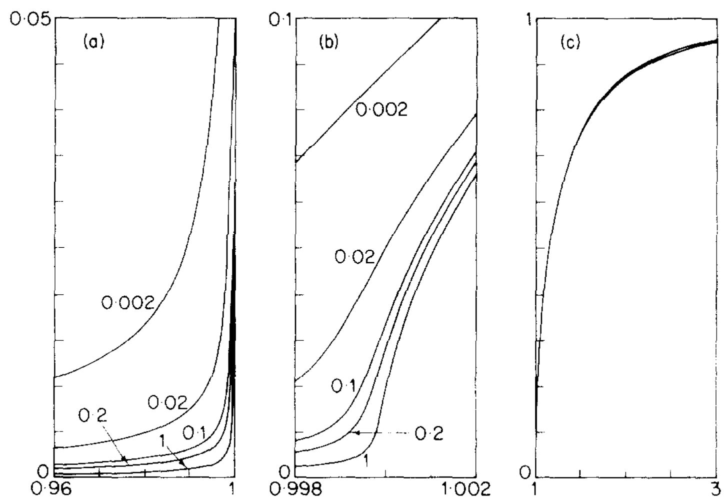

Figure 12. Normalized real part of the wavenumber χ/k of (2, 0) mode versus frequency parameter f/f_c, for several rectangular tubes ($l_y/l_z=1$, $l_z=0.002$ m to 1 m).

the assumption that the rigid wall is motionless leads to $(\mathbf{v}_l+\mathbf{v}_t+\mathbf{v}_v)\cdot\mathbf{n}=0$, where $\mathbf{n}$ is the unit normal to the surface, or

$$A_v\mathbf{v}_v\cdot\mathbf{n}=-\frac{\gamma\beta}{\gamma-1}\frac{i\omega}{\rho c^2}\left[\frac{1}{k_l^2}(1-k_l^2/k_t^2)A_l\mathbf{n}\cdot\boldsymbol{\nabla}\tau_l-\frac{\gamma-1}{k_t^2}A_t\mathbf{n}\cdot\boldsymbol{\nabla}\tau_t\right]. \tag{24}$$

It is now assumed that the solution for the acoustic pressure and the acoustic temperature can be expressed by a sum of modes, and each of these modes may be written in the form

$$e^{-ik_xx}\,e^{-im\varphi}\,J_m[(k_l^2-k_x^2)^{1/2}r], \tag{25}$$

where J_m is the cylindrical Bessel function of the first kind (order m). Thus one can write $(1/r)\partial_\varphi\tau_l=-i(m/r)\tau_l$ and $\partial_x\tau_l=-ik_x\tau_l$. Moreover, the boundary conditions (22) and

(23) *on the wall* ($r=a$) are valid whatever are the values of the co-ordinates x and φ. Hence, the quantities τ_t and $\mathbf{v}_v$ in these equations have an x- and φ-dependent factor which must have the same behavior as in equation (25). As a consequence, equations (17a) and (18a) lead to two Bessel equations and their solutions, respectively $J_m[(k_t^2-k_x^2)^{1/2}r]$ and $J_m[(k_v^2-k_x^2)^{1/2}r]$, are required for the radial part of τ_t and $\mathbf{v}_v$.

Starting with the set of equations (22), (23) and (24), and recognizing that the radial component of $\mathbf{v}_v$ may be written as (see equation (21))

$$v_{vr}=(k_v^2-k_x^2-m^2/r^2)^{-1}[r^{-2}\partial_\varphi v_{v\varphi}+r^{-1}\partial^2_{r\varphi}v_{v\varphi}+\partial^2_{rx}v_{vx}]$$

yields, for $r=a$,

$$\left(1-\gamma\frac{k_1^2}{k_t^2}\right)\left(\frac{\partial_r\tau_1}{\tau_1}\right)+(\gamma-1)\frac{k_1^2}{k_t^2}\left(\frac{\partial_r\tau_t}{\tau_t}\right)+\frac{1-k_1^2/k_t^2}{k_v^2-(k_x^2+m^2/a^2)}\left[k_x^2\left(\frac{\partial_r v_{vx}}{v_{vx}}\right)+\frac{m^2}{a^2}\left(\frac{\partial_r v_{v\varphi}}{v_{v\varphi}}\right)+\frac{m^2}{a^3}\right]=0. \tag{26}$$

Note that $\partial_r\tau_1/\tau_1$ is proportional to the specific admittance ε of the walls. Consequently, upon disregarding the second order terms in $(kl_v)^{1/2}$ or $(kl_h)^{1/2}$, this equation (26) gives equation (1). Letting $(\partial_r v_{vx}/v_{vx})_{r=a}=(\partial_r v_{v\varphi}/v_{v\varphi})_{r=a}=[\partial_r J_m[(k_v^2-k_x^2)^{1/2}r]/J_m[(k_v^2-k_x^2)^{1/2}r]]_{r=a}$ in equation (26) leads to

$$D_1\left(1-\gamma\frac{k_1^2}{k_t^2}\right)+D_t(\gamma-1)\frac{k_1^2}{k_t^2}+\frac{(k_x^2+m^2/a^2)D_v+m^2/a^3}{k_v^2-k_x^2-m^2/a^2}\left(1-\frac{k_1^2}{k_t^2}\right)=0, \tag{27}$$

where $D_i=(\partial_r(J_m[(k_i^2-k_x^2)^{1/2}r]/J_m[(k_i^2-k_x^2)^{1/2}r])_{r=a}$. By putting $m=0$, one obtains Kirchhoff's equation.

It is now supposed that kl_v and kl_t are small: only boundary layer effects are to be considered, and the classical visco-thermal effects in free space are to be ignored (see, e.g., references [8–10]). Moreover, effects due to rotational and vibrational relaxation are to be neglected (see, e.g., references [2] and [11]). Then equation (17b) reduces to $k_1\simeq k$, $k_t=(\gamma-1)k/\varepsilon_t$ and

$$D_1+(\gamma-1)D_t\frac{k^2}{k_t^2}+\frac{(k_x^2+m^2/a^2)D_v+m^2/a^3}{k_v^2-k_x^2-m^2/a^2}=0. \tag{28}$$

The solution of equation (28) for the fundamental mode leads to the well-known "Zwikker-Kosten-Daniels" (Z.K.D.) solution (see, e.g., reference [12] that is valid if $(\omega a/c)(\omega l/c)^{1/2}\ll 1$ and $l/a\ll 1$, where l is l_v or l_t. The extension to higher modes is more complicated, because one needs the assumption $k_x^2\ll k_v^2$ (for the calculation of D_v), which leads to $k_x^2\simeq k^2-\gamma_{mn}^2/a^2\ll k_v^2$. At higher frequencies, the condition is simply $kl\ll 1$, but at lower frequencies it becomes $k_v a\gg\gamma_{mn}$. This new condition is satisfied neither by capillary tubes (except for the fundamental mode), nor for very high order modes in any tube. Nevertheless, one may calculate an expansion of D_t and D_v with respect to k_x^2/k_v^2, by assuming $k_v a\gg\gamma_{mn}$ and by restricting the calculation to large tubes (i.e., non-capillary tubes).

The approximate method for solving equation (16) is the following. One writes $k_x^2=k^2-(\gamma_{mn}/a)^2-q$, where q is the unknown, and obtains the following expansions:

$$D_1=-k_v\frac{g^2w}{2s}\left[X+\frac{1}{2}\frac{1-w}{w}X^2+O(X^3)\right],$$

where $w=1-m^2/\gamma_{mn}^2$, $s=1/k_v a$, $g=\gamma_{mn}/k_v a$, and $X=q(a^2/\gamma_{mn}^2)$,

$$D_v=k_v\left[\frac{J'_m(k_v a)}{J_m(k_v a)}-\frac{k_x^2}{k_v^2}G_m(k_v a)+O\left(\frac{k_x^4}{k_v^4}\right)\right],$$

where $G_m(x)=(x/2)[(m^2/x^2)-1-J_m'^2(x)/J_m^2(x)]$, or, if $\omega l/c \ll 1$,

$$D_v=-(k_v/2s)[K_m(k_v a)-2sg^2(w+X)G_m(k_v a)+O(sg^4(w+X)^2)],$$

where $K_m(x)=-(2/x)(J_m'/J_m)-(2m^2/x^2)G_m(x)$.

By using asymptotic expansions of Kelvin functions (see the handbook by Abramowitz and Stegun [13]), one can write $G_m(x)=\mathrm{i}/2+O(1/x^2)$ and $K_m(x)=-(2\mathrm{i}/x)+(1/x^2)+O(1/x^3)$.

There are similar equations for D_t and $k_t a$. Then, equation (28) can be rewritten as follows (if $\omega l/c \ll 1$):

$$X+\frac{1}{2}\frac{1-w}{w}X^2+O(X^3)=-\frac{k^2}{k_v^2}\frac{K_m(k_v a)+(\gamma-1)K_m(k_t a)}{g^2 w}$$
$$+\left(1+\frac{X}{w}\right)K_m(k_v a)\left[1-\left(1+2s\frac{G_m}{K_m}\right)g^2(w+X)+O(g^4(w+X)^2)\right]+2s^2\frac{(1-w)}{w},$$

or

$$X[1-K_m(1/s)(1/w-2ug^2)+O(sg^4)]+X^2\left[\frac{1}{2}\frac{1-w}{w}+\frac{ug^2}{w}K_m+O(sg^4)\right]+O(X^3)$$
$$=A_0+K_m(1/s)[1-wug^2+O(g^4)]+2s^2(1-w)/w,$$

where

$$A_0=-\frac{[(\gamma-1)K_m(k_t a)+K_m(k_v a)]}{g^2 w}\frac{k^2}{k_v^2}, \qquad u=1+2s\frac{G_m(1/s)}{K_m(1/s)}.$$

Because the expansion is limited to terms of second order in X, it is convenient to write the unknown as a combination of the following terms: A, s, As, A^2, s^2, sg^2, s^2g^2, where $A=(s/g^2w)(k^2/k_v^2)$. Other terms, such as A^3, s^3, sg^4, As^2, A^2s, A^2s^2, and Asg^2, can be neglected. As examples, $As^2 \ll s$ because $\omega l/c$ and $\gamma_{mn}^2-m^2 \geqslant 2{\cdot}4\ \forall m+n\neq 0$; $Asg^2 \ll s$, because one can show that $s^2/w \ll 1$ if $g \ll 1$. One can also summarize these assumptions as follows: A^2, s^2 and g^2 need to be small quantities. Then, knowing that g^2 is always larger than s^2, one obtains

$$\frac{\omega l}{c}\ll 1, \qquad \frac{\omega}{c}\frac{a^2}{l}\gg \gamma_{mn}^2, \qquad \frac{\omega a}{c}\left(\frac{\omega l}{c}\right)^{1/2}\ll \gamma_{mn}^2-m^2. \tag{29}$$

One can then write the final result:

$$X=2\mathrm{i}QA+[2(1-w)/w]Q^2A^2+[(4Q/w)-Q']As-2\mathrm{i}s-3s^2+\mathrm{i}wsg^2+(4+w)s^2g^2,$$

where $Q=1+(\gamma-1)(l_t/l_v)^{1/2}$ and $Q'=1+(\gamma-1)l_t/l_v$, which can be developed as

$$k_x^2=k^2\left[1-\frac{2\mathrm{i}Q}{(1-m^2/\gamma_{mn}^2)k_v a}-\frac{1}{k_v^2a^2}\left(\frac{4Q}{1-m^2/\gamma_{mn}^2}-Q'\right)-\frac{2k^2}{k_v^2}\frac{m^2\gamma_{mn}^2Q^2}{(\gamma_{mn}^2-m^2)^3}\right]$$
$$-\left(\frac{\gamma_{mn}}{a}\right)^2\left[1-\frac{2\mathrm{i}}{k_v a}-\frac{3}{k_v^2a^2}+(\gamma_{mn}^2-m^2)\frac{1}{k_v^3a^3}+(5\gamma_{mn}^2-m^2)\frac{1}{k_v^4a^4}\right], \tag{30a}$$

or

$$k_x^2=k^2-\left(\frac{\gamma_{mn}}{a}\right)^2-2\mathrm{i}\frac{k}{a}\frac{\varepsilon_{mn}}{1-m^2/\gamma_{mn}^2}-\frac{\varepsilon_v^2}{a^2}\left(\frac{4Q}{1-m^2/\gamma_{mn}^2}-Q'\right)$$
$$-2k^2\varepsilon_v^2\frac{m^2\gamma_{mn}^2Q^2}{(\gamma_{mn}^2-m^2)^3}-\left(\frac{\gamma_{mn}}{a}\right)^2\left[-\frac{3\varepsilon_v^2}{k^2a^2}+(\gamma_{mn}^2-m^2)\frac{\varepsilon_v^3}{k^3a^3}+(5\gamma_{mn}^2-m^2)\frac{\varepsilon_v^4}{k^4a^4}\right]. \tag{30b}$$

Equation (30b) is more general than equation (7) (for cylindrical tube). It is more convenient when conditions (29) are only approximately satisfied. At very low frequencies or for very high modes ($\gamma_{mn} > a/l_v$), one can calculate the solution of equation (28) by assuming $k_x^2 \gg k_v^2$, and obtain the following result:

$$k_x^2 = k^2 - (\gamma_{mn}/a)^2 - (\gamma - 1)(k_t^3 k_v / k^2).$$

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