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GENERAL FORMULATION OF THE DISPERSION EQUATION IN BOUNDED VISCO-THERMAL FLUID, AND APPLICATION TO SOME SIMPLE GEOMETRIES

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The purpose of this paper is to give a new set of equations derived from the basic classical theory of the acoustic propagation in visco-thermal fluid and valid in the time domain, and to provide us with a general dispersion equation for harmonic waves in several boundary problems of interest. It is shown that this dispersion equation generalizes some known results as the equivalent specific impedance of plane boundaries and resonance frequencies of spherical resonators, and that it provides us with a new general equation giving the propagation constant of waves for all kind of modes in rigid walled cylindrical tubes.

1. Introduction

One hundred and twenty years ago (1868), the famous scientist G. Kirchhoff proposed a very fertile description of sound propagation in gases [1]. This theory was mainly based on the Navier–Stokes equation and the Fourier equation of heat conduction as modified to account for linear acoustics approximations. The author derived an algebraic equation (dispersion equation) and he found solutions to it, that is, propagation constants, for plane waves and spherical outgoing waves in an unbounded medium and for waves propagating along the axis of a circular tube. In the latter case, he assumed the boundary conditions of zero particle velocity and acoustic temperature at the tube wall. Then he calculated the attenuation factors and speed of propagations from the real and imaginary parts of the propagation constants respectively, which are solutions of the dispersion equation, for the wide tube case and the upper wavelength range (compared with the thickness of the boundary layers), making approximations to the lowest orders possible. Lord Rayleigh has included a detailed account of this theory in his book "Theory of Sound" [2].

Eighty years later (1948), L. Cremer [3] showed that, for the reflection of a harmonic plane-wave on an infinite rigid plane wall, the acoustic behavior of the medium, in the neighborhood of the boundary but outside the boundary layer, is adequately described by a simple propagational acoustic mode in a perfect gas, most of the power losses through viscosity and thermal conductivity occurring within the boundary layer (see for example [4]). As a consequence, assuming the temperature fluctuations and the particle velocity to be nearly equal to zero at the boundaries, the properties of plane-wave reflection on a plane surface are described by the ratio of the normal component of the acoustic particle velocity to the acoustic pressure at the boundary, called apparent specific admittance and depending on the shear and bulk viscosities, the coefficient of thermal conduction, and also the angle of incidence θ. Making use of this innovative result, the problem of extending the classical Kirchhoff result for the attenuation of harmonic plane waves in rigid walled tubes to the case of higher order propagating modes was given by Beatty [5], but the theory is not valid near the adiabatic cutoff frequency and for the evanescent modes.

Attempts to solve this problem have been made recently [6]: useful results have been given at the cutoff frequency and for the evanescent modes, but unfortunately the extension to all kind of modes of the above-mentioned Kirchhoff theory given in eq. (27) of that paper is not correct. Finally, in another paper [7], an extension for the Kirchhoff theory to the spherical shell leads to an equation which determines the complex resonance frequencies of the gas in the shell.

Our purpose is to definitely provide a correct dispersion equation for all kind of modes in cylindrical tubes. In fact, the main feature of our work is a new set of equations, derived from the basic classical theory of the acoustic propagation in visco-thermal fluid and valid not only in the frequency domain but also in the time domain; as a consequence, all the results mentioned above for harmonic waves, i.e. the equivalent specific impedance of a rigid plane wall (equivalent to the viscosity and thermal conduction effects inside the boundary layers), the propagation constant of waves for all kind of modes in a rigid walled circular tube, and the resonant frequencies of a rigid walled sphere, are three particular solutions of a unique general dispersion equation. (Note that only differentiable boundary surfaces, that is, piece-wise $\mathscr{C}^1$ boundary, can be taken into account.)

2. The basic equations

In this section, we give the basic linear equations governing the acoustic disturbances, in the presence of viscosity and thermal conduction. In most classical cases, the effects of vibrational molecular relaxation can be taken into account by changing nothing but the specific heat ratio γ in γ_r, which depends on the frequency and the relaxation time (see for example [8], [9]). But this last approach is strictly restricted to the frequency domain.

The variables describing the dynamical and thermodynamical state of the fluid are the acoustic pressure p, the particle velocity $\boldsymbol{v}$, the variable part of the density ρ', the entropy variation σ, and the temperature variation τ. The parameters which specify the properties and the nature of the fluid are the ambient values of the pressure P, the ambient values of the temperature T, the ambient values of the density ρ, the viscosity μ, the bulk viscosity η, the coefficient of thermal conductivity λ, the heat coefficients at constant pressure and constant volume per unit of mass C_p and C_v, the specific heat ratio γ, and the increase in pressure per unit increase in temperature at constant density β.

A complete set of linear homogeneous equations governing small amplitude disturbances [8, 9] includes the following:

- The Navier–Stokes equations

$$\frac{1}{c}\,\partial_t \boldsymbol{v} + \frac{1}{\rho c}\,\mathbf{grad}\, p = \ell_v\,\mathbf{grad}\,\mathrm{div}\,\boldsymbol{v} - \ell_v'\,\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{v} \tag{1}$$

where the characteristic lengths ℓ_v and ℓ_v' are defined as follows, with c the velocity of sound:

$$\ell_v = (\tfrac{4}{3}\mu + \eta)/\rho c, \qquad \ell_v' = \mu/\rho c.$$

- The conservation of mass equation, taking into account that the equation expressing the acoustic part of the density (regarded as a function of the independent variables p and τ) is considered as total differential

$$\rho c\,\mathrm{div}\,\boldsymbol{v} + \gamma\frac{1}{c}\,\partial_t(p - \beta\tau) = 0. \tag{2}$$

- The Fourier equation for conduction of heat, taking into account that the equation expressing the entropy variations (regarded as a function of the independent variables p and τ) is considered as total differential

$$\left[\frac{1}{c}\partial_t - \ell_h\Delta\right]\tau = \frac{\gamma-1}{\beta\gamma}\frac{1}{c}\partial_t p \tag{3}$$

where the characteristic length ℓ_h is defined as $\ell_h = \lambda/\rho c C_p$.

Under the usual gauge conditions, the particle velocity $\boldsymbol{v}$ of any disturbances governed by this system of linear equations can be considered as a superposition of a rotational velocity $\boldsymbol{v}_v$ (due to viscosity effects) and a solenoïdal velocity $\boldsymbol{v}_\ell$ (due to acoustic and heat conduction effects):

$$\boldsymbol{v} = \boldsymbol{v}_\ell + \boldsymbol{v}_v.$$

Consequently, eq. (1) can be split into two equations in such a way that eqs. (1), (2) and (3) yield:

$$\frac{1}{c}\partial_t\tau - \frac{\rho c}{\gamma\beta}\operatorname{div}\boldsymbol{v}_\ell = \frac{1}{\beta}\frac{1}{c}\partial_t p, \tag{4}$$

$$\left(\frac{1}{c}\partial_t - \ell_h\Delta\right)\tau = \frac{\gamma-1}{\beta\gamma}\frac{1}{c}\partial_t p, \tag{5}$$

$$\left(\frac{1}{c}\partial_t - \ell_v\Delta\right)\boldsymbol{v}_\ell = -\frac{1}{\rho c}\,\mathbf{grad}\,p, \tag{6}$$

$$\left(\frac{1}{c}\partial_t - \ell'_v\Delta\right)\boldsymbol{v}_v = 0 \tag{7}$$

$$\text{with}\ \operatorname{div}\boldsymbol{v}_v = 0 \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{v}_\ell = \mathbf{0}. \tag{8a, b}$$

Associated with boundary conditions (see Section 3), this set of equations is the basis for the calculation of the acoustic fields outside the sources.

It is convenient, as regards the calculation of the acoustic propagation, to find out the homogeneous wave equations for p, τ, and $\boldsymbol{v}_\ell$.

Combining eq. (4) and the divergence of eq. (6) to eliminate the term div $\boldsymbol{v}_\ell$, then eliminating the terms $\partial_t p$ and Δp in the resulting equation, and making use of eq. (5) and its Laplacian, yields:

$$\ell_h\left(1+\gamma\ell_v\frac{1}{c}\partial_t\right)\Delta\Delta\tau - \left[1+(\ell_v+\gamma\ell_h)\frac{1}{c}\partial_t\right]\frac{1}{c}\partial_t\Delta\tau + \frac{1}{c^3}\partial^3_{ttt}\tau = 0. \tag{9}$$

It is easy to verify that this equation can be written formally as follows, after multiplication by the factor $(1/c)\partial_t$:

$$\left[\frac{1}{c^2}\partial^2_{tt} - \frac{\Gamma+R}{2}\Delta\right]\left[\frac{1}{c^2}\partial^2_{tt} - \frac{\Gamma-R}{2}\Delta\right]\tau = 0 \tag{10}$$

where

$$\Gamma = 1+(\ell_v+\gamma\ell_h)\frac{1}{c}\partial_t,$$

$$R = \left\{1+2[\ell_v-(2-\gamma)\ell_h]\frac{1}{c}\partial_t + (\ell_v-\gamma\ell_h)^2\frac{1}{c^2}\partial^2_{tt}\right\}^{1/2}.$$

Note that an equivalent equation was previously given by Truesdell [10], only in the frequency domain, but has never to our knowledge been used in acoustic until now.

This equation shows clearly the well known result that the total temperature τ can be written as the sum of an acoustic temperature $A_a\tau_a$ and an entropic temperature $A_h\tau_h$, which are respective solutions of the homogeneous equations (A_a and A_h are arbitrary constants):

$$\left(\frac{1}{c^2}\partial_{tt}^2 - \frac{\Gamma+R}{2}\Delta\right)\tau_a = 0, \qquad \left(\frac{1}{c^2}\partial_{tt}^2 - \frac{\Gamma-R}{2}\Delta\right)\tau_h = 0 \tag{11a, b}$$

where eq. (11b) is a diffusion equation because the Taylor expansion of the function $(\Gamma - R)$ shows that the operator $(1/c)\partial_t$ can be factorized. That is one reason why τ_h is associated with the heat transfer due to thermal conduction.

Starting from the set of equations (4) to (7), one can demonstrate easily that the total pressure p and the solenoïdal particle velocity $\boldsymbol{v}_\ell$ satisfy the same equations (9) and (10) as τ. To show this result for p, one combines eq. (4) and the divergence of eq. (6) to eliminate the term div $\boldsymbol{v}_\ell$, then one applies the operator $[(1/c)\partial_t - \ell_h\Delta]$ to the resulting equation and one eliminates τ directly making use of eq. (5). To obtain the same result for $\boldsymbol{v}_\ell$, first one applies the operator **grad** $[(1/c)\partial_t - \ell_h\Delta]$ to eq. (4) and one eliminates directly τ making use of eq. (5); then one eliminates the term **grad** p in the resulting equation making use of eq. (6).

Last, combining eqs. (5) and (11) along with the relation $\tau = A_a\tau_a + A_h\tau_h$ yields:

$$p = p_a + p_h \tag{12}$$

with

$$p_a = \frac{\gamma\beta}{\gamma-1}\left[1 - 2\ell_h(\Gamma+R)^{-1}\frac{1}{c}\partial_t\right]A_a\tau_a, \qquad p_h = \frac{\gamma\beta}{\gamma-1}\left[1 - 2\ell_h(\Gamma-R)^{-1}\frac{1}{c}\partial_t\right]A_h\tau_h$$

and combining eqs. (6) and (11) (which are also verified for $\boldsymbol{v}_{\ell a}$ and $\boldsymbol{v}_{\ell h}$) along with the relations (12) yields:

$$\boldsymbol{v}_\ell = \boldsymbol{v}_{\ell a} + \boldsymbol{v}_{\ell h}$$

with

$$\left[1 - 2\ell_v(\Gamma+R)^{-1}\frac{1}{c}\partial_t\right]\rho\partial_t\boldsymbol{v}_{\ell a} = -\frac{\gamma\beta}{\gamma-1}\left[1 - 2\ell_h(\Gamma+R)^{-1}\frac{1}{c}\partial_t\right]\mathbf{grad}\, A_a\tau_a, \tag{13a}$$

$$\left[1 - 2\ell_v(\Gamma-R)^{-1}\frac{1}{c}\partial_t\right]\rho\partial_t\boldsymbol{v}_{\ell h} = -\frac{\gamma\beta}{\gamma-1}\left[1 - 2\ell_h(\Gamma-R)^{-1}\frac{1}{c}\partial_t\right]\mathbf{grad}\, A_h\tau_h. \tag{13b}$$

Hence, the general solutions of the problem can be derived from the next set of formal equations where all divisions by ∂_t formally mean integration:

$$\left(\frac{1}{c}\partial_t + \ell_v'\,\mathbf{curl\,curl}\right)\boldsymbol{v}_v = 0, \qquad \mathrm{div}\,\boldsymbol{v}_v = 0, \tag{14a, b}$$

$$\left(\frac{1}{c^2}\partial_{tt}^2 - \frac{\Gamma+R}{2}\Delta\right)\tau_a = 0, \qquad \left(\frac{1}{c^2}\partial_{tt}^2 - \frac{\Gamma-R}{2}\Delta\right)\tau_h = 0, \qquad \tau = A_a\tau_a + A_h\tau_h, \tag{15a, b, c}$$

$$p = p_a + p_h = \frac{\gamma\beta}{\gamma-1}\left\{\left[1 - \frac{2\ell_h}{\Gamma+R}\frac{1}{c}\partial_t\right]A_a\tau_a + \left[1 - \frac{2\ell_h}{\Gamma-R}\frac{1}{c}\partial_t\right]A_h\tau_h\right\}, \tag{16}$$

$$\boldsymbol{v}_\ell = \boldsymbol{v}_{\ell a} + \boldsymbol{v}_{\ell h} = \alpha_a\,\mathbf{grad}\,A_a\tau_a + \alpha_h\,\mathbf{grad}\,A_h\tau_h \tag{17}$$

where

$$\alpha_a = -\frac{\gamma\beta}{\gamma-1}\frac{1}{\rho\partial_t}\frac{1-\frac{2\ell_h}{\Gamma+R}\frac{1}{c}\partial_t}{1-\frac{2\ell_v}{\Gamma+R}\frac{1}{c}\partial_t}, \qquad \alpha_h = -\frac{\gamma\beta}{\gamma-1}\frac{1}{\rho\partial_t}\frac{1-\frac{2\ell_h}{\Gamma-R}\frac{1}{c}\partial_t}{1-\frac{2\ell_v}{\Gamma-R}\frac{1}{c}\partial_t}$$

and Γ and R are given in eq. (10).

For an harmonic motion ($\partial_t \equiv \mathrm{i}\omega$), this set of equations gives the following equations:

$$\mathbf{curl\,curl}\,\boldsymbol{v}_v - k_v^2\boldsymbol{v}_v = 0, \qquad \operatorname{div}\boldsymbol{v}_v = 0, \tag{18a, b}$$

$$(\Delta + k_a^2)\tau_a = 0, \qquad (\Delta + k_h^2)\tau_h = 0, \qquad \tau = A_a\tau_a + A_h\tau_h, \tag{19a, b, c}$$

$$p = \frac{\gamma\beta}{\gamma-1}\left\{\left[1-\mathrm{i}\frac{c}{\omega}\ell_h k_a^2\right]A_a\tau_a + \left[1-\mathrm{i}\frac{c}{\omega}\ell_h k_h^2\right]A_h\tau_h\right\}, \tag{20}$$

$$\boldsymbol{v}_\ell = \boldsymbol{v}_{\ell a} + \boldsymbol{v}_{\ell h} = \alpha_a\,\mathbf{grad}\,A_a\tau_a + \alpha_h\,\mathbf{grad}\,A_h\tau_h \tag{21}$$

where

$$k_v^2 = -\frac{\mathrm{i}\omega}{c\ell_v'}, \qquad k_a^2 = \frac{\omega^2}{c^2}\left(1+\mathrm{i}\frac{\omega}{c}\ell_{vh} - \frac{\omega^2}{c^2}\ell_{vh}'\ell_h\right)^{-1}, \qquad k_h^2 = -\frac{\mathrm{i}\omega}{c\ell_h}\left(1-\mathrm{i}\frac{\omega}{c}\ell_{vh}'\right)^{-1}, \tag{22a, b, c}$$

$$\alpha_a = \frac{\mathrm{i}}{\rho\omega}\frac{\gamma\beta}{\gamma-1}\frac{1-\mathrm{i}\frac{c}{\omega}\ell_h k_a^2}{1-\mathrm{i}\frac{c}{\omega}\ell_v k_a^2}, \qquad \alpha_h = \frac{\mathrm{i}}{\rho\omega}\frac{\gamma\beta}{\gamma-1}\frac{1-\mathrm{i}\frac{c}{\omega}\ell_h k_h^2}{1-\mathrm{i}\frac{c}{\omega}\ell_v k_h^2} \tag{22d, e}$$

with

$$\ell_{vh} = \ell_v + (\gamma-1)\ell_h, \qquad \ell_{vh}' = (\gamma-1)(\ell_h - \ell_v).$$

3. Boundary conditions

In order to extend the formalism derived so far to bounded media, let us define generalized coordinates on a neighborhood of the boundary surfaces (assumed to be piece-wise $\mathscr{C}^1$). At any point located on the boundary, let $\boldsymbol{e}$ be an outward normal vector to the boundary surface and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ two orthogonal vectors defining the tangent plane. To each vector $\boldsymbol{e}$, $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ can be associated a generalized coordinate u, w_1 and w_2 respectively, also noted $(u, \boldsymbol{w})$. Hence the variation vector $\mathrm{d}\boldsymbol{M}$ of a point M located in the neighborhood is given by:

$$\mathrm{d}\boldsymbol{M} = \mathrm{d}u\,\boldsymbol{e} + \mathrm{d}w_1\,\boldsymbol{e}_1 + \mathrm{d}w_2\,\boldsymbol{e}_2, \tag{23}$$

and the geometrical properties of the neighborhood of the boundary are contained in its linear element:

$$\mathrm{d}\ell^2 = \nu^2\,\mathrm{d}u^2 + \nu_1^2\,\mathrm{d}w_1^2 + \nu_2^2\,\mathrm{d}w_2^2$$

where ν, ν_1 and ν_2 are the modulus of vectors $\boldsymbol{e}$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ respectively [13]. In other words, if $\boldsymbol{n}$ is the normal unit vector to the boundary, $\boldsymbol{e} = \nu\boldsymbol{n}$. We then define the boundary surface by a constant value s taken by the coordinate u on it, and introduce the notation ∂_s for the partial derivative ∂_u taken at $u = s$.

In the case of a spherical or a cylindrical boundary surface, the role of the normal coordinate is naturally played by the radial coordinate and the constant s simply represents the value of the radius of the boundary surface.

The remainder of this paper gives an extension of earlier works in order to obtain a general dispersion equation valid for common shapes of boundaries. Thus the next sections aim at establishing theoretical results in the frequency domain only, the analysis being based on eqs. (18) to (21) and the boundary conditions we now develop.

3.1. Thermal boundary conditions

Under low density conditions, it is more exact to assume a slight temperature jump at the gas-wall interface, instead of the usual continuity of temperature (see for example [11]). The temperature jump, equal to the gas temperature at the interface $\tau(s, w)$ minus the wall surface temperature $\tau_0(s, w)$, is assumed to be proportional to the normal derivative of the gas temperature near the wall, i.e.:

$$\tau(s, w) - \tau_0(s, w) = -L_{\mathrm{j}} \frac{1}{\nu} \partial_s \tau(s, w) \tag{24}$$

where L_{j} is a coefficient depending on thermal accommodation coefficients (see for example [12]), and $1/\nu$ accounts for the non-unity modulus of vector $\boldsymbol{e}$.

In addition, the normal heat flux must be continuous at the interface. Then, the total, outwardly directed, heat flux can be written either as $[-\lambda(1/\nu)\partial_s\tau]$ or $[-\lambda_0(1/\nu)\partial_s\tau_0]$. If we neglect heat-transfer parallel to the wall on the account that the temperature varies slowly in that direction, the diffusion equation for heat inside the wall takes the following form:

$$(\Delta_u - \mathrm{i}\omega\rho_0 C_0/\lambda_0)\tau_0(u) = \left[\frac{1}{\nu\nu_1\nu_2}\partial_u\left(\frac{\nu_1\nu_2}{\nu}\partial_u\right) - \mathrm{i}\omega\rho_0 C_0/\lambda_0\right]\tau_0(u) = 0 \tag{25}$$

where ρ_0 and C_0 are the density and the heat coefficients of the wall respectively. In all usual cases (including most capillary tubes), the curvature radius R of the boundary satisfies the inequality $R \gg (\lambda_0/\omega\rho_0 C_0)^{1/2}$, and eq. (25) can be separated into

$$\left[\frac{\nu_1\nu_2}{\nu}\partial_u + \nu_1\nu_2\sqrt{\mathrm{i}\omega\rho_0 C_0/\lambda_0}\right]\left[\frac{\nu_1\nu_2}{\nu}\partial_u - \nu_1\nu_2\sqrt{\mathrm{i}\omega\rho_0 C_0/\lambda_0}\right]\tau_0(u) = 0, \tag{26}$$

and since the heat sources are all located on the boundary surfaces, the only equation admissible in the context of the problem at the interface is:

$$\frac{1}{\nu}\partial_s\tau_0 = -\sqrt{\mathrm{i}\omega\rho_0 C_0/\lambda_0}\,\tau_0. \tag{27}$$

Finally, combining (24) and (27) along with the continuity of the heat flux at the interface yields:

$$\left(1 + L_{\mathrm{h}}\frac{1}{\nu}\partial_s\right)\tau(s, w) = 0 \quad \text{where} \quad L_{\mathrm{h}} = L_{\mathrm{j}} + \lambda/\sqrt{\mathrm{i}\omega\rho_0 C_0 \lambda_0} \tag{28}$$

which is an impedance-like boundary condition.

It is convenient, as regards the calculation of the constant A_{h} (see eq. 19(c)), to explicit (28) with the generalized coordinates (u, w). Since in most applications we consider single eigenmode solutions, the field quantities can be factorized, leading to:

$$\left(1 + L_{\mathrm{h}}\frac{1}{\nu}\partial_s\right)[A_{\mathrm{a}}\tau_{\mathrm{a}}(s)\psi_{\mathrm{a}}(w) + A_{\mathrm{h}}\tau_{\mathrm{h}}(s)\psi_{\mathrm{h}}(w)] = 0 \tag{29}$$

which involves the acoustic and entropic temperature in the gas, expressed as the product of a function of u (still denoted τ_a and τ_h) and a function of w (denoted ψ_a and ψ_h).

Since this equation must be satisfied for arbitrary values of w all over the boundary surface, we must write

$$\psi_a(w) = \psi_h(w) \triangleq \psi(w), \quad \forall w, \tag{30}$$

and, consequently, eq. (29) can be written as follows:

$$\frac{A_h}{A_a} = -\frac{\left(1 + L_h \frac{1}{\nu} \partial_s\right) \tau_a(s)}{\left(1 + L_h \frac{1}{\nu} \partial_s\right) \tau_h(s)} \tag{31}$$

providing us with a relationship between the amplitude A_h of the entropic temperature and the amplitude A_a of the acoustic one.

In most applications, it is well known that the temperature jump is nearly null at the interface ($L_j \sim 0$), and that the product of the thermal conductivity and specific heat per unit volume of the shell material greatly exceeds the corresponding quantity of the gas; so, the condition (28) is practically equivalent to the requirement that the temperature be constant on the wall, i.e. $L_h \sim 0$ (see for example [7] and [11]). Then eq. (31) reduces to

$$\frac{A_h}{A_a} = -\frac{\tau_a(s)}{\tau_h(s)}. \tag{32}$$

(Note that the amplitude A_a is fixed by the strength of sources.)

3.2. Velocity boundary conditions

The boundary conditions for the particle velocity assume on one hand a slight velocity slip at the interface, proportional to the normal derivative of the tangential component of the velocity of the gas [11], and, on the other hand, include a slight normal velocity written in terms of a boundary specific impedance. We write these conditions as:

$$\left(1 + \zeta \frac{1}{\nu} \partial_s\right) v_u(s, w) = 0, \tag{33}$$

$$\left(1 + \xi \frac{1}{\nu} \partial_s\right) \boldsymbol{v}_w(s, w) = 0 \tag{34}$$

where $v_u(s, w)$ and $\boldsymbol{v}_w(s, w)$ are the normal (outwardly directed) and the tangential components to the interface respectively of the particle velocity, ξ is a coefficient depending on the tangential momentum accommodation coefficient [11], and ζ is linked to the usual boundary specific impedance

$$\mathrm{i}(\omega/c)\zeta = p/(\rho c v_u).$$

Note that eqs. (33) and (34) are impedance-like boundary conditions, as in eq. (28).

In most applications, perfectly rigid walls and nonrarefied gases are considered, and the quantities ζ and ξ are nearly equal to zero. So, we assume henceforth that the total particle velocity $\boldsymbol{v}$ is equal to zero on the boundaries:

$$v_u(s, w) = 0, \qquad \boldsymbol{v}_w(s, w) = 0. \tag{35}$$

It is convenient, as regards the relationship needed between the rotational velocity $\boldsymbol{v}_{\rm v}$ and the temperature on the boundaries, to replace eqs. (35) by

$$\left[A_{\rm a}\alpha_{\rm a}\frac{1}{\nu}\,\partial_s\tau_{\rm a}(s)+A_{\rm h}\alpha_{\rm h}\frac{1}{\nu}\,\partial_s\tau_{\rm h}(s)\right]\psi(\boldsymbol{w})=-v_{{\rm v}u}(s)\,\psi_{{\rm v}u}(\boldsymbol{w}), \tag{36}$$

$$[A_{\rm a}\alpha_{\rm a}\tau_{\rm a}(s)+A_{\rm h}\alpha_{\rm h}\tau_{\rm h}(s)]\boldsymbol{\nabla}_w\psi(\boldsymbol{w})=-v_{{\rm v}w}(s)\,\boldsymbol{\psi}_{{\rm v}w}(\boldsymbol{w}) \tag{37}$$

which involve the acoustic, entropic, and vorticity velocities in the gas (given in eqs. (7) and (21)), expressed as the product of a function of u and a function of $\boldsymbol{w}$.

Since these equations must be satisfied for arbitrary values of $\boldsymbol{w}$ all over the boundary surface, we must write:

$$\psi_{{\rm v}u}(\boldsymbol{w})=\psi(\boldsymbol{w}),\qquad \boldsymbol{\psi}_{{\rm v}w}(\boldsymbol{w})=\boldsymbol{\nabla}_w\psi(\boldsymbol{w}) \tag{38a, b}$$

and this implies in turn the independence of $v_{{\rm v}w}$ on $\boldsymbol{w}$, i.e.:

$$v_{{\rm v}w_1}(s)=v_{{\rm v}w_2}(s)\triangleq v_{{\rm v}w}(s). \tag{39}$$

Finally, combining eqs. (36) and (37) along with eqs. (32), (38) and (39) yields a general dispersion equation:

$$\left(1-\frac{\alpha_{\rm h}}{\alpha_{\rm a}}\right)\frac{v_{{\rm v}u}(s)}{v_{{\rm v}w}(s)}=\frac{1}{\nu}\left[\frac{\partial_s\tau_{\rm a}(s)}{\tau_{\rm a}(s)}-\frac{\alpha_{\rm h}}{\alpha_{\rm a}}\frac{\partial_s\tau_{\rm h}(s)}{\tau_{\rm h}(s)}\right]. \tag{40}$$

Note that, as the square of the acoustic wavenumber $k_{\rm a}$ can be written $k_{\rm a}^2=k_{{\rm a}u}^2+k_{{\rm a}w}^2$, the square of the entropic and vorticity wavenumber can be written $k_{\rm h}^2=k_{{\rm h}u}^2+k_{{\rm a}w}^2$ and $k_{\rm v}^2=k_{{\rm v}u}^2+k_{{\rm a}w}^2$, because the behavior of the $\boldsymbol{w}$-dependence of each kind of movement is given by the same function $\psi(\boldsymbol{w})$.

Various applications of these results (especially eq. (40)) are presented in the next section, for the case of cartesian, cylindrical and spherical coordinates.

4. Applications

For further developments, the general dispersion equation (40) must be particularized for each boundary problem. This last section demonstrates that this equation includes some known results as the equivalent specific impedance of plane boundaries and resonance frequencies of spherical resonators, and provides a new general equation giving the propagation constant of waves for all kind of modes in rigid walled circular tubes.

4.1. The equivalent specific impedance of plane boundaries

Let a semi-infinite medium bounded by an infinite plane rigid wall set at $x=0$ (the x-axis being inwardly directed). As a consequence of eq. (40), we will show that the effects of the thermal and shear modes on the boundary conditions can be treated by using the concept of boundary specific admittance $\rho c/Z_{\rm a}=\rho c v_{\ell{\rm a}}/p_{\rm a}$ (see for example [3] or [4]):

$$\frac{\rho c}{Z_{\rm a}}=\sqrt{\rm i}\sqrt{\frac{\omega}{c}}\left[\left(1-\frac{k_{{\rm a}x}^2}{k_{\rm a}^2}\right)\sqrt{\ell_{\rm v}'}+(\gamma-1)\sqrt{\ell_{\rm h}}\right] \tag{41}$$

where the acoustic wavenumber has been written as usual:

$$k_{\rm a}^2=k_{{\rm a}x}^2+k_{{\rm a}y}^2+k_{{\rm a}z}^2.$$

To demonstrate that, we solve the set of equations (18) to (21), assuming that the function $\psi(y, z)$ (see eqs. (30) and (38)) has the form:

$$e^{-ik_{ay}y}\, e^{-ik_{az}z}. \tag{42}$$

Then, upon disregarding the first order terms in $[(\omega/c)\ell_h]$ and $[(\omega/c)\ell'_v]$, we can solve eq. (19), i.e. $(\Delta+k_h^2)\tau_h=0$, to obtain

$$\frac{-\partial_x\tau_h}{\tau_h}=i\sqrt{k_h^2-k_{ay}^2-k_{az}^2}\sim ik_h\sim i\sqrt{\frac{-i\omega}{c\ell_h}} \tag{43}$$

and, on the other hand, one can write (see eq. (22))

$$\alpha_h/\alpha_a\sim -i\frac{\omega}{c}(\gamma-1)\ell_h \tag{44}$$

and (see eqs. (20) and (21))

$$\partial_x\tau_a/\tau_a\sim i\frac{\omega}{c}\rho c v_{\ell a}/p_a=i\frac{\omega}{c}\frac{\rho c}{Z_a} \tag{45}$$

where Z_a is the acoustic impedance.

At this step, in order to demonstrate the result (41) from eq. (40), we have to solve eq. (18), assuming that $\boldsymbol{v}_v$ has the form (see eq. (38)):

$$\boldsymbol{v}_v=(v_{vx}(x)\, e^{-ik_{ay}y}\, e^{-ik_{az}z},\quad v_{vy}(x)\,\partial_y\, e^{-ik_{ay}y}\, e^{-ik_{az}z},\quad v_{vz}(x)\, e^{-ik_{ay}y}\partial_z\, e^{-ik_{az}z}). \tag{46}$$

The incoming wave (i.e. coming into the fluid from the plane boundary) is the solution of the first equation (18)

$$[\partial_{xx}^2+k_v^2-k_{ay}^2-k_{az}^2]v_{vx}(x)=0, \tag{47}$$

and has the following property

$$\partial_x v_{vx}=-i\sqrt{k_v^2-k_{ay}^2-k_{az}^2}\,v_{vx}\sim -ik_v v_{vx}\sim -i\sqrt{\frac{-i\omega}{c\ell'_v}}\,v_{vx}\sim\frac{v_{vx}}{i\sqrt{i}\sqrt{c/\omega}\sqrt{\ell'_v}}. \tag{48}$$

Combining this last result with the equation

$$\operatorname{div}\boldsymbol{v}_v=\partial_x v_{vx}(x)+k_{ay}^2 v_{vy}(x)+k_{az}^2 v_{vz}(x)=0$$

along with the relation (39), i.e. $v_{vy}(x=0)=v_{vz}(x=0)$, yields:

$$\frac{v_{vx}}{v_{vw}}=i\sqrt{i}k_a^2\left(1-\frac{k_{ax}^2}{k_a^2}\right)\sqrt{\frac{c}{\omega}}\sqrt{\ell'_v}. \tag{49}$$

Substituting the results (43), (44), (45), and (49) into eq. (40) gives the specific acoustic admittance (41) on the boundary in the form that we were looking for. In terms of this admittance, boundary conditions can be conveniently stated for many practical problems in acoustics.

4.2. Acoustic resonance frequencies of spherical resonator

The spherical acoustic resonator is a remarkably accurate tool for measuring several properties of gases, because the speed of sound can be obtained with an accuracy better than 10^{-5} through the acoustic resonance frequencies. In fact, a detailed model is needed, which was given for the first time in the literature in 1986 [7]. We will show here that the dispersion equation used by the authors to calculate the resonant frequencies is simply eq. (40), expressed in spherical coordinates.

We assume a spherical coordinate system with the origin at the center of a geometrically perfect rigid spherical shell. The solutions of eq. (19) can be written as follows:

$$\tau_a(r) = j_n(k_a r), \qquad \tau_h(r) = j_n(k_h r), \qquad \psi(\theta, \varphi) = Y_{nm}(\theta, \varphi) \tag{50a, b, c}$$

where $j_n(z)$ is the nth order spherical Bessel function, and $Y_{nm}(\theta, \varphi)$ is a spherical harmonic.

On the other hand, a quite lengthy but simple calculation, taking into account eq. (38), gives the solution of the four equations (18), which are redundant:

$$v_{vr}(r) = B\frac{j_n(k_v r)}{r}, \qquad v_{v\theta}(r) = v_{v\varphi}(r) = \frac{B}{n(n+1)}\partial_r[r j_n(k_v r)] \tag{51a, b}$$

where B is arbitrary constant which can be calculated in using any equation from among the three equations (36) and (37). We obtain:

$$B = -n(n+1)A_a(\alpha_a - \alpha_h)\, j_n(k_a R)/\partial_R[R j_n(k_v R)].$$

Note that the condition on the boundary surface given by eq. (39) is automatically satisfied because $v_{v\theta}(r) = v_{v\varphi}(r)$ for all values of r (including $r = R$).

Using eqs. (50) and (51), eq. (40) can now be expressed as:

$$\left(1 - \frac{\alpha_h}{\alpha_a}\right)\frac{n(n+1)}{1 + k_v R\dfrac{j_n'(k_v R)}{j_n(k_v R)}} = \frac{k_a R j_n'(k_a R)}{j_n(k_a R)} - \frac{\alpha_h}{\alpha_a}\frac{k_h R j_n'(k_h R)}{j_n(k_h R)} \tag{52}$$

where $j_n'(z)$ is the derivative of $j_n(z)$ with respect to z.

This 'exact' equation can be used to determine the wavenumber k_a, that is, the resonance frequencies.

Note that starting from eq. (34) instead of the second equation (35) permits us to take into account a shell admittance (as the cited authors did). On the other hand, other effects such as molecular vibrational relaxation can be taken into account by an approximate model for k_a (see Section 2).

4.3. Axial wavenumbers in infinite cylindrical waveguides

As we saw in Section 1, no 'exact' equation giving the complex axial wavenumbers for all kinds of modes (propagative or evanescent ones) in infinite cylindrical waveguides has been available yet. This last section provides such a result.

We assume a cylindrical coordinate system (r, φ, z), the axis of the infinite waveguide considered being the z-axis. The solutions of eq. (19) can be written as follows:

$$\tau_a(r) = J_m(\sqrt{k_a^2 - k_{az}^2}\, r), \qquad \tau_h(r) = J_m(\sqrt{k_h^2 - k_{az}^2}\, r), \qquad \psi(\varphi, z) = e^{\pm i k_{az} z}\, e^{\pm i m\varphi} \tag{53a, b, c}$$

where $J_m(z)$ is the mth order cylindrical Bessel function, and where the sign + or − depends on the traveling direction of the waves considered.

On the other hand, a quite lengthy but simple calculation, taking into account eq. (38), gives the solutions of the four (redundant) equations (18), which must also satisfy the boundary conditions (39). Hence, we can write:

$$v_{vz} = \alpha\frac{\mathscr{C}_m(R)}{k_v^2} J_m(\sqrt{k_v^2 - k_{az}^2}\, r), \qquad v_{v\varphi} = \frac{\alpha}{k_v^2 - k_{az}^2}\left[r\partial_r - \frac{k_{az}^2}{k_v^2}\mathscr{C}_m(R)\right] J_m(\sqrt{k_v^2 - k_{az}^2}\, r) \tag{54a, b}$$

$$v_{vr} = \frac{\alpha}{k_v^2 - k_{az}^2}\left[\frac{m^2}{r} - \frac{k_{az}^2}{k_v^2}\mathscr{C}_m(R)\partial_r\right] J_m(\sqrt{k_v^2 - k_{az}^2}\, r) \tag{54c}$$

where

$$\mathscr{C}_m(R) = R\partial_R J_m(\sqrt{k_\mathrm{v}^2 - k_\mathrm{az}^2}\, R)/J_m(\sqrt{k_\mathrm{v}^2 - k_\mathrm{az}^2}\, R).$$

The constant α can be calculated in using any equation from among the three equations (36) and (37). We obtain:

$$\alpha = -A_\mathrm{a}(\alpha_\mathrm{a} - \alpha_\mathrm{h}) \frac{k_\mathrm{v}^2 J_m(\sqrt{k_\mathrm{a}^2 - k_\mathrm{az}^2}\, R)}{R\partial_R J_m(\sqrt{k_\mathrm{v}^2 - k_\mathrm{az}^2}\, R)}. \tag{55}$$

Using eqs. (53) and (54), eq. (40) can now be expressed as:

$$\left(1 - \frac{\alpha_\mathrm{h}}{\alpha_\mathrm{a}}\right)\frac{k_\mathrm{v}^2}{\chi_\mathrm{v}^2}\left[m^2 \frac{J_m(\chi_\mathrm{v}R)}{\chi_\mathrm{v}R\, J'_m(\chi_\mathrm{v}R)} - \frac{k_\mathrm{az}^2}{k_\mathrm{v}^2}\frac{\chi_\mathrm{v}R\, J'_m(\chi_\mathrm{v}R)}{J_m(\chi_\mathrm{v}R)}\right] = \frac{\chi_\mathrm{a}R\, J'_m(\chi_\mathrm{a}R)}{J_m(\chi_\mathrm{a}R)} - \frac{\alpha_\mathrm{h}}{\alpha_\mathrm{a}}\frac{\chi_\mathrm{h}R\, J'_m(\chi_\mathrm{h}R)}{J_m(\chi_\mathrm{h}R)} \tag{56}$$

with

$$\chi_\mathrm{a} = \sqrt{k_\mathrm{a}^2 - k_\mathrm{az}^2}, \qquad \chi_\mathrm{h} = \sqrt{k_\mathrm{h}^2 - k_\mathrm{az}^2}, \qquad \chi_\mathrm{v} = \sqrt{k_\mathrm{v}^2 - k_\mathrm{az}^2}.$$

This 'exact' equation can be used for determining the complex axial wavenumber k_az, i.e. for determining the axial speeds and attenuations of all kinds of modes in circular waveguides, no matter what the transverse dimensions of the tubes are.

For the wide tube case and the upper wavelength range compared with the thickness of the boundary layers (roughly speaking), making approximations to the lowest orders possible, this last equation leads to the solution given in table 1 in the reference mentioned [6]. In this paper, the solution starts with the concept of equivalent boundary specific admittance, which appears to be appropriate only when the surface is 'locally planar' on the scale of an acoustic wavelength.

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