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DISSIPATIVE SYSTEMS, CONSERVATION LAWS AND SYMMETRIES

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Abstract—In a recent note "On Conservation Laws for Dissipative Systems", a new method of constructing conservation laws applicable to dissipative systems was proposed. It is the purpose of this present paper to explore how this new method, called the "Neutral Action Method", is related to the concept of symmetry, and how it embodies the classical methods for obtaining conservation laws of Noether and Bessel-Hagen which are applicable only to Lagrangian systems. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Conservation laws, i.e. divergence-free forms, are of the utmost importance in many fields of physics and mechanics. Mathematically, a conservation law of a physical system with four independent variables x, y, z, and t, for example, is an equation of the form

$$\operatorname{Div} \mathbf{P} = D_x P^x + D_y P^y + D_z P^z + D_t P^t = 0, \tag{1}$$

where $\mathbf{P} = (P^x, P^y, P^z, P^t)$ is a vector function that can depend on the independent variables, the dependent variables and derivatives of the dependent variables of the system. Physically, a conservation law states that the rate of change of P^t inside any domain is equal to the net flux of (P^x, P^y, P^z) through the surface of the domain. Uses and implications of these divergence-free forms are widely appreciated as in the case of the J, L, and M integrals of fracture mechanics. A systematic approach for constructing conservation laws was by use of the classical Noether's theorem (Noether, 1918). While providing a direct procedure for obtaining divergence-free expressions, Noether's approach is applicable only to Lagrangian systems, i.e., to systems possessing a Lagrangian function and governed by the Euler–Lagrange equations obtained variationally. Bessel-Hagen (1921) extended Noether's work by introducing the concept of divergence symmetries. Nonetheless, his generalization of Noether's theorem also operated only in the realm of Lagrangian systems.

Recently, a brief note entitled "On Conservation Laws for Dissipative Systems" (Honein *et al.*, 1991) introduced a new method for constructing conservation laws. This newly proposed method, subsequently referred to as the "Neutral Action (NA) Method", offers a systematic procedure for obtaining conservation laws valid for both dissipative systems (systems without a Lagrangian) as well as for Lagrangian systems. In fact, the NA method allows one to construct systematically divergence-free expressions that are valid for any system governed by a set of differential equations, regardless of whether they are Euler–Lagrange equations or not.

The purpose of this present paper is to establish how the NA method embodies the classical methods of Noether, with extension by Bessel-Hagen, and how it is related to the concept of symmetry.

As widely recognized, conservation laws are intimately related to symmetries. Within the classical framework of Noether, there is a one-to-one correspondence between conservation laws and symmetries (Olver, 1986). In this light, it can be expected that conservation laws obtained via the NA method are also symmetry-related. Utilizing the concept of a Gâteaux derivative, directional derivative of a function or functional referred to by Olver as the Fréchet derivative, it will be shown that there exists an "adjoint" relation between the condition for existence of conservation laws via the NA method and the condition for finding generalized symmetries of the governing equation of any system.

In addition to the relation between conservation laws derivable by the NA method and symmetries, it will also be shown that this method is related to classical methods of constructing conservation laws. If one transforms the condition for existence of a divergence-free expression, as required by Noether and Bessel-Hagen, into a slightly different form, this condition can be shown to be mathematically equivalent to that of the NA method when the governing equations are the Euler–Lagrange equations. Therefore, one can conclude that the NA method not only allows one to obtain conservation laws for non-Lagrangian systems, but also yields identical results as Noether and Bessel-Hagen if applied to Lagrangian systems.

In order to describe how the NA method is related to symmetries and to classical methods of constructing conservation laws, concepts such as infinitesimal generator, prolongations, symmetries, Gâteaux derivatives, and Noether's theorem will be briefly introduced first. A thorough presentation of these concepts is available in Olver (1986) or Bluman and Kumei (1989).

2. SYMMETRIES

Symmetry, by definition, is a map of the object into itself which leaves the object invariant. The symmetry of an object is the set of all transformations leaving the object invariant. In this paper, we are interested in two types of symmetries, namely, variational symmetries and symmetries of the differential equations. In a variational symmetry, the object which is left invariant is the action integral (integral of the Lagrangian density function over material space). In symmetries of differential equations, the solution space of the differential equations is left invariant. The symmetries of a set of differential equations is the set of all transformations which transforms solutions of the system into other solutions. These concepts will be illustrated in the sequel.

Symmetries can also be categorized as being geometric or generalized, depending on the character of the transformation functions. Given a system of m independent variables x^i $(i = 1, 2, \ldots, m)$ and n dependent variables u^k $(k = 1, 2, \ldots, n)$, we can subject this system to an infinitesimal transformation

$$x^i \to x^{i*} = x^i + \varepsilon\xi^i, \tag{2}$$

$$u^k \to u^{k*} = u^k + \varepsilon\phi^k, \tag{3}$$

where ε is an infinitesimal parameter.

If the transformation functions ξ^i and ϕ^k are functions of the independent and dependent variables only, the symmetries generated are called geometric or point symmetries. On the other hand, if ξ^i and ϕ^k are also dependent on derivatives of the dependent variables, we speak of generalized or Lie–Bäcklund symmetries. (It might also be relevant to note that if the transformations are functions of the independent variables, the dependent variables and first derivatives of the dependent variables, they are termed "contact symmetries".) All references to symmetries that follow in this paper will implicitly refer to generalized symmetries.

2.1. *Infinitesimal transformations*

If one applies an infinitesimal transformation as described by eqn (2) to a system with m independent variables x^i and n dependent variables u^k, one must realize that derivatives of the dependent variables in this system will also be transformed.

Using the multi-index notation introduced by Olver (1986), with $J = (j_1, j_2, \ldots, j_p)$ as an unordered p-tuple of integers, $1 \leqslant j_\alpha \leqslant m$ indicating which derivatives are being taken, $\#J = p$ indicating how many derivatives are being taken, a formula for all possible pth order partial derivatives of u^k can be given as

$$u_J^k \equiv \frac{\partial^p u^k}{\partial x^{j_1}\,\partial x^{j_2}\dots\partial x^{j_p}}.$$

Subjected to an infinitesimal transformation, u_J^k is transformed into u_J^{k*},

$$u_J^{k*} \equiv \frac{\partial^p u^{k*}}{\partial x^{j_1^*}\,\partial x^{j_2^*}\dots\partial x^{j_p^*}},$$

by

$$u_J^k \to u_J^{k*} = u_J^k + \varepsilon\Phi_k^J, \tag{4}$$

with

$$\Phi_k^J = D_J(\phi^k - \xi^i u_i^k) + \xi^i u_{J,i}^k,$$

$$u_{J,i}^k \equiv \frac{\partial u_J^k}{\partial x^i},$$

and $D_J \equiv D_{j_1}D_{j_2}\dots D_{j_p}$ being a total differential operator of pth order. Here D_{j_α} indicates a total differentiation with respect to x^{j_α}.

Having established how derivatives transform, any functional $f = f(x^i, u^k, u_J^k)$ can be shown to transform into $f^* = f(x^{i*}, u^{k*}, u_J^{k*})$ by means of the relation

$$f \to f^* = f + \varepsilon pr^{(p)}v(f), \tag{5}$$

where $pr^{(p)}v$ is the pth order prolongation given by

$$pr^{(p)}v = \xi^i\frac{\partial}{\partial x^i} + \phi^k\frac{\partial}{\partial u^k} + \Phi_k^J\frac{\partial}{\partial u_J^k}, \quad 1 \leqslant \#J \leqslant p, \tag{6}$$

and p is the highest order derivative of u^k appearing in f.

Alternatively, $pr^{(p)}v$ can also be written in its evolutionary form (Olver, 1986, p. 297)

$$pr^{(p)}v = pr^{(p)}v_Q + \xi^i D_i, \tag{7}$$

with

$$pr^{(p)}v_Q = (D_{J'}Q^k)\frac{\partial}{\partial u_{J'}^k}, \quad 0 \leqslant \#J' \leqslant p, \tag{8}$$

where

$$Q^k = \phi^k - u_i^k\xi^i \tag{9}$$

2.2. *Symmetries of differential equations*

For symmetries of a set of q differential equations

$$\Delta^\alpha(x^i, u^k, u_J^k) = 0, \quad \alpha = (1, 2, \dots, q),$$

we seek a transformation such that

$$\Delta^{\alpha}(x^{i^*}, u^{k^*}, u_J^{k^*}) = 0$$

From eqn (5), it follows immediately that the condition for symmetries of the set of differential equations is simply

$$pr^{(p)}v(\Delta^{\alpha}) = 0, \tag{10}$$

or in its evolutionary form

$$pr^{(p)}v_Q(\Delta^{\alpha}) + \xi^i D_i(\Delta^{\alpha}) = 0. \tag{11}$$

However, since Δ^{α} must be set equal to zero along the solutions of the system, the symmetry condition can also be written as

$$pr^{(p)}v_Q(\Delta^{\alpha}) = 0, \tag{12}$$

for all u^k satisfying $\Delta^{\alpha} = 0$.

Therefore, in order to determine the symmetry group of Δ^{α}, one only needs to solve eqn (12) for the unknown functions Q^k. In this sense, Q^k can be regarded as the characteristic functions for symmetries of differential equations.

2.3. *Variational symmetries*

As previously stated, a transformation group defines a variational symmetry of a Lagrangian functional $L(x^i, u^k, u_J^k)$ if the transformation leaves the action integral A invariant for an arbitrary domain Ω. Mathematically, this condition is written as

$$A = \int_{\Omega} L\,\mathrm{d}V = \int_{\Omega^*} L^*\,\mathrm{d}V^*, \tag{13}$$

with $\mathrm{d}V$ and $\mathrm{d}V^*$ being volume differentials in Ω and Ω^*, respectively.

By eqn (5),

$$L^* = L + \varepsilon pr^{(p)}v(L), \tag{14}$$

and it is known that

$$\mathrm{d}V^* = J\,\mathrm{d}V, \tag{15}$$

where J is the Jacobian of the transformation. Given the infinitesimal transformation, eqn (2), the Jacobian is found to be

$$J = 1 + \varepsilon D_i \xi^i. \tag{16}$$

With eqn (14) and eqn (16), it follows immediately that the condition for finding variational symmetries, eqn (13), can be stated as

$$pr^{(p)}v(L) + LD_i\xi^i = 0, \tag{17}$$

or in its evolutionary form

$$pr^{(p)}v_Q(L) + D_i(L\xi^i) = 0. \tag{18}$$

3. CLASSICAL METHODS FOR ESTABLISHING CONSERVATION LAWS

3.1. *Noether's first theorem*

Noether's first theorem provides the classical procedure for obtaining conservation laws for Lagrangian systems. Given a Lagrangian function L, Noether asserted that if the action integral remains invariant under a set of infinitesimal transformations of both the dependent and independent variables, then a divergence-free expression exists in the space of the independent variables. Noether's requirement for existence of conservation laws is identical to that for variational symmetries of the Lagrangian function L.

In short, Noether requires that

$$pr^{(p)}v_Q(L) + D_i(L\xi^i) = 0. \tag{19}$$

Using eqn (8) and after some manipulations (Olver, 1986, p. 278), the above can be rewritten as

$$pr^{(p)}v_Q(L) = Q^k E^k(L) + D_i A^i, \tag{20}$$

where

$$\begin{aligned} E^k(L) &= (-D)_J \frac{\partial L}{\partial u_J^k}, && 0 \leqslant \#J \leqslant p, \\ (-D)_J &= D_J, && \text{for}\,\#J = \text{even}, \\ (-D)_J &= -D_J, && \text{for}\,\#J = \text{odd}, \end{aligned}$$

and A^i is some known function of L and Q^k.

Noether's conditions for existence of conservation laws, eqn (19), can now be stated as

$$Q^k E^k(L) + D_i(A^i + L\xi^i) = 0. \tag{21}$$

Upon closer inspection, the term $E^k(L)$ is the Euler operator operating on L, i.e., $E^k(L) = 0$ yields the Euler–Lagrange equations that govern the system. Therefore, if the condition of eqn (19) holds for some infinitesimal transformations, then we can always construct conservation laws in the form

$$D_i(A^i + L\xi^i) = 0. \tag{22}$$

Since the condition for existence of conservation laws, as required by Noether, is identical to that of variational symmetries, every transformation group that yields a variational symmetry of the Lagrangian will also provide the associated conservation law for the system, and vice versa. This indicates the one-to-one correspondence of conservation laws and symmetries within the framework of Noether.

3.2. *Bessel-Hagen's extension*

Bessel-Hagen (1921) extended Noether's theorem by inclusion of the so-called divergence symmetries. Instead of the requirement of Noether, eqn (19), he requires that

$$pr^{(p)}v_Q(L) + D_i(L\xi^i) = D_i B^i, \tag{23}$$

where B^i is a set of arbitrary functions.

Following developments similar to those for Noether's first theorem, Bessel-Hagen's condition can be stated as

$$Q^k E^k(L) + D_i(A^i + L\xi^i - B^i) = 0, \tag{24}$$

and the corresponding conservation law reads

$$D_i(A^i + L\xi^i - B^i) = 0. \tag{25}$$

4. THE NEUTRAL ACTION (NA) METHOD

Having introduced all the background material on symmetries and classical methods for constructing conservation laws, we are now ready to explore how the NA method proposed in the brief note "On Conservation Laws for Dissipative Systems" (Honein *et al.*, 1991) relates to these concepts.

4.1. *Conservation laws*

Given a system of q differential equations,

$$\Delta^\alpha(x^i, u^k, u^k_J) = 0, \quad \alpha = (1, 2, \ldots, q), \tag{26}$$

the NA method states that it is possible to construct conservation laws valid for the system governed by this set of differential equations in the form

$$f^\alpha \Delta^\alpha = D_i P^i = 0, \tag{27}$$

if

$$E^k(f^\alpha \Delta^\alpha) = 0, \tag{28}$$

where E^k is the Euler operator, and $f^\alpha = f^\alpha(x^i, u^k, u^k_J)$ are called the characteristics of conservation laws.

Since our objective is to construct some divergence-free expressions out of $f^\alpha\Delta^\alpha$, and since the Euler operator acting on any total divergence always gives a null result by calculus of variations, it follows that we should require the product $f^\alpha\Delta^\alpha$ to be a null Lagrangian. Equation (28) implies that $f^\alpha\Delta^\alpha$ is a null Lagrangian, i.e., it requires that the action integral of $f^\alpha\Delta^\alpha$,

$$A = \int_\Omega f^\alpha \Delta^\alpha \, \mathrm{d}V, \tag{29}$$

has zero variation, $\delta A = 0$. In other words, for existence of conservation laws, we try to construct a product of $f^\alpha\Delta^\alpha$ whose action integral has vanishing variation for any dependent variable u^k. Hence the name "Neutral Action" method given to this procedure.

In practice, given any set of differential equations, one only needs to solve eqn (28) for the unknown characteristics f^α, and then proceed to construct the conserved currents P^i valid for the system governed by this set of differential equations. Examples on application of this method have been given in Honein *et al.* (1991).

4.2. *Relation to symmetries*

In order to show how the proposed method is related to the concept of symmetry, the idea of a Gâteaux derivative will be useful.

A Gâteaux derivative of a differential functional is the directional derivative of that functional in jet-bundle space (the space of the independent variables, the dependent variables, and the derivatives of the dependent variables). Details on this subject can be found in Olver (1986).

In short, the Gâteaux derivative, $D_{\mathbf{P}}(\mathbf{Q})$, of a set of q differential functionals $\mathbf{P}[u]$,

$$\mathbf{P}[u] = P^{\alpha}(x^i, u^k, u^k_J), \quad \alpha = (1, 2, \ldots, q), \tag{30}$$

in the direction of another set of n (n being the number of dependent variables) differential functionals $\mathbf{Q}[u]$,

$$\mathbf{Q}[u] = Q^{\beta}(x^i, u^k, u^k_J), \quad \beta = (1, 2, \ldots, n), \tag{31}$$

is defined by its differential operator, $D_{\mathbf{P}}$, such that

$$D_{\mathbf{P}}(\mathbf{Q}) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathbf{P}[u + \varepsilon \mathbf{Q}[u]]. \tag{32}$$

It can be shown that the Gâteaux derivative can also be written as

$$D_{\mathbf{P}}(\mathbf{Q}) = (D_{\mathbf{P}})_{\alpha\beta} Q^{\beta} = \frac{\partial P^{\alpha}}{\partial u^{\beta}_J} D_J Q^{\beta}, \quad 0 \leqslant \# J \leqslant p, \tag{33}$$

with the differential operator being

$$(D_{\mathbf{P}})_{\alpha\beta} = \frac{\partial P^{\alpha}}{\partial u^{\beta}_J} D_J. \tag{34}$$

To define a (formal) adjoint differential operator for the Gâteaux derivative, $D^*_{\mathbf{P}}$, this adjoint operator must satisfy

$$\int_{\Omega} \mathbf{F} \cdot D_{\mathbf{P}}(\mathbf{Q})\, \mathrm{d}x = \int_{\Omega} \mathbf{Q} \cdot D^*_{\mathbf{P}}(\mathbf{F})\, \mathrm{d}x, \tag{35}$$

where $\mathbf{F}$ is any set of q differential functionals. The adjoint Gâteaux derivative $D^*_{\mathbf{P}}(\mathbf{F})$ is found to be

$$D^*_{\mathbf{P}}(\mathbf{F}) = (D^*_{\mathbf{P}})_{\beta\alpha} \mathbf{F}^{\alpha} = (-D)_J \left[\frac{\partial P^{\alpha}}{\partial u^{\beta}_J} F^{\alpha}\right], \quad 0 \leqslant \# J \leqslant p, \tag{36}$$

and the adjoint operator is given by

$$(D^*_{\mathbf{P}})_{\beta\alpha} = (-D)_J \cdot \frac{\partial P^{\alpha}}{\partial u^{\beta}_J}. \tag{37}$$

For any system governed by a set of differential equations, $\Delta^{\alpha} = 0$, the necessary and sufficient condition for existence of conservation laws is given by eqn (28), which must hold for all u^k. This condition can be written explicitly as

$$(-D)_J \Delta^{\alpha} \frac{\partial f^{\alpha}}{\partial u^k_J} + (-D)_J f^{\alpha} \frac{\partial \Delta^{\alpha}}{\partial u^k_J} = 0, \tag{38}$$

and in terms of Gâteaux derivatives

$$D^*_{\mathbf{f}}(\boldsymbol{\Delta}) + D^*_{\boldsymbol{\Delta}}(\mathbf{f}) = 0. \tag{39}$$

In Section 2.2, it is shown that for symmetries of a set of differential equations (in this case, the governing equations of the system of interest), the characteristics of symmetries Q^k must satisfy eqn (12),

$$pr^{(p)}v_Q(\Delta^\alpha) = D_J\left[Q^k \frac{\partial \Delta^\alpha}{\partial u_J^k}\right] = 0, \quad 0 \leqslant \#J \leqslant p, \tag{40}$$

for all u^k satisfying $\Delta^\alpha = 0$. In terms of the Gâteaux derivative, this condition on Q^k can be written as

$$D_\Delta(\mathbf{Q}) = 0. \tag{41}$$

Since the condition for symmetries of differential equations, eqn (41), exists only in the space of the solutions; and since we are interested in obtaining divergence-free expressions that are valid along the solutions of the system $\Delta^\alpha = 0$, a connection might exist between conservation laws via the NA method and symmetries of the governing differential equations in the solution space.

In the solution space, where $\Delta^\alpha = 0$, the term $D_{\mathbf{f}}^*(\Delta)$ appearing in eqn (39) is identically equal to zero. The condition for existence of conservation laws thus reduces to

$$D_\Delta^*(\mathbf{f}) = 0, \tag{42}$$

which must hold for all u^k satisfying $\Delta^\alpha = 0$.

Upon inspection of eqn (42) and eqn (41), with f^α being the characteristics of conservation laws, and Q^k being the characteristics of symmetries of the governing equations, it is apparent that the condition for existence of conservation laws and the condition for finding symmetries of the governing differential equations are adjoint to each other in the solution space. In other words, if we restrict ourselves to the solution space, there is a one-to-one correspondence between the set of all f^α that satisfies the condition for existence of conservation laws and the set of all Q^k that characterizes symmetries of the governing equations. This establishes the connection between conservation laws via the NA method and the symmetries of the governing equations for any system of interest.

4.3. *Relation to classical methods*

Classical methods of constructing conservation laws as discussed in Section 3 are based on concepts of variational symmetry. On the surface, there seems to be no relation between the NA method and the classical methods. However, it has been noted in Olver (1986) that for Lagrangian systems, there is a correspondence between conservation laws and the symmetries of the governing differential equations similar to that discussed for conservation laws derived via the NA method. Olver noted that the Gâteaux derivative for any Euler–Lagrange equation is a self-adjoint operator. Thus, for Lagrangian systems governed by such equations, there is a direct correspondence between conservation laws and symmetries of the governing equations in the solution space. Since both classical methods and the NA method can be shown to be related to symmetries of the governing equations, it is expected that these two methods should themselves be related. It is the purpose of this sub-section to show directly how the NA method relates to classical methods of constructing divergence-free expressions valid for Lagrangian systems.

As discussed in Section 3, for existence of conservation laws, both classical methods of Noether and Bessel-Hagen require that

$$Q^k E^k(L) = D_i P^i, \tag{43}$$

with

$$P^i = A^i + L\xi^i, \quad \text{for Noether (eqn 21)}, \tag{44}$$

$$P^i = A^i + L\xi^i - B^i, \quad \text{for Bessel-Hagen (eqn 24)}, \tag{45}$$

where Q^k, in the present context, can be regarded as the characteristics of conservation laws.

Since the Euler operator acting on any total divergence will always yield a null result, eqn (43) can also be written as

$$E^l(Q^k E^k(L)) = E^l(D_i P^i) = 0, \quad l = (1, 2, \ldots, n), \tag{46}$$

or explicitly

$$(-D)_J\left[E^k(L)\frac{\partial Q^k}{\partial u_J^l}\right] + (-D)_J\left[Q^k\frac{\partial E^k(L)}{\partial u_J^l}\right] = 0, \quad 0 \leqslant \#J \leqslant p, \tag{47}$$

and in terms of the Gâteaux derivative

$$D_{\mathbf{Q}}^*(\mathbf{E}(L)) + D_{\mathbf{E}(L)}^*(\mathbf{Q}) = 0, \tag{48}$$

which is a differential equation for the characteristics Q^k. Equation (48) is also the necessary and sufficient condition for generating conservation laws within the framework of Noether and Bessel-Hagen.

On comparison of the necessary and sufficient condition for generating conservation laws, eqn (48) for classical methods and eqn (39) for the NA method, it is obvious that these equations take the same form, with $\mathbf{\Delta} = \mathbf{E}(L)$ for Lagrangian systems and $\mathbf{Q} = \mathbf{f}$ being the characteristics of conservation laws.

Therefore, for Lagrangian systems, the requirement for existence of conservation laws by the NA method is mathematically equivalent to that of Noether and Bessel-Hagen. All conservation laws obtainable via the classical methods can also be obtained by the NA method. In short, the NA method not only extends systematic construction of conservation laws to non-Lagrangian systems, it also encompasses classical results of Noether and Bessel-Hagen for systems governed by Euler–Lagrange equations.

5. CONCLUSIONS

In this paper, it has been shown that the Neutral Action (NA) Method of constructing conservation laws as proposed by Honein *et al.* (1991) is related to the concept of symmetry. This new method is also shown to embody classical methods of obtaining divergence-free expressions based on Noether and Bessel-Hagen.

For any system governed by a set of differential equations, the condition for obtaining conservation laws is expressible in terms of Gâteaux derivatives. In the solution space for the system, this condition of existence, as imposed by the NA method, is adjoint to the condition for symmetries of the governing differential equations. In the space of solutions, characteristics of conservation laws by the NA method are adjointly related to the characteristics of the symmetry for the governing equations of the system of interest. This reveals the connection between conservation laws and symmetries in the present context.

It is also shown in this paper that the NA method of constructing conservation laws is related to the classical methods of Noether and Bessel-Hagen. The condition for construction of a divergence-free expression as imposed by Noether and Bessel-Hagen can be transformed into such a form that it is identical to the necessary and sufficient condition as required by the NA method. Therefore, for Lagrangian systems governed by the associated Euler–Lagrange equations, the classical method and the NA method of constructing conservation laws will yield identical results. The NA method not only extends systematic construction of conservation laws beyond Lagrangian systems, it also encompasses the classical procedures of Noether and Bessel-Hagen.

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REFERENCES

Bluman, G. W. and Kumei S. (1989). *Symmetries and Differential Equations*, Springer-Verlag, New York.

Bessel-Hagen, E. (1921). Über die Erhaltungssätze der Electrodynamik. *Math. Ann.* **84,** 258–276.

Honein, T., Chien, N. and Herrmann, G. (1991). On conservation laws for dissipative systems. *Physics Letter A.* **155,** 223–224.

Noether, E. (1918). Invariante Variationsprobleme. *Nachr. König. Gessell. Wissen. Göttingen, Math.-Phys. Kl.* 235–257. English translation: (1971) *Transport Theory and Stat. Phys.* **1,** 186–207.

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York.